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ASYMPTOTIC REPRESENTATIONS OF THE SOLUTIONS OF A CLASS OF THE SECOND ORDER NONAUTONOMOUS DIFFERENTIAL EQUATIONS

Abstract. For the second order nonlinear nonautonomous differential equation

$$y'' = \alpha_0 p(t) y |\ln |y||^{\sigma},$$

where $\alpha_0 \in \{-1, 1\}$, $\sigma \in \mathbb{R}$ and $p : [a, \omega[\to]0, +\infty[$ is a continuous function, asymptotic representations of solutions are established as $t \to +\infty$.

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[illegible]

$$y'' = \alpha_0 p(t) y |\ln |y||^{\sigma},$$

[illegible] $\alpha_0 \in \{-1, 1\}$, $\sigma \in \mathbb{R}$, [illegible] $p : [a, \omega[\to]0, +\infty[$ [illegible], [illegible] $t \to +\infty$.

The differential equation

$$y'' = \alpha_0 p(t) y |\ln |y||^\sigma \tag{1}$$

is considered, where $\alpha_0 \in \{-1, 1\}$, $\sigma \in \mathbb{R}$, $p : [a, \omega[\to]0, +\infty[$ is a continuous function, $a < \omega \le +\infty$, $a > 1$ if $\omega = +\infty$ and $a > \omega - 1$ if $\omega < +\infty$.

Here the function $|\ln |y||^\sigma$ is slowly varied as $y \to 0$ and as $y \to \pm\infty$ [1]. Therefore we are to think that the equation (1) is in some sense similar to the linear differential equation of the second order

$$y'' = \alpha_0 p(t) y \tag{2}$$

in the investigation of the asymptotics of disappearing and boundless as $t \uparrow \omega$ solutions of the equation (1). Perhaps due to this fact it is not covered by the results of the works [2]–[5] that are devoted to the establishing of the asymptotics of the essentially nonlinear differential equations of the type

$$y'' = \alpha_0 p(t) \varphi(y),$$

where $\varphi : \Delta_Y \to]0, +\infty[$ (Δ_Y is one-sided neighbourhood of Y, Y is either zero, or $\pm\infty$) is a twice continuously differentiable function.

We call a defined and different from zero on $[t_y, \omega[\subset [a, \omega[$ solution y of the equation (1) a $P_\omega(\lambda_0)$-solution if it satisfies the conditions

$$\lim_{t\uparrow\omega} y^{(k)}(t) = \begin{cases} \text{either } \ 0, \\ \text{or } \ \pm\infty \end{cases} \quad (k = 0, 1), \quad \lim_{t\uparrow\omega} \frac{(y'(t))^2}{y''(t) y(t)} = \lambda_0. \tag{3}$$

The aim of the work is to establish necessary and sufficient conditions for the existence of $P_\omega(0)$-solutions of the equation (1), and to find the asymptotic representations as $t \uparrow \omega$ for all such solutions and their derivatives of the first order.

Introduce the subsidiary notation

$$\pi_\omega(t) = \begin{cases} t, & \text{if } \omega = +\infty, \\ t - \omega, & \text{if } \omega < +\infty, \end{cases} \quad P_1(t) = \int_{A_1}^{t} p(\tau)\, d\tau, \quad P_2(t) = \int_{A_2}^{t} P_1(\tau)\, d\tau,$$

where

$$A_1 = \begin{cases} a, & \text{if } \displaystyle\int_a^\omega p(\tau)\, d\tau = +\infty, \\ \omega, & \text{if } \displaystyle\int_a^\omega p(\tau)\, d\tau < \infty, \end{cases} \qquad A_2 = \begin{cases} a, & \text{if } \displaystyle\int_a^\omega |P_1(\tau)|\, d\tau = +\infty, \\ \omega, & \text{if } \displaystyle\int_a^\omega |P_1(\tau)|\, d\tau < \infty. \end{cases}$$

Theorem 1. *If $\sigma \ne 1$, the following conditions are necessary and sufficient for the existence of $P_\omega(0)$-solutions of the equation* (1) :

$$\lim_{t\uparrow\omega} |P_2(t)|^{\frac{1}{1-\sigma}} = +\infty, \quad \lim_{t\uparrow\omega} \frac{P_1^2(t) |P_2(t)|^{\frac{\sigma}{1-\sigma}}}{p(t)} = 0. \tag{4}$$

For any such solution, the asymptotic representations as $t \uparrow \omega$

$$\begin{aligned} \ln|y(t)| &= \mu|(1-\sigma)P_2(t)|^{\frac{1}{1-\sigma}}[1+o(1)], \\ \frac{y'(t)}{y(t)} &= \alpha_0 P_1(t)|(1-\sigma)P_2(t)|^{\frac{\sigma}{1-\sigma}}[1+o(1)], \end{aligned} \tag{5}$$

where $\mu = \alpha_0 \operatorname{sign}[(1-\sigma)P_2(t)]$*, take place. Moreover, in the case* $A_1 = \omega$ $(A_1 = a)$ *there exists a one-parameter (two-parameter) family of such solutions.*

Remark 1. If $\omega = +\infty$, the conditions (4) are satisfied a fortiori, for example, in the case

$$p(t) = t^{-2}\ln^{\gamma} t, \quad \frac{1+\gamma}{1-\sigma} \geq 0, \quad \frac{\gamma+\sigma}{1-\sigma} < 0.$$

Proof of Theorem 1. *Necessity.* Let $y : [t_y, \omega[\to \mathbb{R} \setminus \{0\}$ be a $P_\omega(0)$-solution of the equation (1). Then the first of the conditions (3) holds and

$$\lim_{t\uparrow\omega} \frac{(y'(t))^2}{y''(t)y(t)} = 0. \tag{6}$$

Without loss of generality we can assume here that $y'(t)$ and $\ln|y(t)|$ are different from zero if $t \in [t_y, \omega[$. From this it follows that

$$\begin{aligned} &\Big(\frac{y'(t)}{y(t)|\ln|y(t)||^{\sigma}}\Big)' = \\ &\quad = \frac{y''(t)}{y(t)|\ln|y(t)||^{\sigma}}\Big(1 - \frac{(y'(t))^2}{y''(t)y(t)} - \sigma\frac{(y'(t))^2}{y''(t)y(t)\ln|y(t)|}\Big) \sim \\ &\qquad \sim \frac{y''(t)}{y(t)|\ln|y(t)||^{\sigma}} \ \text{ as } \ t \uparrow \omega. \end{aligned}$$

Therefore, according to (1) we have

$$\Big(\frac{y'(t)}{y(t)|\ln|y(t)||^{\sigma}}\Big)' = \alpha_0 p(t)[1+o(1)] \ \text{ as } \ t \uparrow \omega.$$

If we integrate this ratio from t_y to t, we will get

$$\frac{y'(t)}{y(t)|\ln|y(t)||^{\sigma}} = \alpha_0 P_1(t)[1+o(1)] + c \ \text{ as } \ t \uparrow \omega,$$

where c is a constant. If $A_1 = a$ (i. e. when $\lim\limits_{t\uparrow\omega} P_1(t) = +\infty$), then from here the asymptotic representation

$$\frac{y'(t)}{y(t)|\ln|y(t)||^{\sigma}} = \alpha_0 P_1(t)[1+o(1)] \ \text{ as } \ t \uparrow \omega \tag{7}$$

follows. We will show that this representation holds also if $A_1 = \omega$ (i.e., when $\lim\limits_{t\uparrow\omega} P_1(t) = 0$). Indeed, if it is not the case, then the representation

$$\frac{y'(t)}{y(t)|\ln|y(t)||^{\sigma}} = c + o(1) \ \text{ as } \ t \uparrow \omega,$$

where the constant $c \neq 0$, will hold. Then by virtue of (1) the representation

$$\frac{y''(t)}{y'(t)} = \frac{\alpha_0}{c} p(t)[1 + o(1)] \quad \text{as} \quad t \uparrow \omega$$

will hold too. From this representation, according to the condition $\int\limits_a^\omega p(\tau)\, d\tau < +\infty$, it follows that

$$\ln |y'(t)| = c_1 + o(1) \quad \text{as} \quad t \uparrow \omega,$$

where c_1 is some constant. But this is impossible because, by virtue of (3), the left side of this representation has an infinite limit as $t \uparrow \omega$.

Then, integrating (7) from t_y to t, we get, due to the conditions (3) and $\sigma \neq 1$, that

$$|\ln |y(t)||^{1-\sigma} = \Big[\alpha_0(1-\sigma)\,\mathrm{sign}(\ln |y(t)|)\Big] P_2(t)[1 + o(1)] \quad \text{as} \quad t \uparrow \omega.$$

From this it follows that as $t \uparrow \omega$ the first of the asymptotic representations (5) take place and, regarding (3), the first of the conditions (4) is valid. The second of the asymptotic representations (5) follows from the first one and (7).

At last note that by virtue of (1) and (5),

$$\frac{y''(t)y(t)}{(y'(t))^2} = \frac{\alpha_0 p(t)[1 + o(1)]}{P_1^2(t)|(1-\sigma)P_2(t)|^{\frac{\sigma}{1-\sigma}}} \quad \text{as} \quad t \uparrow \omega.$$

Therefore, due to (6), we have that the second of the conditions (4) holds.

Sufficiency. Let $\sigma \neq 1$ and the conditions (4) be satisfied. We will show that in this case there exist $P_\omega(0)$-solutions of the equation (1) that admit asymptotic representations (5) as $t \uparrow \omega$.

Using the transformation

$$\begin{aligned} &\ln |y(t)| = \mu|(1-\sigma)P_2(t)|^{\frac{1}{1-\sigma}}[1 + v_1(\tau)],\\ &\frac{y'(t)}{y(t)} = \alpha_0 P_1(t)|(1-\sigma)P_2(t)|^{\frac{\sigma}{1-\sigma}}[1 + v_2(\tau)], \quad \tau = \tau(t), \end{aligned} \tag{8}$$

to the equation (1), where

$$\tau(t) = \beta \ln |P_2(t)|, \quad \beta = \begin{cases} 1, & \text{if } A_2 = a,\\ -1, & \text{if } A_2 = \omega, \end{cases} \quad \mu = \alpha_0\, \mathrm{sign}[(1-\sigma)P_2(t)], \tag{9}$$

we will get the system of differential equations

$$\begin{cases} v_1' = \dfrac{\beta}{1-\sigma}(v_2 - v_1),\\ v_2' = \beta h(\tau)\Big[-\alpha_0 q(\tau)(1+v_2)^2 - \Big(1 + \dfrac{\sigma}{(1-\sigma)h(\tau)}\Big)(1+v_2) + (1+v_1)^\sigma\Big], \end{cases} \tag{10}$$

where

$$h(\tau(t)) = \frac{p(t)P_2(t)}{P_1^2(t)}, \quad q(\tau(t)) = \frac{P_1^2(t)|(1-\sigma)P_2(t)|^{\frac{\sigma}{1-\sigma}}}{p(t)}.$$

Here, according to the conditions (4) and the choice of the limits of integration A_i $(i = 1, 2)$,

$$\begin{aligned} \lim_{\tau\to+\infty} q(\tau) &= \lim_{t\uparrow\omega} q(\tau(t)) = 0, \\ \lim_{\tau\to+\infty} h(\tau)q(\tau) &= \lim_{t\uparrow\omega} h(\tau(t))q(\tau(t)) = \infty \end{aligned} \tag{11}$$

and

$$\int\limits_{\tau_0}^{+\infty} \beta h(\tau)\, d\tau = \int\limits_{t_0}^{\omega} \frac{p(t)}{P_1(t)}\, dt = \ln|P_1(t)|\Big|_{t_0}^{\omega} = \begin{cases} +\infty, & \text{if } A_1 = a, \\ -\infty, & \text{if } A_1 = \omega, \end{cases} \tag{12}$$

where $\tau_0 = \beta \ln|P_2(t_0)|$, $t_0 \in]a, \omega[$. Using (11) and (12), we reduce the system (10) with the help of the additional transformation

$$v_1 = z_1, \quad v_2 = z_2 + \sigma z_1 \tag{13}$$

to an almost triangle form

$$\begin{cases} z_1' = \dfrac{\beta}{1-\sigma}[z_2 + (\sigma - 1)z_1], \\ z_2' = \beta h(\tau)\Big[f(\tau) + c_1(\tau)z_1 + c_2(\tau)z_2 + Z(\tau, z_1, z_2)\Big], \end{cases}$$

where

$$f(\tau) = -\alpha_0 q(\tau) - \frac{\sigma}{(1-\sigma)h(\tau)}, \quad c_1(\tau) = -2\alpha_0\sigma q(\tau) - \frac{\sigma^2 + \sigma - 1}{(1-\sigma)h(\tau)},$$
$$c_2(\tau) = -1 - 2\alpha_0 q(\tau) - \frac{\sigma + 1}{(1-\sigma)h(\tau)},$$
$$Z(\tau, z_1, z_2) = -\alpha_0 q(\tau)(z_2 + \sigma z_1)^2 + \big[(1 + z_1)^\sigma - 1 - \sigma z_1\big].$$

By virtue of the conditions (11), (12) and the type of nonlinearity of Z in the case $A_1 = \omega$ $(A_1 = a)$, we get due to Theorem 1.3 and Remarks 1.1, 1.4, 1.5 from the work [6] that there exists a one-parameter (two-parameter) family of solutions $(z_1, z_2) : [\tau_1, +\infty[\to \mathbb{R}^2$ of this system of equations, where $\tau_1 \ge \tau_0$, that tend to zero as $\tau \to +\infty$. According to the transformations (13) and (8), for every such solution there exists a corresponding solution $y : [t_1, \omega[\to \mathbb{R}$ $(t_1 \in [t_0, \omega[)$ that admits as $t \uparrow \omega$ the asymptotic representations (5). Using this representations and the conditions (4), we get that it is a $P_\omega(0)$-solution. □

Remark 2. In the case $\sigma = 0$, from the second equation of the system (10) it follows that for the function $v_2(\tau(t))$ tending to zero as $t \uparrow \omega$ it is true that

$$v_2(\tau(t)) = \frac{1}{P_1(t)}\left[c - \alpha_0 \int\limits_{A_1}^{t} P_1^2(s)[1 + v_2(\tau(s))]^2\, ds\right],$$

where $c = 0$ if $A_1 = a$ and $c \in \mathbb{R}$ if $A_1 = \omega$. Due to the trasformations (9), this representation can be used for the improvement of the asymptotic representations (5) if $\sigma = 0$. For example, if the function p is from Remark 1, it helps to get the asymptotic representation of the form

$$y(t) = \exp\Big(-\frac{\alpha_0}{1+\gamma}\ln^{1+\gamma} t\Big)[c_1 + o(1)] \ \text{ as } \ t \to +\infty$$

for $P_\omega(0)$-solutions of the linear differential equation (2) if $-1 < \gamma < -\frac{1}{2}$, and the representation

$$y(t) = (\ln t)^{-\alpha_0}[c_1 + o(1)] \ \text{ as } \ t \to +\infty$$

if $\gamma = -1$, where $c_1 \in \mathbb{R}$. It is in concordance with the results from [7] (look Chapter 1, § 6, p. 184–185).

Theorem 2. *Let $\sigma \in \mathbb{R} \setminus \{0; 1\}$, the conditions* (4) *hold and there exist a finite limit*

$$\lim_{t\uparrow\omega} \frac{P_1^2(t)|(1-\sigma)P_2(t)|^{\frac{1+\sigma}{1-\sigma}}}{p(t)} = \rho, \ \ \text{where} \ \ \rho \neq -\frac{\sigma}{2}\operatorname{sign}[(1-\sigma)P_2(t)]. \tag{14}$$

Then the differential equation (1) *has at least one $P_\omega(0)$-solution that admits the following asymptotic representations as $t \uparrow \omega$*

$$\begin{aligned} y(t) &= \exp\big(\mu|(1-\sigma)P_2(t)|^{\frac{1}{1-\sigma}}\big)[C + o(1)],\\ y'(t) &= \alpha_0 P_1(t)|(1-\sigma)P_2(t)|^{\frac{\sigma}{1-\sigma}}\exp\big(\mu|(1-\sigma)P_2(t)|^{\frac{1}{1-\sigma}}\big)[C + o(1)], \end{aligned} \tag{15}$$

where $\mu = \alpha_0 \operatorname{sign}[(1-\sigma)P_2(t)]$ and $C = \pm\exp\big(\frac{\alpha_0\mu\rho}{\sigma}\big)$. Moreover, there exists a one-parameter family of such solutions in the case $A_1 = a$.

Proof. Using the transformation

$$\begin{aligned} \tau=\tau(t), \quad y(t)&=\exp\big(\mu|(1-\sigma)P_2(t)|^{\frac{1}{1-\sigma}}\big)[C+v_1(\tau)],\\ y'(t)=\alpha_0 P_1(t)|(1-\sigma)P_2(t)|^{\frac{\sigma}{1-\sigma}}&\exp\big(\mu|(1-\sigma)P_2(t)|^{\frac{1}{1-\sigma}}\big)[C+v_2(\tau)] \end{aligned} \tag{16}$$

of the equation (1), where

$$\tau(t) = \beta\ln|P_1(t)|, \quad \beta = \begin{cases} 1, & \text{if } A_1 = a,\\ -1, & \text{if } A_1 = \omega, \end{cases}$$
$$\mu = \alpha_0 \operatorname{sign}[(1-\sigma)P_2(t)], \quad C = \pm\exp\Big(\frac{\alpha_0\mu\rho}{\sigma}\Big),$$

we get the system of differential equations

$$\begin{cases} v_1' = \beta\varepsilon(\tau)(v_2 - v_1),\\ v_2' = \beta\Big[(C+v_1)(1+\delta(\tau))^\sigma\Big|1 + \dfrac{\ln|1+\frac{v_1}{C}|}{q(\tau)[1+\delta(\tau)]}\Big|^\sigma -\\ \qquad -(C+v_2)\Big(1+\varepsilon(\tau)+\dfrac{\alpha_0\sigma\varepsilon(\tau)}{q(\tau)}\Big)\Big], \end{cases} \tag{17}$$

where

$$\varepsilon(\tau(t)) = \frac{\alpha_0 P_1^2(t)|(1-\sigma)P_2(t)|^{\frac{\sigma}{1-\sigma}}}{p(t)},$$
$$q(\tau(t)) = \mu|(1-\sigma)P_2(t)|^{\frac{1}{1-\sigma}}, \quad \delta(\tau) = \frac{\alpha_0\mu\rho}{\sigma q(\tau)}.$$

Here

$$\tau'(t) > 0 \quad \text{if} \quad t \in]a,\omega[, \quad \lim_{t\uparrow\omega} \tau(t) = +\infty$$

and therefore, by virtue of (4) and (14), we have

$$\begin{gathered} \lim_{\tau\to+\infty} \varepsilon(\tau) = 0, \quad \lim_{\tau\to+\infty} \delta(\tau) = 0, \\ \lim_{\tau\to+\infty} |q(\tau)| = +\infty, \quad \lim_{\tau\to+\infty} \varepsilon(\tau)q(\tau) = \alpha_0\mu\rho. \end{gathered} \tag{18}$$

Consider the differential equation

$$r' = \beta\Big[\varepsilon(\tau)q(\tau) + \Big(1 + \frac{(\alpha_0\sigma+1)\varepsilon(\tau)}{q(\tau)}\Big)r - \\ - \frac{1}{q(\tau)}(1+\delta(\tau))^{\sigma-1}\Big(1 + \frac{\alpha_0\mu\rho+\sigma^2}{\sigma q(\tau)}\Big)r^2\Big].$$

Since the conditions (18) are valid due to Theorem 1.3 and Remark 1.4 from the work [6], this equation has at least one solution $r : [\tau_1, +\infty[\to \mathbb{R}\setminus\{0\}$, $\tau_1 \ge \tau_0$, that tends to $\alpha_0\mu\rho$ as $\tau \to +\infty$. Let $r(\tau)$ be such a solution. Consider the function $h(\tau) = \frac{r(\tau)}{q(\tau)}$. By such choice of the function h, the conditions

$$\lim_{\tau\to+\infty} h(\tau)q(\tau) = \alpha_0\mu\rho, \quad \int_{\tau_1}^{+\infty} \Big|\frac{h(\tau)}{q(\tau)}\Big| d\tau = +\infty \tag{19}$$

hold and the system (17), using the transformation

$$v_1 = z_2 + h(\tau)z_1, \quad v_2 = z_1, \tag{20}$$

can be reduced to the form

$$\begin{cases} z_1' = \beta\Big(f_1(\tau) + c_1(\tau)z_1 + c_2(\tau)z_2 + \dfrac{1}{q(\tau)}Z(\tau, z_1, z_2)\Big), \\ z_2' = \beta h(\tau)\Big(-f_1(\tau) + c_3(\tau)z_2 - \dfrac{1}{q(\tau)}Z(\tau, z_1, z_2)\Big), \end{cases} \tag{21}$$

where

$$f_1(\tau) = C\Big[(1+\delta(\tau))^{\sigma} - 1 - \varepsilon(\tau) - \frac{\alpha_0\sigma\varepsilon(\tau)}{q(\tau)}\Big],$$

$$c_1(\tau) = h(\tau)(1+\delta(\tau))^{\sigma-1}\Big(1+\delta(\tau)+\frac{\sigma h(\tau)}{q(\tau)}\Big) - 1 - \varepsilon(\tau) - \frac{\alpha_0\sigma\varepsilon(\tau)}{q(\tau)},$$

$$c_2(\tau) = (1+\delta(\tau))^{\sigma-1}\Big(1+\delta(\tau)+\frac{\sigma}{q(\tau)}\Big),$$

$$c_3(\tau) = -\beta\frac{h'(\tau)}{h(\tau)} + 1 + \frac{\alpha_0\sigma\varepsilon(\tau)}{q(\tau)} - (1+\delta(\tau))^{\sigma-1}\Big(1+\delta(\tau)+\frac{\sigma}{q(\tau)}\Big)(1+h),$$

$$Z(\tau,z_1,z_2) = q(\tau)(1+\delta(\tau))^{\sigma}\bigg[(C+z_2+h(\tau)z_1)\Big(1+\frac{\ln|1+\frac{z_2+h(\tau)z_1}{C}|}{q(\tau)(1+\delta(\tau))}\Big)^{\sigma} - \\ - C - (z_2+h(\tau)z_1)\Big(1+\frac{\sigma}{q(\tau)(1+\delta(\tau))}\Big)\bigg].$$

By virtue of the conditions (18) and the first of the conditions (19), we have that the asymptotic representations

$$f_1(\tau) = o\Big(\frac{1}{q(\tau)}\Big), \quad c_1(\tau) = -1 + o(1), \quad c_2(\tau) = 1 + o(1),$$
$$c_3(\tau) = -\beta\frac{h'(\tau)}{h(\tau)} + \frac{2\alpha_0\mu\rho + \sigma + o(1)}{q(\tau)}$$

hold as $\tau \to +\infty$ and

$$\lim_{|z_1|+|z_2|\to 0} \frac{\partial Z(\tau,z_1,z_2)}{\partial z_i} = 0 \ \ (i=1,2) \ \text{ uniformly on } \ \tau \in [\tau_2,+\infty[\,.$$

Here $\tau_2 \geq \tau_1$ is some sufficiently large number. Furthermore, here $2\alpha_0\mu\rho + \sigma \neq 0$ and the second of the conditions (19) holds. Therefore, by the Theorem 1.3 and Remarks 1.4, 1.5 of the work [6], the system of differential equations (21) has at least one solution $(z_1, z_2) : [\tau_3, +\infty[\to \mathbb{R}^2$ $(\tau_3 \geq \tau_2)$ that tends to zero as $\tau \to +\infty$, and there exists a one-parameter family of such solutions in the case $A_1 = a$. According to the transformations (20) and (16), for every such solution there exists a corresponding solution y of the differential equation (1) that admits the asymptotic representations (15) as $t \uparrow \omega$. □

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