Mem. Differential Equations Math. Phys. 44 (2008), 143–150

Malkhaz Ashordia and Shota Akhalaia

ON THE SOLVABILITY OF THE PERIODIC TYPE BOUNDARY VALUE PROBLEM FOR LINEAR IMPULSIVE SYSTEMS

Abstract. Effective necessary and sufficient conditions are established for the unique solvability of periodic type boundary value problems for linear impulsive systems.

რეზიუმე. წრფივი იმპულსური სისტემებისათვის მიღებულია პერიოდული ტიპის სასაზღვრო ამოცანების ცალსახად ამოხსნადობის ეფექტური აუცილებელი და საკმარისი პირობები.

2000 Mathematics Subject Classification: 34K13, 34K45.
Key words and phrases: Linear systems, impulsive equations, periodic type boundary value problems, unique solvability, effective necessary and sufficient conditions.

Let ω be a positive number and let $\tau_{ik} \in [(i-1)\omega, i\omega]$ $(i = 0, \pm 1, \pm 2, \dots;$ $k = 1, 2, \dots)$ be points such that $\tau_{ik} \le \tau_{ik+1}$, $\tau_{ik} = \tau_{i-1k} + \omega$. For the linear system of impulsive equations

$$\frac{dx}{dt} = P(t)x + q(t) \quad \text{for almost all} \quad t \in \mathbb{R} \setminus T, \tag{1}$$

$$x(\tau_{ik}+) - x(\tau_{ik}-) = G(\tau_{ik})x(\tau_{ik}) + g(\tau_{ik}) \qquad (i = 0, \pm 1, \pm 2, \dots;\ k = 1, 2, \dots) \tag{2}$$

we investigate the periodic boundary value problem

$$x(t + \omega) = x(t) \quad \text{for} \quad t \in \mathbb{R}, \tag{3}$$

where $T = \{\tau_{ik} : i = 0, \pm 1, \pm 2, \dots;\ k = 1, 2, \dots\}$, and $P = (p_{il})_{i,l=1}^n \in L_{loc}(\mathbb{R}, \mathbb{R}^{n\times n})$, $G : T \to \mathbb{R}^{n\times n}$ and $q = (q_i)_{i=1}^n \in L_{loc}(\mathbb{R}, \mathbb{R}^n)$, $g : T \to \mathbb{R}^n$ are ω-periodic matrix- and vector-functions, respectively, i.e.,

$$\begin{gathered} P(t + \omega) = P(t) \quad \text{for almost all} \quad t \in \mathbb{R} \setminus T, \\ G(\tau_{ik} + \omega) = G(\tau_{ik}) \quad (i = 0, \pm 1, \pm 2, \dots;\ k = 1, 2, \dots) \end{gathered} \tag{4}$$

and

$$\begin{gathered} q(t + \omega) = q(t) \quad \text{for almost all} \quad t \in \mathbb{R} \setminus T, \\ g(\tau_{ik} + \omega) = g(\tau_{ik}) \quad (i = 0, \pm 1, \pm 2, \dots;\ k = 1, 2, \dots). \end{gathered} \tag{5}$$

Reported on the Tbilisi Seminar on Qualitative Theory of Differential Equations on December 17, 2007.

Note that the following ω-type boundary value problem

$$x(t+\omega) = x(t) + c \quad \text{for} \quad t \in \mathbb{R},$$

where $c \in \mathbb{R}^n$, is reduced to the problem (3) by transformation

$$y(t) = x(t) - \frac{t}{\omega}\, c \quad \text{for} \quad t \in \mathbb{R},$$

so that we consider only the problem (3).

Along with the system (1), (2) we consider the corresponding homogeneous system

$$\frac{dx}{dt} = P(t)x \quad \text{for almost all} \quad t \in \mathbb{R} \setminus T, \tag{1_0}$$

$$x(\tau_{ik}+) - x(\tau_{ik}-) = G(\tau_{ik})x(\tau_{ik}) \quad (i = 0, \pm 1, \pm 2, \dots;\ k = 1, 2, \dots). \tag{2_0}$$

In the paper, we establish effective necessary and sufficient conditions for unique solvability of the problem (1), (2); (3). Analogous results are contained in [12]–[15] for general linear boundary value problems and ω-periodic boundary problems for systems of ordinary differential equations and functional differential equations.

Quite a number of issues of the theory of systems of differential equations with impulsive effect (both linear and nonlinear) have been studied sufficiently well (for survey of the results on impulsive systems see, e.g., [5]–[7], [10], [11], [16]–[20] and references therein).

Using the theory of so called generalized ordinary differential equations (see, e.g., [1]–[9]), we extend these results to systems of impulsive equations.

Throughout the paper the following notation and definitions will be used.

$R =]-\infty, +\infty[$, $R_+ = [0, +\infty[$; $[a, b]$ $(a, b \in R)$ is a closed segment.

$R^{n\times m}$ is the space of all real $n \times m$-matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm

$$\|X\| = \max_{j=1,\dots,m} \sum_{i=1}^{n} |x_{ij}|;$$

$O_{n\times m}$ (or O) is the zero $n \times m$ matrix.

If $X = (x_{ij})_{i,j=1}^{n,m} \in \mathbb{R}^{n\times m}$, then

$$|X| = (|x_{ij}|)_{i,j=1}^{n,m}\,.$$

$$R_+^{n\times m} = \Big\{(x_{ij})_{i,j=1}^{n,m} :\ x_{ij} \geq 0\ (i = 1, \dots, n;\ j = 1, \dots, m)\Big\}.$$

$R^n = R^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^n$; $R_+^n = R_+^{n\times 1}$.

If $X \in \mathbb{R}^{n\times n}$, then X^{-1}, $\det X$ and $r(X)$ are, respectively, the matrix inverse to X, the determinant of X and the spectral radius of X; I_n is the identity $n \times n$-matrix.

The inequalities between the matrices are understood componentwise.

$\mathop{\vee}\limits_a^b(X)$ is the total variation of the matrix-function $X : [a, b] \to R^{n\times m}$, i.e., the sum of total variations of the latter's components.

$X(t-)$ and $X(t+)$ are the left and the right limit of the matrix-function $X : \mathbb{R} \to R^{n\times m}$ at the point t;

$$d_1X(t) = X(t) - X(t-), \quad d_2X(t) = X(t+) - X(t).$$

$\mathrm{BV}([a,b], R^{n\times m})$ is the set of all matrix-functions of bounded variation $X : [a,b] \to R^{n\times m}$ (i.e., such that $\overset{b}{\underset{a}{\vee}}(X) < +\infty$);

$\mathrm{BV}_{loc}(\mathbb{R}, R^{n\times m})$ is the set of all matrix-functions $X : \mathbb{R} \to R^{n\times m}$ such that $\overset{b}{\underset{a}{\vee}}(X) < +\infty$ for every $a < b$; $a, b \in \mathbb{R}$;

$\widetilde{C}([a,b], D)$, where $D \subset R^{n\times m}$, is the set of all absolutely continuous matrix-functions $X : [a,b] \to D$;

$\widetilde{C}_{loc}(\mathbb{R} \setminus T, D)$ is the set of all matrix-functions $X : \mathbb{R} \to D$ whose restrictions to an arbitrary closed interval $[a,b]$ from $\mathbb{R} \setminus T$ belong to $\widetilde{C}([a,b], D)$;

$L([a,b], D)$ is the set of all measurable and integrable matrix-functions $X : [a,b] \to D$;

$L_{loc}(\mathbb{R}, D)$ is the set of all measurable and locally integrable matrix-functions $X : [a,b] \to D$.

A matrix-function is said to be continuous, integrable, nondecreasing, etc., if each of its components is such.

By a solution of the impulsive system (1), (2) we understand a continuous from the left vector-function $x \in \widetilde{C}_{loc}(\mathbb{R} \setminus T, \mathbb{R}^n) \cap \mathrm{BV}_{loc}(\mathbb{R}, \mathbb{R}^n)$ satisfying both the system (1) for a.a. $t \in \mathbb{R} \setminus T$ and the relation (2) for every $i \in \{0, \pm1, \pm2, \dots\}$ and $k \in \{1, 2, \dots\}$.

For every ω-periodic matrix-functions $X \in L_{loc}(\mathbb{R}, \mathbb{R}^{n\times n})$ and $Y : T \to \mathbb{R}^{n\times n}$ we put

$$\big[(X,Y)(t+\omega)\big]_l = \big[(X,Y)(t)\big]_l \quad \text{for} \quad t \in \mathbb{R} \quad (l = 1, 2, \dots),$$

where

$$\big[(X,Y)(t)\big]_0 = I_n \ \text{ for } \ 0 \le t \le \omega,$$
$$\big[(X,Y)(0)\big]_l = O_{n\times n} \ \ (l = 1, 2, \dots),$$
$$\big[(X,Y)(t)\big]_{l+1} = \int_0^t X(\tau) \cdot \big[(X,Y)(\tau)\big]_l \, d\tau +$$
$$+ \sum_{0 \le \tau_{1k} < t} Y(\tau_{1k}) \cdot \big[(X,Y)(\tau_{1k})\big]_l \ \text{ for } \ 0 < t \le \omega \ \ (l = 1, 2, \dots).$$

We assume that

$$\sum_{k=1}^{\infty} (\|G(\tau_{1k})\| + \|g(\tau_{1k})\|) < \infty \quad (k = 1, 2, \dots) \tag{6}$$

and

$$\det(I_n + G(\tau_{1k})) \ne 0 \quad (k = 1, 2, \dots). \tag{7}$$

The condition (7) guarantees the unique solvability of the Cauchy problem for the corresponding impulsive systems.

Theorem 1. *Let the conditions* (4)–(7) *hold. Then the system* (1), (2) *has a unique ω-periodic solution if and only if the corresponding homogeneous problem* $(1_0), (2_0)$ *has only the trivial ω-periodic solution, i.e.,*

$$\det(Y(0) - Y(\omega)) \neq 0, \tag{8}$$

where Y is a fundamental matrix of the system $(1_0), (2_0)$.

Corollary 1. *Let the conditions* (4)–(7) *hold and*

$$P(t)G(\tau_{1k}) = G(\tau_{1k})P(t) \quad \textit{for almost all} \quad t \in [0, \omega] \quad (k = 1, 2, \dots).$$

Let, moreover, there exists $t_0 \in [0, \omega]$ such that

$$P(t)\left(\int_{t_0}^{t} P(s)\,ds\right) = \left(\int_{t_0}^{t} P(s)\,ds\right)P(t) \quad \textit{for almost all} \quad t \in [0, \omega].$$

Then the system (1), (2) *has a unique ω-periodic solution if and only if*

$$\det\left[\exp\left(\int_{t_0}^{\omega} P(s)\,ds\right) \prod_{t_0 \leq \tau_{1k} < \omega} (I_n + G(\tau_{1k})) - \right.$$
$$\left. - \exp\left(-\int_{0}^{t_0} P(s)\,ds\right) \prod_{0 \leq \tau_{1k} < t_0} (I_n + G(\tau_{1k}))^{-1}\right] \neq 0.$$

Remark 1. If the homogeneous system $(1_0), (2_0)$ has a nontrivial ω-periodic solution, then for every vector-function $q \in L_{loc}(\mathbb{R}, \mathbb{R}^n)$ satisfying the condition (5) there exists a vector $c \in \mathbb{R}^n$ such that the system

$$\frac{dx}{dt} = P(t)x + q(t) - c \quad \text{for almost all} \quad t \in \mathbb{R} \setminus T,$$
$$x(\tau_{ik}+) - x(\tau_{ik}-) = G(\tau_{ik})x(\tau_{ik}) + g(\tau_{ik}) \quad (i = 0, \pm 1, \pm 2, \dots;\ k = 1, 2, \dots)$$

has no ω-periodic solution.

Definition 1. A matrix-function $\mathcal{G}_\omega : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{n \times n}$ is said to be the Green matrix of the problem $(1_0), (2_0); (3)$ if

$$\mathcal{G}_\omega(t+\omega, \tau+\omega) = \mathcal{G}_\omega(t, \tau), \quad \mathcal{G}_\omega(t, t+\omega) - \mathcal{G}_\omega(t, t) = I_n \quad \text{for} \quad t \quad \text{and} \quad \tau \in \mathbb{R},$$

and the matrix-function $\mathcal{G}_\omega(\cdot, \tau) : \mathbb{R} \to \mathbb{R}^{n \times n}$ is a fundamental matrix of the system $(1_0), (2_0)$.

If the conditions (4),(6)(for $g \equiv 0$) and (7) hold and the system $(1_0), (2_0)$ has only the trivial ω-periodic solution, then there exists a unique Green matrix and it admits the following representation

$$\mathcal{G}_\omega(t, \tau) = Y(t)\big(Y^{-1}(\omega)Y(0) - I_n\big)^{-1}Y^{-1}(\tau) \quad \text{for} \quad t \quad \text{and} \quad \tau \in \mathbb{R},$$

where Y is a fundamental matrix of the system $(1_0), (2_0)$.

Theorem 2. *The system* (1), (2) *has a unique* ω*-periodic solution if and only if the corresponding homogeneous problem* $(1_0), (2_0)$ *has only the trivial* ω*-periodic solution. If the latter condition holds, then the* ω*-periodic solution* x *of the system* (1), (2) *admits the representation*

$$x(t) = \int\limits_t^{t+\omega} d_s\mathcal{G}_\omega(t,s)\cdot f(s) + \sum_{t\le\tau_{ik}<t+\omega} (\mathcal{G}_\omega(t,\tau_{ik}+) - \mathcal{G}_\omega(t,\tau_{ik}))\cdot g(\tau_{ik}) \quad \text{for} \quad t\in\mathbb{R},$$

where $\mathcal{G}_\omega$ *is the Green matrix of the problem* $(1_0), (2_0); (3)$.

In general, it is quite difficult to verify the condition (8) directly even in the case where one is able to write out the fundamental matrix of the system $(1_0), (2_0)$ explicitly. Therefore it is important to seek for effective conditions which would guarantee the absence of nontrivial solutions of the homogeneous problem $(1_0), (2_0); (3)$. Analogous results for general linear boundary value problems for systems of ordinary differential equations belong to T. Kiguradze [15], and for ω-periodic boundary problem for systems of ordinary differential equations and functional differential equations they belong to I. Kiguradze [12]–[14].

Theorem 3. *Let the conditions* (4)–(7) *hold. Then the system* (1), (2) *has a unique* ω*-periodic solution if and only if there exist natural numbers* k *and* m *such that the matrix*

$$M_k = (-1)^l\Big(I_n - \sum_{i=0}^{k-1}\big[(P,G)(c_l)\big]_i\Big)$$

is nonsingular for some $l\in\{1,2\}$ *and*

$$r(M_{k,m}) < 1,$$

where $c_l = \omega(2-l)$ *and*

$$M_{k,m} = \big[(|P|,|G|)(c_l)\big]_m + \\ +\Big(\sum_{i=0}^{m-1}\big[(|P|,|G|)(c_l)\big]_i\Big)\cdot|M_k^{-1}|\big[(|P|,|G|)(c_l)\big]_k.$$

Corollary 2. *Let there exists a natural number* k *such that*

$$\big[(P,G)(c_l)\big]_i = 0 \quad (i=0,\dots,k-1)$$

and

$$\det\Big(\big[(P,G)(c_l)\big]_k\Big) \ne 0$$

for some $l\in\{1,2\}$, *where* $c_l = \omega(2-l)$. *Then there exists* $\varepsilon_0 > 0$ *such that the system*

$$\frac{dx}{dt} = \varepsilon P(t)x + q(t) \quad \text{for almost all} \quad t\in\mathbb{R}\setminus T,$$

$$x(\tau_{ik}+) - x(\tau_{ik}-) = \varepsilon G(\tau_{ik})x(\tau_{ik}) + g(\tau_{ik}) \quad (i=0,\pm1,\pm2,\dots;\ k=1,2,\dots)$$

has one and only one ω-periodic solution for every $\varepsilon \in]0, \varepsilon_0[$.

Corollary 3. *Let*

$$\det\Big(\int_0^\omega P(\tau)\,d\tau + \sum_{0\le\tau_{1k}<\omega} G(\tau_{1k})\Big) \neq 0.$$

Then the conclusion of Corollary 2 is true.

Theorem 4. *Let $P_0 \in L_{loc}(\mathbb{R}, \mathbb{R}^{n\times n})$ and $G_0 : T \to \mathbb{R}^{n\times n}$ be ω-periodic matrix-functions such that*

$$\sum_{k=1}^{\infty} (\|G_0(\tau_{1k})\|) < \infty,$$

$$\det(I_n + G_0(\tau_{1k})) \neq 0 \quad (k = 1, 2, \dots)$$

and the homogeneous system

$$\frac{dx}{dt} = P_0(t)x \quad \textit{for almost all} \quad t \in \mathbb{R} \setminus T, \tag{9}$$

$$x(\tau_{ik}+) - x(\tau_{ik}-) = G_0(\tau_{ik})x(\tau_{ik}) \quad (i = 0, \pm1, \pm2, \dots;\ k = 1, 2, \dots) \tag{10}$$

has only the trivial ω-periodic solution. Let, moreover, the ω-periodic matrix-functions $P \in L_{loc}(\mathbb{R}, \mathbb{R}^{n\times n})$ and $G : T \to \mathbb{R}^{n\times n}$ admit the estimate

$$\int_0^\omega |\mathcal{G}_{0\omega}(t,\tau)|\cdot\big|P(\tau)-P_0(\tau)\big|\,d\tau+\sum_{k=1}^{\infty}\big|\mathcal{G}_{0\omega}(t,\tau_{1k}+)\cdot(G_0(\tau_{1k})-G_0(\tau_{1k}))\big| \le M,$$

where $\mathcal{G}_{0\omega}$ is the Green matrix of the problem $(9), (10); (3)$, and $M \in \mathbb{R}_+^{n\times n}$ is a constant matrix such that

$$r(M) < 1.$$

Then the system $(1), (2)$ has one and only one ω-periodic solution.

To establish the results dealing with the boundary value problems for the impulsive system $(1), (2)$, we use the following concept.

It is easy to show that the vector-function x is a solution of the impulsive system $(1), (2)$ if and only if it is a solution of the linear system of so called generalized ordinary differential equations of the following form

$$dx(t) = dA(t)\cdot x(t) + df(t) \quad \text{for} \quad t \in \mathbb{R},$$

where the matrix-function A and vector-function f are defined by the equalities

$$A(0) = O_{n\times n}, \quad f(0) = 0;$$

$$A(t) = \int_{(i-1)\omega}^{t} P(\tau)\,d\tau + \sum_{(i-1)\omega\le\tau_{ik}<t} G(\tau_{ik})$$

$$\text{for} \quad t \in [(i-1)\omega, i\omega] \quad (i = 0, \pm1, \pm2, \dots),$$

$$f(t) = \int\limits_{(i-1)\omega}^{t} q(\tau)\,d\tau + \sum_{(i-1)\omega\le\tau_{ik}<t} g(\tau_{ik})$$

$$\text{for} \quad t \in [(i-1)\omega, i\omega] \quad (i = 0, \pm1, \pm2, \dots).$$

From this definitions it is evident that A and f are continuous from the left matrix- and vector- functions, respectively,

$$d_2A(\tau_{ik}) = G(\tau_{ik}), \quad d_2f(\tau_{ik}) = g(\tau_{ik}) \quad (i = 0, \pm1, \pm2, \dots;\ k = 1, 2, \dots),$$

$$d_2A(t) = O_{n\times n}, \quad d_2f(t) = 0 \quad \text{for} \quad t \in \mathbb{R}\setminus T,$$

and

$$A(t+\omega) = A(t) + A(\omega), \quad f(t+\omega) = f(t) + f(\omega) \quad \text{for} \quad t \in \mathbb{R}$$

because P, G and q, g are ω-periodic matrix- and vector- functions, respectively. Moreover, by the conditions (6) and (7) the matrix- and vector-functions A and f have bounded total variations on every closed interval from $\mathbb{R}$ and the condition

$$\det\big(I_n + (-1)^j d_jA(t)\big) \neq 0 \ \text{ for } \ t \in \mathbb{R} \ \ (j = 1, 2)$$

holds.

So that, the above given results follow from analogous results obtained in [8], [9] for generalized linear systems.

Acknowledgement

This work is supported by the Georgian National Science Foundation (Grant No. GNSF/ST06/3-002)

References

1. M. Ashordia, On the question of solvability of the periodic boundary value problem for a system of linear generalized ordinary differential equations. *Mem. Differential Equations Math. Phys.* **6** (1995), 121–123.
2. M. T. Ashordia, A criterion for the solvability of a multipoint boundary value problem for a system of generalized ordinary differential equations. (Russian) *Differ. Uravn.* **32** (1996), No. 10, 1303–1311, 1437; English transl.: *Differential Equations* **32** (1996), No. 10, 1300–1308 (1997)
3. M. Ashordia, Conditions of existence and uniqueness of solutions of the multipoint boundary value problem for a system of generalized ordinary differential equations. *Georgian Math. J.* **5** (1998), No. 1, 1–24.
4. M. Ashordia, On the solvability of linear boundary value problems for systems of generalized ordinary differential equations. *Funct. Differ. Equ.* **7** (2000), No. 1-2, 39–64 (2001).
5. M. Ashordia, On the general and multipoint boundary value problem for linear systems of generalized ordinary differential equations, linear impulsive and linear difference systems. *Mem. Differential Equations Math. Phys.* **36**(2005), 1–80.
6. M. Ashordia, On Lyapunov stability of a class of linear systems of generalized ordinary differential equations and linear impulsive systems.*Mem. Differential Equations Math. Phys.* **31**(2004), 139–144.
7. M. Ashordia, Lyapunov stability of systems of linear generalized ordinary differential equations. *Computers and Mathematics with Applications.* **50** (2005), 957–982.

8. M. Ashordia, On the solvability of a multipoint boundary value problem for systems of nonlinear generalized ordinary differential equations. *Mem. Differential Equations Math. Phys.* **43**(2008), 143–152.
9. M. Ashordia, On the solvability of the periodic type boundary value problem for linear systems of generalized ordinary differential equations. *Mem. Differential Equations Math. Phys.* **44**(2008), 133–142.
10. M. Ashordia and G. Ekhvaia, Criteria of correctness of linear boundary value problems for systems of impulsive equations with finite and fixed points of impulses actions. *Mem. Differential Equations Math. Phys.* **37**(2008), 154–157.
11. M. Ashordia and G. Ekhvaia, On the solvability of a multipoint boundary value problem for systems of nonlinear impulsive equations with finite and fixed points of impulses actions. *Mem. Differential Equations Math. Phys.* **43**(2008), 153–158.
12. I. Kiguradze, Boundary value problems for systems of ordinary differential equations. (Russian) *Itogi Nauki Tekh., Ser. Sovrem. Probl. Mat., Novejshie Dostizh.* **30** (1987), 3–103; English transl.: *J. Sov. Math.* **43** (1988), No. 2, 2259–2339.
13. I. Kiguradze, The initial value problem and boundary value problems for systems of ordinary differential equations. Vol. I. Linear theory. (Russian) *Metsniereba, Tbilisi,* 1997.
14. I. T. Kiguradze and B. Puža, On the some boundary value problems for the system of ordinary differential equations. (Russian) *Differentsial'nye Uravneniya* **12**(1976), No. 12, 2139–2148.
15. T. Kiguradze, Some boundary value problems for systems of linear partial differential equations of hyperbolic type. *Mem. Differential Equations Math. Phys.* **1** (1994), 144 pp.
16. V. Lakshmikantham, D. D. Baĭnov, and P. S. Simeonov, Theory of impulsive differential equations. *Series in Modern Applied Mathematics*, 6. *World Scientific Publishing Co., Inc., Teaneck, NJ,* 1989.
17. A. Samoilenko, S. Borysenko, C. Cattani, G. Matarazzo, and V. Yasinsky, Differential models. Stability, inequalities & estimates. *Naukova Dumka, Kiev,* 2001.
18. A. M. Samoĭlenko and N. A. Perestyuk, Impulsive differential equations. (Translated from the Russian) *World Scientific Series on Nonlinear Science. Series A: Monographs and Treatises,* 14. *World Scientific Publishing Co., Inc., River Edge, NJ,* 1995.
19. S. T. Zavalishchin and A. N. Sesekin, Impulse processes: models and applications. (Russian) *Nauka, Moscow,* 1991.
20. F. Zhang, Zh. Ma, and J. Yan, Functional boundary value problem for first order impulsive differential equations at variable times. *Indian J. Pure Appl. Math.* **34** (2003), No. 5, 733–742.

(Received 10.01.2008)

Authors' addresses:

M. Ashordia
A. Razmadze Mathematical Institute
1, M. Aleksidze St., Tbilisi 0193, Georgia
E-mail: ashord@rmi.acnet.ge

M. Ashordia and Sh. Akhalaia
Sukhumi State University
12, Jikia St., Tbilisi 0186, Georgia
E-mail: akhshi@posta.ge