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LOCAL VARIATION FORMULAS FOR SOLUTION OF DELAY CONTROLLED DIFFERENTIAL EQUATION WITH MIXED INITIAL CONDITION

Abstract. In this work the variation formulas are proved for solution of non-linear controlled differential equation with variable delays and mixed initial condition.

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Introduction

In the present paper the differential equation

$$\dot{x}(t)=f\big(t,y(\tau_1(t)),\dots,y(\tau_s(t)),z(\sigma_1(t)),\dots,z(\sigma_m(t)),u(t)\big) \tag{1}$$

with the mixed initial condition

$$x(t)=\big(y(t),z(t)\big)^T=\big(\varphi(t),g(t)\big)^T,\ \ t\in[\tau,t_0),\quad x(t_0)=\big(y_0,g(t_0)\big)^T \tag{2}$$

is considered.

The condition (2) is called the mixed initial condition. It consists of two parts: the first one is the discontinuous part, $y(t)=\varphi(t)$, $t\in[\tau,t_0)$, $y(t_0)=y_0$, because in general $\varphi(t_0)\neq y_0$; the second part is the continuous part $z(t)=g(t)$, $t\in[\tau,t_0]$ because, always $z(t_0)=g(t_0)$.

The local formula of variation of solution, that is, a linear representation of variation of the solution of the problem (1)–(2) in a neighborhood of the right end of the main interval with respect to initial data and perturbation of control $u(t)$ is proved by the scheme given in [1].

An analogous formula for the equation

$$\dot{x}(t)=f\Big(t,y(\tau_1(t)),\dots,y(\tau_s(t)),z(\sigma_1(t)),\dots,z(\sigma_m(t))\Big) \tag{3}$$

with the initial condition (2) when variation of initial data and right-hand side of equation occurs is proved in [1].

It is important to note that the formula of variation which is proved in the present work doesn't follow from the formula proved in [1].

Formulas of variation for differential equations with delays for concrete cases of continuous and discontinuous initial conditions are obtained in [2]–[6].

Formulas of variation for controlled differential equations with delays, with continuous and discontinuous initial conditions are proved in [7], [8].

Formulas of variation of solution play an important role in the proof of necessary conditions of optimality [6], [9]–[12].

1. Formulation of Main Results

Let R^n_x be the n-dimensional vector space of points $x=(x^1,\dots,x^n)^T$, T means transpose; $O_1\subset R^k_y$, $O_2\subset R^e_z$, $G\subset R^r_u$ be open sets, $x=(y,z)^T$, $n=k+e$; $\tau_i(t)$, $i=\overline{1,s}$, $\sigma_j(t)$, $j=\overline{1,m}$, $t\in R^1_t$ be absolutely continuous scalar-valued functions and satisfy the following conditions:

$$\tau_i(t)\le t,\ \ \dot{\tau}_i(t)>0;\quad \sigma_j(t)\le t,\ \ \dot{\sigma}_j(t)>0.$$

Let $f(t,y_1,\dots,y_s,z_1,\dots,z_m,u)$ be an n-dimensional function satisfying the following conditions: for almost all $t\in I=[a,b]$ the function $f(t,\cdot):O_1^s\times O_2^m\times G\to R^n_x$ is continuously differentiable; for any

$$(y_1,\dots,y_s,z_1,\dots,z_m,u)\in O_1^s\times O_2^m\times G$$

the functions f, f_{y_i}, $i=\overline{1,s}$, f_{z_j}, $j=\overline{1,m}$, f_u, are measurable on I; for any compacts $K\subset O_1^s\times O_2^m$ and $M\subset G$ there exists a function $m_{K,M}(\cdot)\in$

$L(I,R_+)$, $R_+=[0,\infty)$, such that for any $(y_1,\dots,y_s,z_1,\dots,z_m,u)\in K\times M$ and for almost all $t\in I$ we have

$$\big|f(t,y_1,\dots,y_s,z_1,\dots,z_m,u)\big|+\\ +\sum_{i=1}^{s}\big|f_{y_i}(\,\cdot\,)\big|+\sum_{j=1}^{m}\big|f_{z_j}(\,\cdot\,)\big|+\big|f_u(\,\cdot\,)\big|\le m_{_{K,M}}(t).$$

Let $E_\varphi^{(k)}=E_\varphi(I_1,R_y^k)$ be the space of piecewise continuous functions $\varphi: I_1=[\tau,b]\to R_y^k$ with a finite number of discontinuity points of the first kind, equipped with the norm $\|\varphi\|=\sup\{|\varphi(t)|:\ t\in I_1\}$, $\tau=\min\{\tau_1(a),\dots,\tau_s(a),\sigma_1(a),\dots,\sigma_m(a)\}$.

Next, $\Delta_1=\{\varphi\in E_\varphi^{(k)}:\ \operatorname{cl}\varphi(I_1)\subset O_1\}$, $\Delta_2=\{g\in E_g^{(e)}=E_g^{(e)}(I_1;R_z^e):\ \operatorname{cl}g(I_1)\subset O_2\}$ are sets of initial functions, where $\varphi(I_1)=\{\varphi(t),\ t\in I_1\}$; let E_u be the space of measurable functions $u:I\to R_u^r$, satisfying the following condition: the set $\operatorname{cl}u(I)$ is compact in R_u^r, $\|u\|=\sup\{|u(t)|:\ t\in I\}$, $\Omega=\{u\in E_u:\ \operatorname{cl}u(I)\subset G\}$ is the set of controls.

To any element $\mu=(t_0,y_0,\varphi,g,u)\in A=I\times O_1\times\Delta_1\times\Delta_2\times\Omega$ we put in correspondence the differential equation

$$\dot x(t)=f\big(t,y(\tau_1(t)),\dots,y(\tau_s(t)),z(\sigma_1(t)),\dots,z(\sigma_m(t)),u(t)\big) \tag{1.1}$$

with the mixed initial condition

$$x(t)=\big(y(t),z(t)\big)^T=\big(\varphi(t),g(t)\big)^T,\ \ t\in[\tau,t_0),\ \ x(t_0)=\big(y_0,g(t_0)\big)^T. \tag{1.2}$$

Definition 1.1. Let $\mu=(t_0,y_0,\varphi,g,u)\in A$, $t_0<b$. A function $x(t;\mu)=\big(y(t;\mu),z(t;\mu)\big)^T$, $t\in[\tau,t_1]$, $t_1\in(t_0,b]$, where $y(t,\mu)\in O_1$, $z(t,\mu)\in O_2$, is called a solution, corresponding to the element μ, and defined on the interval $[\tau,t_1]$, if it satisfies the condition (1.2) on the interval $[\tau,t_0]$, it is absolutely continuous on the interval $[t_0,t_1]$ and almost everywhere on $[t_0,t_1]$ satisfies the equation (1.1).

In the space $E_\mu=R\times R_y^k\times E_\varphi^{(k)}\times E_g^{(e)}\times E_u$ we introduce the set of variations

$$V=\Big\{\delta\mu=(\delta t_0,\delta y_0,\delta\varphi,\delta g,\delta u)\in E_\mu:\ |\delta t_0|\le c,\ |\delta y_0|\le c,\ \|\delta\varphi\|\le c,\\ \delta g=\sum_{i=1}^{l}\lambda_i\delta g_i,\ |\lambda_i|\le c,\ i=\overline{1,l},\ \|\delta u\|\le c\Big\},$$

where $c>0$ is a fixed number and $\delta g_i\in E_g^{(e)}$, $i=\overline{1,l}$ are fixed points.

Lemma 1.1. *Let $x_0(t)$ be the solution corresponding to the element $\mu_0=(t_{00},y_{00},\varphi_0,g_0,u_0)\in A$, and defined on the interval $[\tau,t_{10}]$, t_{00}, $t_{10}\in(a,b)$. There exist numbers $\varepsilon_1>0$ and $\delta_1>0$, such that for any $(\varepsilon,\delta\mu)\in[0,\varepsilon_1]\times V$ we have $\mu_0+\varepsilon\delta\mu\in A$. In addition, to this element corresponds a solution $x(t;\mu_0+\varepsilon\delta\mu)$, defined on the interval $[\tau,t_{10}+\delta_1]\subset I_1$.*

This lemma follows from Theorem 1.3.2 (see [6, p. 17]).

Due to uniqueness, the solution $x(t;\mu_0)$, which is defined on $[\tau, t_{10}+\delta_1]$ is a continuation of the solution $x_0(t)$. Therefore we can assume that the solution $x_0(t)$ is defined on the whole interval $[\tau, t_{10}+\delta_1]$.

Lemma 1.1 allows us to introduce the increment of the solution $x_0(t) = x(t;\mu_0)$:

$$\Delta x(t) = \Delta x(t;\varepsilon\delta\mu) = x(t;\mu_0+\varepsilon\delta\mu) - x_0(t),$$
$$(t,\varepsilon,\delta\mu) \in [\tau, t_{10}+\delta_1]\times[0,\varepsilon_1]\times V.$$

In order to formulate main results, consider the following notation:

$$\begin{aligned}
\omega_{0i}^- &= \big(t_{00}, \underbrace{y_{00},\dots,y_{00}}_{i}, \underbrace{\varphi_0(t_{00}-),\dots,\varphi_0(t_{00}-)}_{p-i}, \varphi_0(\tau_{p+1}(t_{00}-)),\dots,\\
&\qquad \varphi_0(\tau_s(t_{00}-)), g_0(\sigma_1(t_{00}-)),\dots,g_0(\sigma_m(t_{00}-))\big),\ \ i=\overline{0,p},\\
\omega_{0i}^- &= \big(\gamma_i, y_0(\tau_1(\gamma_i)),\dots,y_0(\tau_{i-1}(\gamma_i)), y_{00}, \varphi_0(\tau_{i+1}(\gamma_i-)),\dots,\varphi_0(\tau_s(\gamma_i-)),\\
&\qquad z_0(\sigma_1(\gamma_i-)),\dots,z_0(\sigma_m(\gamma_i-))\big),\\
\omega_{1i}^- &= \big(\gamma_i, y_0(\tau_1(\gamma_i)),\dots,y_0(\tau_{i-1}(\gamma_i)), \varphi_0(t_{00}-), \varphi_0(\tau_{i+1}(\gamma_i-)),\dots,\\
&\qquad \varphi_0(\tau_s(\gamma_i-)), z_0(\sigma_1(\gamma_i-)),\dots,z_0(\sigma_m(\gamma_i-))\big),\ \ i=\overline{p+1,s},\\
\gamma_i(t) &= \tau_i^{-1}(t),\quad \gamma_i=\gamma_i(t_{00}),\quad \rho_j(t)=\sigma_j^{-1}(t),\quad \dot\gamma_i^- = \dot\gamma_i(t_{00}-);\\
\omega &= (t, y_1,\dots,y_s, z_1,\dots,z_m),\\
f_0[t] &= f\big(t, y_0(\tau_1(t)),\dots,y_0(\tau_s(t)), z_0(\sigma_1(t)),\dots,z_0(\sigma_m(t)) u_0(t)\big);\\
f_0(\omega) &= f\big(\omega, u_0(t)\big).
\end{aligned}$$

$$\lim_{\omega\to\omega_{0i}^-} f_0(\omega) = f_i^-,\quad \omega\in(t_{00}-\delta, t_{00}]\times O_1^s\times O_2^m,\quad i=\overline{0,p},\ \ \delta>0,$$
$$\lim_{(\omega_1,\omega_2)\to(\omega_{0i}^-,\omega_{1i}^-)} \big[f_0(\omega_1)-f_0(\omega_2)\big] = f_i^-,$$
$$\omega_1,\,\omega_2\in(\gamma_i-\delta,\gamma_i]\times O_1^s\times O_2^m,\ \ i=\overline{p+1,s}.$$

Similarly we can define ω_{0i}^+, ω_{1i}^+, $\dot\gamma_i^+$, f_i^+. In this case we have $t_{00}+$, γ_i+, and the right semi-intervals of points t_{00}, γ_i.

Theorem 1.1. *Let the following conditions hold:*

(1) $\gamma_i = t_{00}$, $i=\overline{1,p}$, $\gamma_{p+1}<\dots<\gamma_s<t_{10}$;
(2) *there exists a number* $\delta>0$ *such that* $\gamma_1(t)\le\dots\le\gamma_p(t)$, $t\in(t_{00}-\delta, t_{00}]$;
(3) *the quantities* $\dot\gamma_i^-$, f_i^-,$i=\overline{1,s}$ *are finite;*
(4) *the function* $g_0(t)$ *is absolutely continuous on the interval* $(t_{00}-\delta, t_{00}]$ *and there exists a finite limit* $\dot g_0^-$.

Then there exist numbers $\varepsilon_2\in(0,\varepsilon_1)$, $\delta_2\in(0,\delta_1)$ *such that for any*

$$(t,\varepsilon,\delta\mu)\in[t_{10}-\delta_2, t_{10}+\delta_2]\times[0,\varepsilon_2]\times V^-,$$

where $V^- = \{\delta\mu \in V:\ \delta t_0 \le 0\}$, *we have*

$$\Delta x(t) = \varepsilon\delta x(t;\delta\mu) + o(t;\varepsilon\delta\mu), \tag{1.3}$$

where

$$\delta x(t;\delta\mu) = Y(t_{00};t)\big[Y_0\delta y_0 + Y_1\delta g(t_{00}-)\big]+ \\ +\Big\{Y(t_{00};t)\Big[Y_1\dot g_0{}^- + \sum_{i=0}^{p}\big(\widehat{\gamma}\,{}^-_{i+1} - \widehat{\gamma}\,{}^-_i\big)f_i^-\Big]- \\ -\sum_{i=p+1}^{s} Y(\gamma_i;t)f_i^-\dot\gamma\,{}^-_i\Big\}\delta t_0 + \beta(t;\delta\mu), \tag{1.4}$$

$$\beta(t;\delta\mu) = \sum_{i=p+1}^{s}\int_{\tau_i(t_{00})}^{t_{00}} Y(\gamma_i(\xi);t)f_{0y_i}[\gamma_i(\xi)]\dot\gamma_i(\xi)\delta\varphi(\xi)\,d\xi+ \\ +\sum_{j=1}^{m}\int_{\sigma_j(t_{00})}^{t_{00}} Y(\rho_j(\xi);t)f_{0z_j}[\rho_j(\xi)]\dot\rho_j(\xi)\delta g(\xi)\,d\xi+ \\ +\int_{t_{00}}^{t} Y(\xi;t)f_{0u}[\xi]\delta u(\xi)\,d\xi, \tag{1.5}$$

$\widehat{\gamma}\,{}^-_0 = 1$, $\widehat{\gamma}\,{}^-_i = \dot\gamma\,{}^-_i$, $i = \overline{1,p}$, $\widehat{\gamma}\,{}^-_{p+1} = 0$; *next,* $\lim\limits_{\varepsilon\to0}\frac{o(t;\varepsilon\delta\mu)}{\varepsilon} = 0$ *uniformly with respect to* $(t,\delta\mu) \in [t_{10}-\delta_2, t_{10}+\delta_2]\times V^-$;

$$f_{0y_i}[t] = f_{y_i}\Big(t, y_0(\tau_1(t)),\dots,y_0(\tau_s(t)), z_0(\sigma_1(t)),\dots,z_0(\sigma_m(t)), u_0(t)\Big);$$

$Y(\xi;t)$ *is an* $n\times n$ *matrix-valued function satisfying the equation*

$$Y_\xi(\xi;t) = -\sum_{i=1}^{s} Y(\gamma_i(\xi);t)F_{y_i}[\gamma_i(\xi)]\dot\gamma_i(\xi)- \\ -\sum_{j=1}^{m} Y(\rho_j(\xi);t)F_{z_j}[\rho_j(\xi)]\dot\rho_j(\xi),\quad \xi\in[t_{00},t], \tag{1.6}$$

and the condition

$$Y(\xi,t) = \begin{cases} I_{n\times n}, & \xi = t,\\ \Theta_{n\times n}, & \xi > t,\end{cases} \tag{1.7}$$

where $I_{n\times n}$ *and* $\Theta_{n\times n}$ *are the identity and zero* $n\times n$ *matrices,* $F_{y_i} = (f_{0y_i}, \Theta_{n\times e})$, $F_{z_j} = (\Theta_{n\times k}, f_{0z_j})$, $Y_0 = (I_{k\times k}, \Theta_{e\times k})^T$, $Y_1 = (\Theta_{k\times e}, I_{e\times e})^T$.

The function $\delta x(t;\delta\mu)$ *is called the variation of the solution* $x_0(t)$, $t \in [t_{10}-\delta_2, t_{10}+\delta_2]$ *and the formula* (1.4) *is called the variation formula.*

Theorem 1.2. *Let the condition* (1) *and the following conditions hold:*

(5) *there exists a number* $\delta > 0$ *such that* $\gamma_1(t) \le \cdots \le \gamma_p(t)$, $t \in [t_{00}, t_{00}+\delta)$;

(6) *the quantities* $\dot\gamma_i^+$, f_i^+, $i=\overline{1,s}$ *are finite*

(7) *the function* $g_0(t)$ *is absolutely continuous on the interval* $[t_{00}, t_{00}+\delta)$ *and there exists a finite limit* $\dot g_0{}^+$.

Then there exist numbers $\varepsilon_2\in(0,\varepsilon_1)$ *and* $\delta_2\in(0,\delta_1)$ *such that for any* $(t,\varepsilon,\delta\mu)\in[t_{10}-\delta_2, t_{10}+\delta_2]\times[0,\varepsilon_2]\times V^+$, *where* $V^+=\{\delta\mu\in V:\delta t_0\ge 0\}$, *the formula* (1.3) *holds, where*

$$\begin{aligned}\delta x(t;\delta\mu) = {} & Y(t_{00};t)\big[Y_0\delta y_0 + Y_1\delta g(t_{00}+)\big]+\\ & +\Big\{Y(t_{00};t)\Big[Y_1\dot g_0{}^+ + \sum_{i=0}^{p}(\widehat\gamma_{i+1}^{\,+}-\widehat\gamma_i^{\,+})f_i^+\Big]-\\ & -\sum_{i=p+1}^{s}Y(\gamma_i;t)f_i^+\dot\gamma_i^+\Big\}\delta t_0+\beta(t;\delta\mu),\end{aligned}\tag{1.8}$$

$$\widehat\gamma_0^{\,+}=1,\quad \widehat\gamma_i^{\,+}=\dot\gamma_i^+,\quad i=\overline{1,p},\quad \widehat\gamma_{p+1}^{\,+}=0.$$

Theorems 1.1 and 1.2 immediately imply the following assertion.

Theorem 1.3. *Let the conditions* (1)–(7) *and the following conditions hold:*

(8) $$\sum_{i=0}^{p}(\widehat\gamma_{i+1}^{\,-}-\widehat\gamma_i^{\,-})f_i^- + Y_1\dot g_0{}^- = \sum_{i=0}^{p}(\widehat\gamma_{i+1}^{\,+}-\widehat\gamma_i^{\,+})f_i^+ + Y_1\dot g_0{}^+ =: f_0,$$

$$f_i^-\dot\gamma_i^- = f_i^+\dot\gamma_i^+ =: f_i,\quad i=\overline{p+1,s};$$

(9) *the functions* $\delta g_i(t)$, $i=\overline{1,l}$ *are continuous at the point* t_{00}.

Then there exist numbers $\varepsilon_2>0$, $\delta_2>0$ *such that for any* $(t,\varepsilon,\delta\mu)\in[t_{10}-\delta_2,t_{10}+\delta_2]\times[0,\varepsilon_2]\times V$ *the formula* (1.3) *holds, where*

$$\begin{aligned}\delta x(t;\delta\mu) = {} & Y(t_{00};t)\big[Y_0\delta y_0+Y_1\delta g(t_{00})\big]+\\ & +\Big\{Y(t_{00};t)f_0-\sum_{i=p+1}^{s}Y(\gamma_i;t)f_i\Big\}\delta t_0+\beta(t;\delta\mu).\end{aligned}$$

Some comments: Theorems 1.1 and 1.2 correspond to the case where at the point t_{00} right-hand and left-hand variations, respectively, take place. Theorem 1.3 corresponds to the case where at the point t_{00} double-sided variation takes place.

In the formula of variation proved in [1], for the equation (3) instead of the expression

$$\int_{t_{00}}^{t}Y(\xi;t)f_{0u}[\xi]\delta u(\xi)\,d\xi$$

(see (1.5)), we have

$$\int_{t_{00}}^{t}Y(\xi;t)\delta f[\xi]\,d\xi.$$

The formula (1.4) follows from the formula of variation obtained in [1] if the function f additionally satisfies the condition: $f_u(t, y_1, \dots, y_s, z_1, \dots, z_m, u)$ is continuously differentiable with respect to the variables $y_i \in O_1$, $i = \overline{1, s}$ and $z_j \in O_2$, $j = \overline{1, m}$.

In the present work formulas of variation are proved without of these conditions.

2. Auxiliary Lemmas

To any element $\mu = (t_0, y_0, \varphi, g, u) \in A$, let us correspond the functional-differential equation

$$\dot{\omega}(t) = f\Big(t, h(t_0, \varphi, q)(\tau_1(t)), \dots, h(t_0, \varphi, q)(\tau_s(t)), \\ h(t_0, g, v)(\sigma_1(t)), \dots, h_0(t_0, g, v)(\sigma_m(t)), u(t)\Big) \quad (2.1)$$

with the initial condition

$$\omega(t_0) = \big(q(t_0), v(t_0)\big)^T = x_0 = \big(y_0, g(t_0)\big)^T, \quad (2.2)$$

where the operator $h(\cdot)$ is defined by the formula

$$h(t_0, \varphi, q)(t) = \begin{cases} \varphi(t), & t \in [\tau, t_0), \\ q(t), & t \in [t_0, b]. \end{cases} \quad (2.3)$$

Definition 2.1. Let $\mu = (t_0, y_0, \varphi, g, u) \in A$. An absolutely continuous function $\omega(t) = \omega(t; \mu) = (q(t; \mu), v(t; \mu))^T \in (O_1, O_2)^T$, $t \in [r_1, r_2] \subset I$, where $(O_1, O_2)^T = \big\{x = (y, z)^T \in R_x^n : \ y \in O_1, \ z \in O_2\big\}$, is called a solution corresponding to the element $\mu \in A$, defined on the interval $[r_1, r_2]$, if $t_0 \in [r_1, r_2]$, the function $\omega(t)$ satisfies the condition (2.2) and the equation (2.1) almost everywhere on $[r_1, r_2]$.

Remark 2.1. Let $\omega(t; \mu)$, $t \in [r_1, r_2]$ be the solution corresponding to the element $\mu \in A$. Then the function

$$x(t; \mu) = \big(y(t; \mu), z(t; \mu)\big)^T = \\ = \big(h(t_0, \varphi, q(\cdot\,; \mu))(t), h(t_0, g, v(\cdot\,; \mu))(t)\big)^T, \ \ t \in [\tau, r_2] \quad (2.4)$$

is a solution of the equation (1.1) with the initial condition (1.2) (see (2.3)).

Lemma 2.1. *Let $\omega_0(t)$, $t \in [r_1, r_2] \subset (a, b)$ be the solution corresponding to the element $\mu_0 \in A$; let $K \subset (O_1, O_2)^T$ be a compact set containing some neighborhood of the set $((\varphi_0(I_1) \cup q_0([r_1, r_2])), (g_0(I_1) \cup v_0([r_1, r_2])))^T$ and let $M \subset G$ be a compact set containing some neighborhood of the set $\operatorname{cl} u_0(I)$. Then there exist numbers $\varepsilon_1 > 0$, $\delta_1 > 0$ such that for an arbitrary $(\varepsilon, \delta\mu) \in [0, \varepsilon_1] \times V$ to the element $\mu_0 + \varepsilon\delta\mu \in A$ there corresponds a solution*

$\omega(t;\mu_0+\varepsilon\delta\mu)$ *defined on* $[r_1-\delta_1,r_2+\delta_1]\subset I$*. Moreover,*

$$
\begin{gathered}
\big(\varphi(t),g(t)\big)=\big(\varphi_0(t)+\varepsilon\delta\varphi(t),g_0(t)+\varepsilon g(t)\big)\in K,\ \ t\in I_1,\\
u(t)=u_0(t)+\varepsilon\delta u(t)\in M,\ \ t\in I,\\
\omega(t;\mu_0+\varepsilon\delta\mu)\in K,\ \ t\in[r_1-\delta_1,r_2+\delta_1],\\
\lim_{\varepsilon\to 0}\omega(t;\mu+\varepsilon\delta\mu)=\omega(t,\mu_0)\\
\textit{uniformly for}\ \ (t,\delta\mu)\in[r_1-\delta_1,r_2+\delta_1]\times V.
\end{gathered}
\tag{2.5}
$$

This lemma follows from Lemma 1.3.2 (see [6, p. 18]).

Due to uniqueness, the solution $\omega(t;\mu_0)$ on the interval $[r_1-\delta_1,r_2+\delta_1]$ is a continuation of the solution $\omega(t;\mu_0)$, therefore the solution $\omega_0(t)$ is assumed to be defined on the whole interval $[r_1-\delta_1,r_2+\delta_1]$.

Let us define the increment of the solution $\omega_0(t)=\omega(t;\mu_0)$,

$$
\begin{gathered}
\Delta\omega(t)=\big(\Delta q(t),\Delta v(t)\big)^T=\Delta\omega(t;\varepsilon\delta\mu)=\omega(t;\mu_0+\varepsilon\delta\mu)-\omega_0(t),\\
(t,\varepsilon,\delta\mu)\in[r_1-\delta_1,r_2+\delta_1]\times[0,\varepsilon_1]\times V.
\end{gathered}
\tag{2.6}
$$

It is obvious that

$$
\lim_{\varepsilon\to 0}\Delta\omega(t;\varepsilon\delta\mu)=0 \tag{2.7}
$$

uniformly with respect to $(t,\delta\mu)\in[r_1-\delta_1,r_2+\delta_1]\times V$.

Lemma 2.2. *Let* $\gamma_i=t_{00}$*,* $i=\overline{1,p}$*,* $\gamma_{p+1}<\cdots<\gamma_s\le r_2$ *and let the conditions* 2)–4) *of Theorem* 1.1 *hold. Then there exist numbers* $\varepsilon_2>0$ *and* $\delta_2>0$ *such that for any* $(\varepsilon,\delta\mu)\in[0,\varepsilon_2]\times V^-$ *we have*

$$
\max_{t\in[t_{00},r_2+\delta_2]}|\Delta\omega(t)|=O(\varepsilon\delta\mu). \tag{2.8}
$$

Moreover,

$$
\begin{aligned}
\Delta\omega(t_{00})&=\varepsilon\big[Y_0\delta y_0+Y_1\delta g(t_{00}-)\big]+\\
&\quad+\varepsilon\Big[Y_1\dot g_0^{\,-}+\sum_{i=0}^{p}(\widehat\gamma_{\,i+1}^{\,-}-\widehat\gamma_i)f_i^-\Big]\delta t_0+o(\varepsilon\delta\mu).
\end{aligned}
\tag{2.9}
$$

Lemma 2.3. *Let* $\gamma_i=t_{00}$*,* $i=\overline{1,p}$*;* $\gamma_{p+1}<\cdots<\gamma_s\le r_2$*, and let*

conditions (5)–(7) of Theorem 1.2 hold. Then there exist numbers $\varepsilon_2>0$ and $\delta_2>0$ such that for any $(\varepsilon,\delta\mu)\in[0,\varepsilon_2]\times V^+$ we have

$$
\max_{t\in[t_0,r_2+\delta_2]}|\Delta\omega(t)|=O(\varepsilon\delta\mu). \tag{2.10}
$$

In addition,

$$
\Delta\omega(t_0)=\varepsilon\big[Y_0\delta y_0+Y_1\delta g(t_{00}+)+(Y_1\dot g_0^{\,+}-f_p^+)\delta t_0\big]+o(\varepsilon\delta\mu). \tag{2.11}
$$

Lemmas 2.2 and 2.3 are proved in analogue way as Lemmas 2.2 and 3.1, respectively (see [1]).

3. Proof of Theorem 1.1

Let $r_1 = t_{00}$, $r_2 = t_{10}$. Then for an arbitrary element $(\varepsilon, \delta\mu) \in [0, \varepsilon_1] \times V^-$ the corresponding solution $\omega(t; \mu_0 + \varepsilon\delta\mu)$ is defined on the interval $[t_{00} - \delta_1, t_{10} + \delta_1]$ and the solution $x(t; \mu_0 + \varepsilon\delta\mu)$ is defined on the interval $[\tau, t_{10} + \delta_1]$. Moreover,

$$\omega(t; \mu_0 + \varepsilon\delta\mu) = x(t, \mu_0 + \varepsilon\delta\mu), \quad t \in [t_{00}, t_{10} + \delta_1]$$

(see Lemma 1.1 , 2.1 and Remark 2.1).

Therefore

$$\Delta y(t) = \begin{cases} \varepsilon\delta\varphi(t), & t \in [\tau, t_0), \\ q(t; \mu_0 + \varepsilon\delta\mu) - \varphi_0(t), & t \in [t_0, t_{00}), \\ \Delta q(t), & t \in [t_{00}, t_{00} + \delta_1], \end{cases} \tag{3.1}$$

$$\Delta z(t) = \begin{cases} \varepsilon\delta g(t), & t \in [\tau, t_0), \\ v(t; \mu_0 + \varepsilon\delta\mu) - g_0(t), & t \in [t_0, t_{00}), \\ \Delta v(t), & t \in [t_{00}, t_{00} + \delta_1] \end{cases} \tag{3.2}$$

(see(2.6)).

By Lemma 2.2, there exist numbers

$$\varepsilon_2 \in (0, \varepsilon_1), \quad \delta_2 \in \big(0, \min(\delta_1, t_{10} - \gamma_s)\big) \tag{3.3}$$

such that the following inequalities hold

$$|\Delta y(t)| \le O(\varepsilon\delta\mu), \;\; \forall\, (t, \varepsilon, \delta\mu) \in [t_{00}, t_{10} + \delta_2] \times [0, \varepsilon_2] \times V^-, \tag{3.4}$$

$$|\Delta z(t)| \le O(\varepsilon\delta\mu), \;\; \forall\, (t, \varepsilon, \delta\mu) \in [\tau, t_{10} + \delta_2] \times [0, \varepsilon_2] \times V^- \tag{3.5}$$

(see (2.8), (3.1), (3.2)),

$$\Delta x(t_{00}) = \Delta\omega(t_{00}) = \varepsilon\bigg(Y_0\delta y_0 + Y_1\delta g(t_{00}-) + \\ + \Big[Y_1\,\dot g_{\,0}^{\,-} + \sum_{i=0}^{p}\big(\widehat\gamma_{\,i+1}^{\,-} - \widehat\gamma_{\,i}^{\,-}\big)f_i^-\Big]\delta t_0\bigg) + o(\varepsilon\delta\mu) \tag{3.6}$$

(see (2.9)).

The function $\Delta x(t)$ on the interval $[t_{00}, t_{10} + \delta_2]$ satisfies the equation

$$\frac{d}{dt}\,\Delta x(t) = \sum_{i=1}^{s} f_{0y_i}[t]\Delta y(\tau_i(t)) + \\ \sum_{j=1}^{m} f_{0z_j}[t]\Delta z(\sigma_j(t)) + \varepsilon f_{0u}[t]\delta u(t) + R(t; \varepsilon\delta\mu), \tag{3.7}$$

where

$$R(t; \varepsilon\delta\mu) = f\Big(t, y_0(\tau_1(t)) + \Delta y(\tau_1(t)), \dots, y_0(\tau_s(t)) + \Delta y(\tau_s(t)), \\ z_0(\sigma_1(t)) + \Delta z(\sigma_1(t)), \dots, z_0(\sigma_m(t)) + \Delta z(\sigma_m(t)), u_0(t)\Big) -$$

$$f_0[t]-\sum_{i=1}^{s} f_{0y_i}[t]\Delta y(\tau_i(t))-\sum_{j=1}^{m} f_{0z_j}[t]\Delta z(\sigma_j(t))-\varepsilon f_{0u}[t]\delta u(t). \quad (3.8)$$

We can represent the solution of (3.7) by the Cauchy formula in the following form:

$$\Delta x(t)=Y(t_{00};t)\Delta x(t_{00})+\varepsilon\int_{t_{00}}^{t} Y(\xi;t)f_{0u}[t]\delta u(\xi)\,d\xi+$$
$$+\sum_{i=0}^{2} h_i(t;t_0,\varepsilon\delta\mu),\quad t\in[t_{00},t_{10}+\delta_2], \quad (3.9)$$

where

$$\begin{cases} h_0=\displaystyle\sum_{i=p+1}^{s}\int_{\tau_i(t_{00})}^{t_{00}} Y(\gamma_i(\xi);t)f_{0y_i}[\gamma_i(\xi)]\dot\gamma_i(\xi)\Delta y(\xi)\,d\xi,\\ h_1=\displaystyle\sum_{j=1}^{m}\int_{\tau_i(t_{00})}^{t_{00}} Y(\rho_j(\xi);t)f_{0z_j}[\rho_j(\xi)]\dot\rho_j(\xi)\Delta z(\xi)\,d\xi,\\ h_2=\displaystyle\int_{t_{00}}^{t} Y(\xi;t)R(\xi;\varepsilon\delta\mu)\,d\xi. \end{cases} \quad (3.10)$$

$Y(\xi,t)$ is a matrix-valued function satisfying (1.6) and the condition (1.7).

The function $Y(\xi,t)$ is continuous on the set $\Pi=\{(\xi,t):\ a\le\xi\le t\le b\}$. Therefore

$$Y(t_{00},t)\Delta x(t_{00})=\varepsilon Y(t_{00};t)\Big\{Y_0\delta y_0+Y_1\delta g(t_{00}-)+$$
$$+\Big[Y_1\,\dot g_0^{\,-}+\sum_{i=0}^{p}\big(\widehat\gamma_{\,i+1}^{\,-}-\widehat\gamma_{\,i}^{\,-}\big)f_i^-\Big]\delta t_0\Big\}+o(t;\varepsilon\delta\mu) \quad (3.11)$$

(see (3.6)).

For $h_0(t;t_0,\varepsilon\delta\mu)$ we have

$$h_0(t;t_0,\varepsilon\delta\mu)=\sum_{i=p+1}^{s}\Big[\varepsilon\int_{\tau_i(t_{00})}^{t_0} Y(\gamma_i(\xi);t)f_{0y_i}[\gamma_i(\xi)]\dot\gamma_i(\xi)\delta\varphi(\xi)\,d\xi+$$
$$+\int_{t_0}^{t_{00}} Y(\gamma_i(\xi);t)f_{0y_i}[\gamma_i(\xi)]\dot\gamma_i(\xi)\Delta y(\xi)\,d\xi\Big]=$$
$$=\varepsilon\sum_{i=p+1}^{s}\int_{\tau_i(t_{00})}^{t_{00}} Y(\gamma_i(\xi);t)f_{0y_i}[\gamma_i(\xi)]\dot\gamma_i(\xi)\delta\varphi(\xi)\,d\xi+$$

$$+\sum_{i=p+1}^{s}\int\limits_{\gamma_i(t_0)}^{\gamma_i} Y(\xi;t)f_{0y_i}[\xi]\Delta y(\tau_i(\xi))\,d\xi+o(t;\varepsilon\delta\mu), \qquad (3.12)$$

where

$$o(t;\varepsilon\delta\mu)=-\varepsilon\sum_{i=p+1}^{s}\int\limits_{t_0}^{t_{00}} Y(\gamma_i(\xi);t)f_{0y_i}[\gamma_i(\xi)]\dot{\gamma}_i(\xi)\delta\varphi(\xi)\,d\xi.$$

Further, for $h_1(t;t_0,\varepsilon\delta\mu)$ we have

$$\begin{aligned}
h_1(t;t_0,\varepsilon\delta\mu)&=\sum_{j\in I_1\cup I_2}\int\limits_{\tau_j(t_{00})}^{t_{00}} Y(\rho_j(\xi);t)f_{0z_j}[\rho_j(\xi)]\dot{\rho}_j(\xi)\Delta z(\xi)\,d\xi=\\
&=\sum_{j\in I_1\cup I_2}\Bigg[\varepsilon\int\limits_{\tau_j(t_{00})}^{t_0} Y(\rho_j(\xi);t)f_{0z_j}[\rho_j(\xi)]\dot{\rho}_j(\xi)\delta g(\xi)\,d\xi+\\
&\qquad+\int\limits_{t_0}^{t_{00}} Y(\rho_i(\xi);t)f_{0z_j}[\rho_j(\xi)]\dot{\rho}_j(\xi)\Delta z(\xi)\,d\xi\Bigg]=\\
&=\sum_{j\in I_1\cup I_2}\big[\varepsilon\alpha_j(t)+\beta_j(t)\big],
\end{aligned}$$

where

$$\begin{aligned}
\alpha_j(t)&=\int\limits_{\sigma_j(t_{00})}^{t_0} Y(\rho_j(\xi);t)f_{0z_j}[\rho_j(\xi)]\dot{\rho}_j(\xi)\delta g(\xi)\,d\xi,\\
\beta_j(t)&=\int\limits_{t_0}^{t_{00}} Y(\rho_j(\xi);t)f_{0z_j}[\rho_j(\xi)]\dot{\rho}_j(\xi)\Delta z(\xi)\,d\xi.
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
\alpha_j(t)&=\int\limits_{\sigma_j(t_{00})}^{t_{00}} Y(\rho_j(\xi);t)f_{0z_j}[\rho_j(\xi)]\dot{\rho}_j(\xi)\delta g(\xi)\,d\xi-\\
&\qquad-\int\limits_{t_0}^{t_{00}} Y(\rho_j(\xi);t)f_{0z_j}[\rho_j(\xi)]\dot{\rho}_j(\xi)\delta g(\xi)\,d\xi,\\
\beta_j(t)&=o(t;\varepsilon\delta\mu)
\end{aligned}$$

(see (3.5)). Therefore

$$h_1(t;t_0,\varepsilon\delta\mu)=\varepsilon\sum_{i=1}^{m}\int\limits_{\sigma_j(t_{00})}^{t_{00}} Y(\rho_j(\xi);t)f_{0z_j}[\rho_j(\xi)]\dot{\rho}_j(\xi)\delta g(\xi)\,d\xi+ \\ +o(t;\varepsilon\delta\mu). \tag{3.13}$$

For $t\in[t_{10}-\delta_2,t_{10}+\delta_2]$ we have

$$h_2(t;t_0,\varepsilon\delta\mu)=\sum_{k=1}^{4}\alpha_k(t;\varepsilon\delta\mu), \tag{3.14}$$

where

$$\alpha_1(t;\varepsilon\delta\mu)=\int\limits_{t_{00}}^{\gamma_{p+1}(t_0)}\overline{\omega}(\xi;t,\varepsilon\delta\mu)\,d\xi,\quad \alpha_2(t;\varepsilon\delta\mu)=\sum_{i=p+1}^{s}\int\limits_{\gamma_i(t_0)}^{\gamma_i}\overline{\omega}(\xi;t,\varepsilon\delta\mu)\,d\xi,$$

$$\alpha_3(t;\varepsilon\delta\mu)=\sum_{i=p+1}^{s-1}\int\limits_{\gamma_i(t_0)}^{\gamma_{i+1}(t_0)}\overline{\omega}(\xi;t,\varepsilon\delta\mu)\,d\xi,\quad \alpha_4(t;\varepsilon\delta\mu)=\int\limits_{\gamma_s}^{t}\overline{\omega}(\xi;t,\varepsilon\delta\mu)\,d\xi$$

(see (3.10)),

$$\overline{\omega}(\xi;t,\varepsilon\delta\mu)=Y(\xi;t)R(\xi;\varepsilon\delta\mu).$$

Let us estimate $\alpha_1(t;\varepsilon\delta\mu)$

$$\big|\alpha_1(t;\varepsilon\delta\mu)\big|\le\|Y\|\int\limits_{t_{00}}^{\gamma_{p+1}(t_0)}\Bigg[\bigg|f\Big(t,y_0(\tau_1(t))+\Delta y(\tau_1(t)),\dots, \\ y_0(\tau_p(t))+\Delta y(\tau_p(t)),\varphi(\tau_{p+1}(t)),\dots,\varphi(\tau_s(t)), \\ z_0(\sigma_1(t))+\Delta z(\sigma_1(t)),\dots,z_0(\sigma_m(t))+\Delta z(\sigma_m(t)),u_0(t)+\varepsilon\delta u(t)\Big)- \\ -f\Big(t,y_0(\tau_1(t)),\dots,y_0(\tau_p(t)),\varphi_0(\tau_{p+1}(t)),\dots,\varphi_0(\tau_s(t)), \\ z_0(\sigma_1(t)),\dots,z_0(\sigma_m(t)),u_0(t)\Big)- \\ -\sum_{i=1}^{p}f_{0y_i}[t]\Delta y(\tau_i(t))-\varepsilon\sum_{i=p+1}^{s}f_{0y_i}[t]\delta\varphi(\tau_i(t))- \\ -\sum_{j=1}^{m}f_{0z_j}[t]\Delta z(\sigma_j(t))-\varepsilon f_{0u}[t]\bigg|\Bigg]\,dt\le$$

$$\le\|Y\|\int\limits_{t_{00}}^{t_{10}+\delta_2}\bigg\{\int\limits_0^1\Big|\frac{d}{d\xi}\,f\Big(t,y_0(\tau_1(t))+\xi\Delta y(\tau_1(t)),\dots,y_0(\tau_p(t))+\xi\Delta y(\tau_p(t)), \\ \varphi(\tau_{p+1}(t))+\xi\varepsilon\delta\varphi_0(\tau_{p+1}(t)),\dots,\varphi_0(\tau_s(t))+\xi\varepsilon\delta\varphi(\tau_s(t)),$$

$$z_0(\sigma_1(t))+\xi\Delta z(\sigma_1(t)),\dots,z_0(\sigma_s(t))+\xi\Delta z(\sigma_s(\xi)),u_0(t)+\xi\varepsilon\delta u(t)\Big)\Big|-$$
$$-\sum_{i=1}^{p}f_{0y_i}[t]\Delta y(\tau_i(t))-\varepsilon\sum_{i=p+1}^{s}f_{0y_i}[t]\delta\varphi(\tau_i(t))-$$
$$-\sum_{j=1}^{m}f_{0z_j}[t]\Delta z(\sigma_j(t))-\varepsilon f_{0u}[t]\delta u(t)\bigg|\bigg]\,d\xi\bigg\}\,dt\le$$
$$\le\|Y\|\int\limits_{t_{00}}^{t_{10}+\delta_2}\bigg\{\int\limits_0^1\bigg[\sum_{i=1}^{p}\Big|f_{y_i}\big(t,y_0(\tau_1(t))+\xi\Delta y(\tau_1(t)),\dots\big)-f_{0y_i}[t]\Big|\,\big|\Delta y(\tau_i(t))\big|+$$
$$+\varepsilon\sum_{i=p+1}^{s}\Big|f_{y_i}\big(t,y_0(\tau_1(t))+\xi\Delta y(\tau_1(t)),\dots\big)-f_{0y_i}[t]\Big|\,\big|\delta\varphi(\tau_i(t))\big|+$$
$$+\sum_{j=1}^{m}\Big|f_{z_j}\big(t,y_0(\tau_1(t))+\xi\Delta y(\tau_1(t)),\dots\big)-f_{0z_j}[t]\Big|\,\big|\delta z(\sigma_j(t))\big|+$$
$$+\varepsilon\Big|f_u\big(t,y_0(\tau_1(t))+\xi\Delta y(\tau_1(t)),\dots\big)-f_{0u}[t]\Big|\,\big|\delta u(t)\big|\bigg]\,d\xi\bigg\}\,dt\le$$
$$\le\|Y\|\bigg[O(\varepsilon\delta\mu)\sum_{i=1}^{p}\vartheta_i(t_{00};\varepsilon\delta\mu)+\varepsilon c\sum_{i=p+1}^{s}\vartheta_i(t_{00};\varepsilon\delta\mu)+$$
$$+O(\varepsilon\delta\mu)\sum_{j=1}^{m}\eta_j(t_{00};\varepsilon\delta\mu)+\varepsilon c\delta(t_{00};\varepsilon\delta\mu)\bigg],\qquad(3.15)$$

where

$$\|Y\|=\sup_{(\xi,t)\in\Pi}|Y(\xi,t)|,$$
$$\vartheta_i(t_{00};\varepsilon\delta\mu)=\int\limits_{t_{00}}^{t_{10}+\delta_2}\bigg[\int\limits_0^1\Big|f_{y_i}\big(t,y_0(\tau_1(t))+\xi\Delta y(\tau_1(t)),\dots\big)-f_{0y_i}[t]\Big|\,d\xi\bigg]\,dt,$$
$$i=\overline{1,s},$$
$$\eta_j(t_{00};\varepsilon\delta\mu)=\int\limits_{t_{00}}^{t_{10}+\delta_2}\bigg[\int\limits_0^1\Big|f_{z_j}\big(t,y_0(\tau_1(t))+\xi\delta y(\tau_1(t)),\dots\big)-f_{0z_j}[t]\Big|\,d\xi\bigg]\,dt,$$
$$j=1,\dots,m,$$
$$\delta(t_{00};\varepsilon\delta\mu)=\int\limits_{t_{00}}^{t_{10}+\delta_2}\bigg[\int\limits_0^1\Big|f_u\big(t,y_0(\tau_1(t))+\xi\Delta y(\tau_1(t)),\dots\big)-f_{0u}[t]\Big|\,d\xi\bigg]\,dt.$$

We have

$$\varphi(t)=\varphi_0(t)+\varepsilon\delta\varphi(t)\to\varphi_0(t);\ \ \Delta y(\tau_i(t))\to0,\ \ i=\overline{1,p},$$
$$\Delta z(\sigma_j(t))\to0,\ \ j=\overline{1,m};$$

$$u_0(t) + \xi\varepsilon\delta u(t) \to u_0(t)$$

as $\varepsilon \to 0$ uniformly with respect to

$$(\xi, t, \delta\mu) \in [0,1] \times [t_{00}, t_{10} + \delta_2] \times V^-.$$

By the Lebesque theorem we obtain that

$$\lim_{\varepsilon\to 0} \vartheta_i(t_{00}; \varepsilon\delta\mu) = 0, \quad i = \overline{1,s}, \quad \lim_{\varepsilon\to 0} \eta_j(t_{00}; \varepsilon\delta\mu) = 0, \quad j = \overline{1,m},$$
$$\lim_{\varepsilon\to 0} \delta(t_{00}; \varepsilon\delta\mu) = 0$$

uniformly with respect to $\delta\mu \in V^-$.

Therefore

$$\alpha_1(t; \varepsilon\delta\mu) = o(t; \varepsilon\delta\mu).$$

Consider $\alpha_2(t; \varepsilon\delta\mu)$. It is easy to see that for $i \in p+1, \dots, s$ and $t \in [\gamma_i(t_0), \gamma_i]$ we have

$$\begin{aligned} \big|\Delta y(\tau_j(t))\big| &\le O(\varepsilon\delta\mu), \quad j = \overline{1, i-1}; \\ \Delta y(\tau_j(t)) &= \varepsilon\delta\varphi(\tau_j(t)), \quad j = \overline{i+1, s} \end{aligned} \tag{3.16}$$

(see (3.1), (3.4)). Therefore

$$\int\limits_{\gamma_i(t_0)}^{\gamma_i} \overline{\omega}(\xi; t, \varepsilon\delta\mu)\, d\xi = \int\limits_{\gamma_i(t_0)}^{\gamma_i} Y(\xi; t)\beta_i(\xi)\, d\xi -$$
$$- \int\limits_{\gamma_i(t_0)}^{\gamma_i} Y(\xi; t) f_{0y_i}[\xi] \Delta y(\tau_i(\xi))\, d\xi + o(t; \varepsilon\delta\mu),$$

where

$$\begin{aligned} \beta_i(\xi) = f\Big(&\xi, y_0(\tau_1(\xi)) + \Delta y(\tau_1(\xi)), \dots, y_0(\tau_i(\xi)) + \Delta y(\tau_i(\xi)), \\ &\varphi(\tau_{i+1}(\xi)), \dots, \varphi(\tau_s(\xi)), z_0(\sigma_1(\xi)) + \Delta z(\sigma_1(\xi)), \dots, \\ &z_0(\sigma_m(\xi)) + \Delta z(\sigma_m(\xi)), u_0(\xi) + \delta u(\xi)\Big) - f_0[\xi], \end{aligned}$$

$$\begin{aligned} o(t; \varepsilon\delta\mu) = &- \sum_{j=1}^{i-1} \int\limits_{\gamma_i(t_0)}^{\gamma_i} Y(\xi; t) f_{0y_j}[\xi] \Delta y(\tau_j(\xi))\, d\xi - \\ &- \varepsilon \sum_{j=i+1}^{s} \int\limits_{\gamma_i(t_0)}^{\gamma_i} Y(\xi; t) f_{0y_j}[\xi] \delta\varphi(\tau_j(\xi))\, d\xi - \\ &- \sum_{j=1}^{m} \int\limits_{\gamma_i(t_0)}^{\gamma_i} Y(\xi; t) f_{0z_j}[\xi] \Delta z(\sigma_j(\xi))\, d\xi - \varepsilon \int\limits_{\gamma_i(t_0)}^{\gamma_i} f_{0u}[\xi] \delta u(\xi)\, d\xi \end{aligned}$$

(see (3.5), (3.16)). Clearly,

$$\int\limits_{\gamma_i(t_0)}^{\gamma_i} Y(\xi;t)\beta_i(\xi)\,d\xi = \alpha_5(t;\varepsilon\delta\mu) + \alpha_6(t;\varepsilon\delta\mu),$$

where

$$\alpha_5(t;\varepsilon\delta\mu) = \int\limits_{\gamma_i(t_0)}^{\gamma_i} Y(\xi;t)[\beta_i(\xi) - f_i^-]\,d\xi, \quad \alpha_6(t;\varepsilon\delta\mu) = \int\limits_{\gamma_i(t_0)}^{\gamma_i} Y(\xi;t)f_i^-\,d\xi.$$

Further, if $i \in \{p+1,\dots,s\}$ and $\xi \in [\gamma_i(t_0),\gamma_i]$, then $\tau_j(\xi) \geq t_{00}$, $j = \overline{1,i-1}$. Hence

$$\lim_{\varepsilon\to0}\big(y_0(\tau_j(\xi)) + \Delta y(\tau_j(\xi))\big) = \lim_{\xi\in\gamma_i-} y_0(\tau_j(\xi)) = y_0(\tau_j(\gamma_i)), \quad j = \overline{1,i-1}.$$

We have $\tau_i(\xi) \in [t_0,t_{00}]$ for $\xi \in [\gamma_i(t_0),\gamma_i]$. Therefore

$$y_0(\tau_i(\xi)) + \Delta y(\tau_i(\xi)) = y(\tau_i(\xi),\mu_0 + \varepsilon\delta\mu) = q_0(\tau_i(\xi)) + \Delta q(\tau_i(\xi))$$

(see (2.4), (2.5)).

Therefore, taking into account the continuity of the function $q_0(t)$, $t \in [t_{00} - \delta_2, t_{10} + \delta_2]$, (2.6), and the condition $q_0(t_{00}) = y_{00}$, we have

$$\lim_{\varepsilon\to0}\big(y_0(\tau_i(\xi)) + \Delta y(\tau_i(\xi))\big) = \lim_{\xi\in\gamma_i-} q_0(\tau_i(\xi)) = y_{00}.$$

Hence, we see that for $\varepsilon \to 0$, $i \in \{p+1,\dots,s\}$ and $\xi \in [\gamma_i(t_0),\gamma_i]$, we have

$$\lim_{\varepsilon\to0}\Big(\xi, y_0(\tau_1(\xi)) + \Delta y(\tau_1(\xi)),\dots, y_0(\tau_i(\xi)) + \Delta y(\tau_i(\xi)), \varphi(\tau_{i+1}(\xi)),\dots,$$
$$\varphi(\tau_s(\xi)), z_0(\sigma_1(\xi)) + \Delta z(\sigma_1(\xi)),\dots, z_0(\sigma_m(\xi)) + \Delta z(\sigma_m(\xi))\Big) = \omega_{0i}^-.$$

On the other hand,

$$\lim_{\varepsilon\to0}\Big(\xi, y_0(\tau_1(\xi)),\dots, y_0(\tau_{i-1}(\xi)),$$
$$\varphi_0(\tau_i(\xi)),\dots,\varphi_0(\tau_s(\xi)), z_0(\sigma_1(\xi)),\dots, z_0(\sigma_m(\xi))\Big) = \omega_{1i}^-.$$

Therefore,

$$\lim_{\varepsilon\to0}\ \sup_{\xi\in[\gamma_i(t_0),\gamma_i]} |\beta_i(\xi) - f_i^-| = 0$$

uniformly with respect to $\delta\mu \in V^-$.

The function $Y(\xi;t)$ is continuous on the set

$$[\gamma_i(t_0),\gamma_i] \times [t_{10} - \delta_2, t_{10} + \delta_2] \subset \Pi$$

and, moreover

$$\gamma_i - \gamma_i(t_0) = -\varepsilon\dot{\gamma}_i^-\delta t_0 + o(\varepsilon\delta\mu).$$

Therefore $\alpha_5(t;\varepsilon\delta\mu) = o(t;\delta\mu)$ and

$$\alpha_6(t;\varepsilon\delta\mu) = -\varepsilon\sum_{i=p+1}^{s} Y(\gamma_i;t)f_i^-\,\dot{\gamma}_i^-\,\delta t_0 + o(t;\varepsilon\delta\mu).$$

Finally,

$$\alpha_2(t;\varepsilon\delta\mu) = -\,\varepsilon \sum_{i=p+1}^{s} Y(\gamma_i;t) f_i^- \dot{\gamma}_i^- \,\delta t_0 - \\ - \sum_{i=p+1}^{s} \int\limits_{\gamma_i(t_0)}^{\gamma_i} Y(\gamma_i;t) f_{0y_i}[\xi] \Delta y(\tau_i(\xi))\,d\xi + o(t;\varepsilon\delta\mu).$$

Similarly, we can prove the relations

$$\alpha_i(t;\varepsilon\delta\mu) = o(t;\varepsilon\delta\mu), \quad i = 3,4$$

(see (3.15)).

For $h_2(t;t_{00},\varepsilon\delta\mu)$ we have the final formula

$$h_2(t;t_{00},\varepsilon\delta\mu) = -\,\varepsilon \sum_{i=p+1}^{s} Y(\gamma_i;t) f_i^- \dot{\gamma}_i^- \delta t_0 - \\ - \sum_{i=p+1}^{s} \int\limits_{\gamma_i(t_0)}^{\gamma_i} Y(\xi;t) f_{0y_i}[\xi] \Delta y(\tau_i(\xi))\,d\xi + o(t;\varepsilon\delta\mu) \quad (3.17)$$

(see (3.14)).

Taking into account (3.9)–(3.13) and (3.17), we obtain (1.3), where $\delta x(t;\varepsilon\delta\mu)$ has the form (1.4).

4. Proof of Theorem 1.2

Assume that in Lemma 2.3 $r_1 = t_{00}$ and $r_2 = t_{10}$. Then for any element $(\varepsilon,\delta\mu) \in [0,\varepsilon_1] \times V^+$, the corresponding solution $\omega(t;\mu_0+\varepsilon\delta\mu)$ is defined on $[t_{10}-\delta_1, t_{10}+\delta_1]$. The solution $x(t;\mu_0+\varepsilon\delta\mu)$ is defined on $[\tau, t_{10}+\delta_1]$ and

$$\omega(t;\mu_0+\varepsilon\delta\mu) = x(t;\mu_0+\varepsilon\delta\mu), \ \ t \in [t_0, t_{10}+\delta_1]$$

(see Lemma 1.1 and 2.1). It is easy to see that

$$\Delta y(t) = \begin{cases} \varepsilon\delta\varphi(t), & t \in [\tau, t_{00}], \\ \varphi(t) - y_0(t), & t \in [t_{00}, t_0), \\ \Delta q(t), & t \in [t_0, t_{10}+\delta_1], \end{cases} \quad (4.1)$$

$$\Delta z(t) = \begin{cases} \varepsilon\delta g(t), & t \in [\tau, t_{00}], \\ g(t) - v_0(t), & t \in [t_{00}, t_0), \\ \Delta v(t), & t \in [t_0, t_{10}+\delta_1]. \end{cases} \quad (4.2)$$

Let numbers $\delta_2 \in (0,\delta_1)$ and $\varepsilon_2 \in (0,\varepsilon_1)$ be sufficiently small so that for an arbitrary $(\varepsilon,\delta\mu) \in [0,\varepsilon_2] \times V^+$ the inequality $\gamma_s(t_0) < t_{10} - \delta_2$ holds. By Lemma 3.1 we have

$$|\Delta y(t)| \le O(\varepsilon\delta\mu), \ \ \forall\,(t,\varepsilon,\delta\mu) \in [t_0, t_{10}+\delta_1] \times [0,\varepsilon_2] \times V^+, \quad (4.3)$$

$$|\Delta z(t)| \le O(\varepsilon\delta\mu), \ \ \forall\,(t,\varepsilon,\delta\mu) \in [\tau, t_{10}+\delta_1] \times [0,\varepsilon_2] \times V^+ \quad (4.4)$$

(see (4.1), (4.2), (2.10)). Moreover,

$$\Delta x(t_0)=\Delta\omega(t_0)=\varepsilon\big[Y_0\delta y_0+Y_1\delta g(t_{00}+)+(Y_1\dot{g}_0{}^{+}-f_p^{+})\delta t_0\big]+o(\varepsilon\delta\mu) \quad (4.5)$$

(see (2.11)).

The function $\Delta x(t)$ on the interval $[t_0,t_{10}+\delta_2]$ satisfies (3.7) and hence it can be represented by the Cauchy formula

$$\Delta x(t)=Y(t_{00},t)\Delta x(t_0)+\varepsilon\int_{t_0}^{t}Y(\xi;t)f_{0u}[t]\delta u(\xi)\,d\xi+\sum_{i=0}^{2}h_i(t;t_0,\varepsilon\delta\mu), \quad (4.6)$$

where

$$h_0(t;t_0,\varepsilon\delta\mu)=\sum_{i=1}^{s}\int_{\tau_i(t_0)}^{t_0}Y(\gamma_i(\xi);t)f_{0y_i}[\gamma_i(\xi)]\dot{\gamma}_i(\xi)\Delta y(\xi)\,d\xi$$

and the functions $h_i(t;t_0,\varepsilon\delta\mu)$, $i=1,2$ are defined by the formulas (3.10).

The function $Y(\xi;t)$ is continuous on the set $[t_{00},\tau_s(t_{10}-\delta_2)]\times[t_{10}-\delta_2,t_{10}+\delta_2]$. Since $t_0\in[t_{00},\tau_s(t_{10}-\delta_2)]$, we have

$$\begin{aligned}Y(t_{00};t)\Delta x(t_0)=\varepsilon Y(t_{00};t)\big[Y_0\delta y_0+Y_1\delta g(t_{00}+)+(Y_1\dot{g}_0^{+}-f_p^{+})\delta t_0\big]+\\+o(t;\varepsilon\delta\mu).\end{aligned} \quad (4.7)$$

(see(4.5)).

Consider $h_0(t;t_0,\varepsilon\delta\mu)$. We have

$$\begin{aligned}h_0(t;t_0,\varepsilon\delta\mu)&=\sum_{i=1}^{p}\int_{\tau_i(t_0)}^{t_0}Y(\gamma_i(\xi);t)f_{0y_i}[\gamma_i(\xi)]\dot{\gamma}_i(\xi)\Delta y(\xi)\,d\xi+\\&\quad+\sum_{i=p+1}^{s}\Big[\varepsilon\int_{\tau_i(t_0)}^{t_{00}}Y(\gamma_i(\xi);t)f_{0y_i}[\gamma_i(\xi)]\dot{\gamma}_i(\xi)\delta\varphi(\xi)\,d\xi+\\&\quad+\int_{t_{00}}^{t_0}Y(\gamma_i(\xi);t)f_{0y_i}[\gamma_i(\xi)]\dot{\gamma}_i(\xi)\Delta y(\xi)\,d\xi\Big]=\\&=\sum_{i=1}^{p}\int_{t_0}^{\gamma_i(t_0)}Y(\xi;t)f_{0y_i}[\xi]\Delta y(\tau_i(\xi))\,d\xi+\\&\quad+\varepsilon\sum_{i=p+1}^{s}\int_{\tau_i(t_{00})}^{t_{00}}Y(\gamma_i(\xi);t)f_{0y_i}[\gamma_i(\xi)]\dot{\gamma}_i(\xi)\delta\varphi(\xi)\,d\xi+\\&\quad+\sum_{i=p+1}^{s}\int_{\gamma_i}^{\gamma_i(t_0)}Y(\xi;t)f_{0y_i}[\xi]\Delta y(\tau_i(\xi))\,d\xi+o(t;\varepsilon\delta\mu),\end{aligned} \quad (4.8)$$

where

$$o(t;\varepsilon\delta\mu) = -\varepsilon\sum_{i=1}^{s}\int\limits_{\tau_i(t_{00})}^{\tau_i(t_0)} Y(\gamma_i(\xi);t)f_{0y_i}[\gamma_i(\xi)]\dot{\gamma}_i(\xi)\delta\varphi(\xi)\,d\xi.$$

This implies

$$\sum_{i=1}^{p}\int\limits_{t_0}^{\gamma_i(t_0)} Y(\xi;t)f_{0y_i}[\xi]\Delta y(\tau_i(\xi))\,d\xi =$$

$$= \sum_{i=1}^{p}\sum_{j=0}^{i-1}\int\limits_{\gamma_j(t_0)}^{\gamma_{j+1}(t_0)} Y(\xi;t)f_{0y_i}[\xi]\Delta y(\tau_i(\xi))\,d\xi =$$

$$= \sum_{i=0}^{p-1}\sum_{j=i+1}^{p}\int\limits_{\gamma_i(t_0)}^{\gamma_{i-1}(t_0)} Y(\xi;t)f_{0y_j}[\xi]\Delta y(\tau_j(\xi))\,d\xi, \quad \gamma_0(t_0) = t_0. \quad (4.9)$$

Further,

$$h_1(t;t_0,\varepsilon\delta\mu) = \sum_{j\in I_1\cup I_2}\Bigg[\varepsilon\int\limits_{\sigma_j(t_0)}^{t_{00}} Y(\rho_j(\xi);t)f_{0z_j}[\rho_j(\xi)]\dot{\rho}_j(\xi)\delta g(\xi)\,d\xi +$$

$$+ \int\limits_{t_{00}}^{t_0} Y(\rho_j(\xi);t)f_{0z_j}[\rho_j(\xi)]\dot{\rho}_j(\xi)\Delta z(\xi)\,d\xi\Bigg] +$$

$$+ \sum_{j\in I_3}\int\limits_{\sigma_j(t_0)}^{t_0} Y(\rho_j(\xi);t)f_{0z_j}[\rho_j(\xi)]\dot{\rho}_j(\xi)\Delta z(\xi)\,d\xi =$$

$$= \sum_{j\in I_1\cup I_2}(\varepsilon\alpha_j(t) + \beta_j(t)) + \sum_{j\in I_3}\eta_j(t),$$

where

$$\alpha_j(t) = \int\limits_{\sigma_j(t_0)}^{t_{00}} Y(\rho_j(\xi);t)f_{0z_j}[\rho_j(\xi)]\dot{\rho}_j(\xi)\delta g(\xi)\,d\xi,$$

$$\beta_j(t) = \int\limits_{t_{00}}^{t_0} Y(\rho_j(\xi);t)f_{0z_j}[\rho_j(\xi)]\dot{\rho}_j(\xi)\Delta z(\xi)\,d\xi,$$

$$\eta_j(t) = \int\limits_{\sigma_j(t_0)}^{t_0} Y(\rho_j(\xi);t)f_{0z_j}[\rho_j(\xi)]\dot{\rho}_j(\xi)\Delta z(\xi)\,d\xi.$$

Obviously $\beta_j(t) = o(t;\varepsilon\delta\mu)$, $\eta_j(t) = o(t;\varepsilon\delta\mu)$, so we have

$$\alpha_j(t) = \int\limits_{\sigma_j(t_{00})}^{t_{00}} Y(\rho_j(\xi);t) f_{0z_j}[\rho_j(\xi)]\dot{\rho}_j(\xi)\delta g(\xi)\,d\xi - \\ - \int\limits_{\sigma_j(t_{00})}^{\sigma_j(t_0)} Y(\rho_j(\xi);t) f_{0z_j}[\rho_j(\xi)]\dot{\rho}_j(\xi)\delta g(\xi)\,d\xi.$$

Therefore

$$h_1(t;t_0,\varepsilon\delta\mu) = \varepsilon \sum_{j=1}^{s} \int\limits_{\sigma_j(t_{00})}^{t_{00}} Y(\rho_j(\xi);t) f_{0z_j}[\rho_j(\xi)]\dot{\rho}_j(\xi)\delta g(\xi)\,d\xi + \\ + o(t;\varepsilon\delta\mu). \tag{4.10}$$

$h_2(t;t_0,\varepsilon\delta\mu)$ for $t \in [t_{10}-\delta_2, t_{10}+\delta_2]$ can be represented by the form

$$h_2(t;t_0,\varepsilon\delta\mu) = \sum_{i=1}^{5} \beta_i(t;\varepsilon\delta\mu), \tag{4.11}$$

where

$$\beta_1(t;\varepsilon\delta\mu) = \sum_{i=1}^{p-1} \int\limits_{\gamma_i(t_0)}^{\gamma_{i+1}(t_0)} \overline{\omega}(\xi;t,\varepsilon\delta\mu)\,d\xi,$$

$$\beta_2(t;\varepsilon\delta\mu) = \int\limits_{\gamma_p(t_0)}^{\gamma_{p+1}} \overline{\omega}(\xi;t,\varepsilon\delta\mu)\,d\xi,$$

$$\beta_3(t;\varepsilon\delta\mu) = \sum_{i=p+1}^{s} \int\limits_{\gamma_i}^{\gamma_i(t_0)} \overline{\omega}(\xi;t,\varepsilon\delta\mu)\,d\xi,$$

$$\beta_4(t;\varepsilon\delta\mu) = \sum_{i=p+1}^{s} \int\limits_{\gamma_i(t_0)}^{\gamma_{i+1}} \overline{\omega}(\xi;t,\varepsilon\delta\mu)\,d\xi,$$

$$\beta_5(t;\varepsilon\delta\mu) = \int\limits_{\gamma_s(t_0)}^{t} \overline{\omega}(\xi;t,\varepsilon\delta\mu)\,d\xi.$$

For $\beta_1(t;\varepsilon\delta\mu)$ we have

$$\beta_1(t;\varepsilon\delta\mu) = \beta_{11}(t;\varepsilon\delta\mu) - \beta_{12}(t;\varepsilon\delta\mu), \tag{4.12}$$

where

$$\beta_{11}(t;\varepsilon\delta\mu)=\sum_{i=0}^{p-1}\int\limits_{\gamma_i(t_0)}^{\gamma_{i+1}(t_0)}Y(\xi;t)\Big[f\Big(\xi,y_0(\tau_1(\xi))+\Delta y(\tau_1(\xi)),\dots,$$
$$y_0(\tau_i(\xi))+\Delta y(\tau_i(\xi)),\varphi(\tau_{i+1}(\xi)),\dots,\varphi(\tau_s(\xi)),$$
$$z_0(\sigma_1(\xi))+\Delta z(\sigma_1(\xi)),\dots,z_0(\sigma_m(\xi))+\Delta z(\sigma_m(\xi)),$$
$$u_0(\xi)+\varepsilon\delta u(\xi)\Big)-$$
$$-f\Big(\xi,y_0(\tau_1(\xi)),\dots,y_0(\tau_p(\xi)),\varphi_0(\tau_{p+1}(\xi)),\dots,\varphi_0(\tau_s(\xi)),$$
$$z_0(\sigma_1(\xi)),\dots,z_0(\sigma_m(\xi)),u_0(\xi)\Big)\Big]\,d\xi,$$

$$\beta_{12}(t;\varepsilon\delta\mu)=\sum_{i=0}^{p-1}\int\limits_{\gamma_i(t_0)}^{\gamma_{i+1}(t_0)}Y(\xi;t)\Big[\sum_{j=1}^{s}f_{0y_j}[\xi]\Delta y(\tau_j(\xi))+$$
$$+\sum_{j=1}^{m}f_{0z_j}[\xi]\Delta z(\tau_j(\xi))\Big]\,d\xi.$$

Let $\xi\in[\gamma_i(t_0),\gamma_{i+1}(t_0)]$. Then

$$\tau_j(\xi)\geq t_0,\ \ j=\overline{1,i},\quad \tau_j(\xi)\leq t_0,\ \ j=\overline{i+1,p},\quad \tau_j(\xi)<t_{00},\ \ j=\overline{p+1,s},$$

and hence

$$\begin{aligned}\big|\Delta y(\tau_j(\xi))\big|&\leq O(\varepsilon\delta\mu),\ \ j=\overline{1,i},\\ \Delta y(\tau_j(\xi))&=\varepsilon\delta\varphi(\tau_j(\xi)),\ \ j=\overline{p+1,s},\end{aligned}\tag{4.13}$$

(see (4.1), (4.3)).

For any $i\in\{0,\dots,p-1\}$, the function $\gamma_{i+1}(t_0)-\gamma_i(t_0)$ tends to zero as $\varepsilon\to 0$. Therefore, taking into account (4.13) and (4.4) we have

$$\beta_{12}(t;\varepsilon\delta\mu)=\sum_{i=0}^{p-1}\sum_{j=i+1}^{p}\int\limits_{\gamma_i(t_0)}^{\gamma_{i+1}(t_0)}Y(\xi;t)f_{0y_i}[\xi]\Delta y(\tau_j(\xi))\,d\xi+o(t;\varepsilon\delta\mu).\tag{4.14}$$

Further

$$\lim_{\varepsilon\to 0}\sup_{\xi\in[\gamma_i(t_0),\gamma_{i+1}(t_0)]}\Big|f\Big(\xi,y_0(\tau_1(\xi))+\Delta y(\tau_1(\xi)),\dots,y_0(\tau_i(\xi))+\Delta y(\tau_i(\xi)),$$
$$\varphi(\tau_{i+1}(\xi)),\dots,\varphi(\tau_s(\xi)),$$
$$z_0(\sigma_1(\xi))+\Delta z(\sigma_1(\xi)),\dots,z_0(\sigma_m(\xi))+\Delta z(\sigma_m(\xi)),u_0(\xi)+\varepsilon\delta u(\xi)\Big)$$
$$-f_i^+ + f_p^+ - f\Big(\xi,y_0(\tau_1(\xi)),\dots,y_0(\tau_p(\xi)),\varphi_0(\tau_{p+1}(\xi)),\dots,\varphi_0(\tau_s(\xi)),$$
$$z_0(\sigma_1(\xi)),\dots,z_0(\sigma_m(\xi)),u_0(\xi)\Big)\Big|=0,\ \ i=\overline{0,p-1},\tag{4.15}$$

uniformly with respect of to $\delta\mu\in V^+$.

The properties of the functions $Y(\xi;t)$ and $\gamma_i(t)$, $i=\overline{1,p}$ imply that

$$\lim_{\varepsilon\to 0}\sup_{\xi\in[\gamma_i(t_0),\gamma_{i+1}(t_0)]}\big|Y(\xi;t)-Y(t_{00};t)\big|=0,\quad i=\overline{0,p-1} \tag{4.16}$$

uniformly with respect to $t\in[t_{10}-\delta_2,t_{10}+\delta_2]$ and

$$\gamma_{i+1}(t_0)-\gamma_i(t_0)=\varepsilon\big(\dot\gamma^{\,+}_{i+1}-\dot\gamma^{\,+}_{i}\big)\delta t_0+o(\varepsilon\delta\mu),\quad i=\overline{0,p-1},\quad \dot\gamma_0=1. \tag{4.17}$$

From (4.13)–(4.15) we have

$$\beta_{11}(t;\varepsilon\delta\mu)=\varepsilon Y(t_{00},t)\sum_{i=0}^{p}(f_i^+-f_p^+)(\dot\gamma^{\,+}_{i+1}-\dot\gamma^{\,+}_{i})\delta t_0+o(t;\varepsilon\delta\mu). \tag{4.18}$$

From (4.12), (4.14) and (4.18) we have

$$\beta_1(t;\varepsilon\delta\mu)=\varepsilon Y(t_{00},t)\Big[\sum_{i=0}^{p}(\dot\gamma^{\,+}_{i+1}-\dot\gamma^{\,+}_{i})f_i^++f_p^+\Big]\delta t_0-$$
$$-\sum_{i=0}^{p-1}\sum_{j=i+1}^{p}\int\limits_{\gamma_i(t_0)}^{\gamma_{i+1}(t_0)}Y(\xi;t)f_{0y_j}[\xi]\Delta y(\tau_j(\xi))\,d\xi+o(t;\varepsilon\delta\mu). \tag{4.19}$$

It is easy to see that

$$\beta_2(t;\varepsilon\delta\mu)=\int\limits_{\gamma_p(t_0)}^{\gamma_{p+1}}Y(\xi;t)\Big[f\Big(\xi,y_0(\tau_1(\xi))+\Delta y(\tau_1(\xi)),\dots,y_0(\tau_p(\xi))+\Delta y(\tau_p(\xi)),$$
$$\varphi(\tau_{p+1}(\xi)),\dots,\varphi(\tau_s(\xi)),z_0(\sigma_1(\xi))+\Delta z(\sigma_1(\xi)),\dots,z_0(\sigma_m(\xi))+\Delta z(\sigma_m(\xi)),$$
$$u_0(\xi)+\varepsilon\delta u(\xi)\Big)-$$
$$-f\Big(\xi,y_0(\tau_1(\xi)),\dots,y_0(\tau_p(\xi)),\varphi_0(\tau_{p+1}(\xi)),\dots,\varphi_0(\tau_s(\xi)),$$
$$z_0(\sigma_1(\xi)),\dots,z_0(\sigma_m(\xi)),u_0(\xi)\Big)-$$
$$-\sum_{j=1}^{p}f_{0y_j}[\xi]\Delta y(\tau_j(\xi))-\varepsilon\sum_{j=p+1}^{s}f_{0y_j}[\xi]\delta\varphi(\tau_j(\xi))-$$
$$-\sum_{j=1}^{m}f_{0z_j}[\xi]\Delta z(\sigma_j(\xi))-\varepsilon f_{0u}[\xi]\delta u(\xi)\Big]\,d\xi.$$

It is easy to prove that

$$\beta_2(t;\varepsilon\delta\mu)=o(t;\varepsilon\delta\mu) \tag{4.20}$$

(see (4.3) and (4.4)).

Consider the other terms of (4.11). We have

$$\beta_3(t;\varepsilon\delta\mu)=\sum_{i=p+1}^{s}\int\limits_{\gamma_i}^{\gamma_i(t_0)}Y(\xi;t)\Big[f\Big(\xi,y_0(\tau_1(\xi))+\Delta y(\tau_1(\xi)),\dots,$$

$$y_0(\tau_{i-1}(\xi)) + \Delta y(\tau_{i-1}(\xi)), \varphi(\tau_i(\xi)), \dots, \varphi(\tau_s(\xi)),$$

$$z_0(\sigma_1(\xi)) + \Delta z(\sigma_1(\xi)), \dots, z_0(\sigma_m(\xi)) + \Delta z(\sigma_m(\xi)), u_0(\xi) + \varepsilon\delta u(\xi)\Big) -$$

$$- f\Big(\xi, y_0(\tau_1(\xi)), \dots, y_0(\tau_i(\xi)), \varphi(\tau_{i+1}(\xi)), \dots, \varphi(\tau_s(\xi)),$$

$$z_0(\sigma_1(\xi)), \dots, z_0(\sigma_m(\xi)), u_0(\xi)\Big)\Big]\, d\xi -$$

$$- \sum_{i=p+1}^{s} \Bigg[\sum_{j=1}^{i-1} \int_{\gamma_i}^{\gamma_i(t_0)} Y(\xi;t) f_{0y_j}[\xi] \Delta y(\tau_j(\xi))\, d\xi +$$

$$+ \int_{\gamma_i}^{\gamma_i(t_0)} Y(\xi;t) f_{0y_i}[\xi] \Delta y(\tau_i(\xi))\, d\xi + \varepsilon \sum_{j=i+1}^{s} \int_{\gamma_i}^{\gamma_i(t_0)} Y(\xi;t) f_{0y_j}[\xi] \delta\varphi(\tau_j(\xi))\, d\xi \Bigg] -$$

$$- \sum_{i=p+1}^{s} \int_{\gamma_i}^{\gamma_i(t_0)} Y(\xi;t) \sum_{j=1}^{m} f_{0z_j}[\xi] \Delta z(\sigma_j(\xi))\, d\xi.$$

By the condition (6) we have

$$\lim_{\varepsilon\to 0} \sup_{\xi\in[\gamma_i,\gamma_i(t_0)]} \Big| f\Big(\xi, y_0(\tau_1(\xi)) + \Delta y(\tau_1(\xi)), \dots, y_0(\tau_{i-1}(\xi)) + \Delta y(\tau_{i-1}(\xi)),$$

$$\varphi(\tau_i(\xi)), \dots, \varphi(\tau_s(\xi)), z_0(\sigma_1(\xi)) + \Delta z(\sigma_1(\xi)), \dots,$$

$$z_0(\sigma_m(\xi)) + \Delta z(\sigma_m(\xi)), u_0(\xi) + \varepsilon\delta u(\xi)\Big) -$$

$$- f\Big(\xi, y_0(\tau_1(\xi)), \dots, y_0(\tau_i(\xi)), \varphi(\tau_{i+1}(\xi)), \dots, \varphi(\tau_s(\xi)),$$

$$z_0(\sigma_1(\xi)), \dots, z_0(\sigma_m(\xi)), u_0(\xi)\Big) + f_i^+ \Big| = 0, \quad i = \overline{p+1, s}$$

uniformly with respect to $\delta\mu \in V^+$.

Further,

$$\big|\Delta y(\tau_j(\xi))\big| \le O(\varepsilon\delta\mu), \quad j = \overline{1, i-1}, \quad \xi \in [\gamma_i, \gamma_i(t_0)],$$

$$\lim_{\varepsilon\to 0} \sup_{\xi\in[\gamma_i,\gamma_i(t_0)]} \big|Y(\xi;t) - Y(\gamma_i;t)\big| = 0, \quad i = \overline{p+1, s}$$

uniformly with respect to $t \in [t_{10} - \delta_2, t_{10} + \delta_2]$.

Now, we obtain for the function $\beta_3(t;\varepsilon\delta\mu)$ the representation

$$\beta_3(t;\varepsilon\delta\mu) = -\,\varepsilon \sum_{i=p+1}^{s} Y(\gamma_i;t) f_i^+ \delta t_0 -$$

$$- \sum_{i=p+1}^{s} \int_{\gamma_i}^{\gamma_i(t_0)} Y(\xi;t) f_{0y_i}[\xi] \Delta y(\tau_j(\xi))\, d\xi + o(t;\varepsilon\delta\mu). \qquad (4.21)$$

Similarly we can prove (see (3.16)) that

$$\beta_i(t;\varepsilon\delta\mu)=o(\varepsilon\delta\mu),\quad i=4,5. \tag{4.22}$$

Taking into account (4.19)–(4.22), we obtain

$$\begin{aligned}h_1(t;t_0,\varepsilon\delta\mu)=\varepsilon\Big\{Y(t_{00},t)\sum_{i=0}^{p}(\dot\gamma^{\,+}_{i+1}-\dot\gamma^{\,+}_{i})-\sum_{i=p+1}^{s}Y(\gamma_i;t)f_i^+\Big\}\delta t_0-\\ -\sum_{i=0}^{p-1}\sum_{j=i+1}^{p}\int\limits_{\gamma_i(t_0)}^{\gamma_{i+1}(t_0)}Y(\xi;t)f_{0y_i}[\xi]\Delta y(\tau_j(\xi))\,d\xi-\\ -\sum_{i=p+1}^{s}\int\limits_{\gamma_i}^{\gamma_i(t_0)}Y(\xi;t)f_{0y_i}[\xi]\Delta y(\tau_i(\xi))\,d\xi+o(t;\varepsilon\delta\mu)\end{aligned} \tag{4.23}$$

(see (4.11)).

From (4.6), taking into account (4.7)–(4.10) and (4.23), we obtain (1.3), where $\delta x(t;\delta\mu)$ has the form (1.8).

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