Memoirs on Differential Equations and Mathematical Physics
Volume 51, 2010, 93–108

Said Kouachi and Belgacem Rebiai

INVARIANT REGIONS AND THE GLOBAL EXISTENCE FOR REACTION-DIFFUSION SYSTEMS WITH A TRIDIAGONAL MATRIX OF DIFFUSION COEFFICIENTS

Abstract. The aim of this study is to prove the global existence of solutions for reaction-diffusion systems with a tridiagonal matrix of diffusion coefficients and nonhomogeneous boundary conditions. In so doing, we make use of the appropriate techniques which are based on invariant domains and Lyapunov functional methods. The nonlinear reaction term has been supposed to be of polynomial growth. This result is a continuation of that by Kouachi [12].

2010 Mathematics Subject Classification. 35K45, 35K57.

Key words and phrases. reaction diffusion systems, invariant domains, Lyapunov functionals, global existence.

რეზიუმე. ნაშრომის მიზანია დამტკიცებულ იქნეს არაერთგვაროვანი სასაზღვრო ამოცანების ამონახსნთა გლობალური არსებობა ისეთი რეაქციულ-დიფუზიური სისტემებისათვის, რომელთა დიფუზიის კოეფიციენტები ქმნიან ტრიდიაგონალურ (იაკობის) მატრიცს. ამისათვის გამოყენებულია შესაბამისი ტექნიკა, რომელიც ეფუძნება ინვარიანტული არეების და ლიაპუნოვის ფუნქციონალის მეთოდებს. არაწრფივი რეაქციის წევრზე დადებულია პოლინომიალური ზრდის პირობა. ნაშრომში მოყვანილი შედეგი წარმოადგენს კუაშის [12] ერთი შედეგის განვრცობას.

1. Introduction

We consider the reaction-diffusion system

$$\frac{\partial u}{\partial t} - a_{11}\Delta u - a_{12}\Delta v - a_{23}\Delta w = f(u,v,w) \text{ in } \mathbb{R}^+ \times \Omega, \tag{1.1}$$
$$\frac{\partial v}{\partial t} - a_{21}\Delta u - a_{22}\Delta v - a_{23}\Delta w = g(u,v,w) \text{ in } \mathbb{R}^+ \times \Omega, \tag{1.2}$$
$$\frac{\partial w}{\partial t} - a_{21}\Delta u - a_{32}\Delta v - a_{11}\Delta w = h(u,v,w) \text{ in } \mathbb{R}^+ \times \Omega, \tag{1.3}$$

with the boundary conditions

$$\begin{aligned} \lambda u + (1-\lambda)\frac{\partial u}{\partial \eta} &= \beta_1, \\ \lambda v + (1-\lambda)\frac{\partial v}{\partial \eta} &= \beta_2 \quad \text{on } \mathbb{R}^+ \times \partial\Omega, \\ \lambda w + (1-\lambda)\frac{\partial w}{\partial \eta} &= \beta_3, \end{aligned} \tag{1.4}$$

and the initial data

$$u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad w(0,x) = w_0(x) \text{ in } \Omega, \tag{1.5}$$

where:

(i) $0 < \lambda < 1$ and $\beta_i \in \mathbb{R}$, $i = 1,2,3$, for nonhomogeneous Robin boundary conditions.

(ii) $\lambda = \beta_i = 0$, $i = 1,2,3$, for homogeneous Neumann boundary conditions.

(iii) $1 - \lambda = \beta_i = 0$, $i = 1,2,3$, for homogeneous Dirichlet boundary conditions.

Ω is an open bounded domain of the class $\mathbb{C}^1$ in $\mathbb{R}^N$ with boundary $\partial\Omega$, and $\frac{\partial}{\partial \eta}$ denotes the outward normal derivative on $\partial\Omega$. The diffusion terms a_{ij} ($i,j = 1,2,3$ and $(i,j) \neq (1,3), (3,1)$) are supposed to be positive constants with $a_{11} = a_{33}$ and $(a_{12}+a_{21})^2 + (a_{23}+a_{32})^2 < 4a_{11}a_{22}$, which reflects the parabolicity of the system and implies at the same time that the matrix of diffusion

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{11} \end{pmatrix}$$

is positive definite. The eigenvalues λ_1, λ_2 and λ_3 ($\lambda_1 < \lambda_2$, $\lambda_3 = a_{11}$) of A are positive. If we put

$$\underline{a} = \min\{a_{11}, a_{22}\} \quad \text{and} \quad \overline{a} = \max\{a_{11}, a_{22}\},$$

then the positivity of a_{ij}'s implies that

$$\lambda_1 < \underline{a} \leq \lambda_3 \leq \overline{a} < \lambda_2.$$

The initial data are assumed to be in the domain

$$\Sigma=\begin{cases}\Big\{(u_0,v_0,w_0)\in\mathbb{R}^3:\mu_2 v_0\le a_{21}u_0+a_{23}w_0\le\mu_1 v_0,\ a_{32}u_0\le a_{12}w_0\Big\}\\ \quad\text{if } \mu_2\beta_2\le a_{21}\beta_1+a_{23}\beta_3\le\mu_1\beta_2,\ a_{32}\beta_1\le a_{12}\beta_3,\\ \Big\{(u_0,v_0,w_0)\in\mathbb{R}^3:\mu_2 v_0\le a_{21}u_0+a_{23}w_0\le\mu_1 v_0,\ a_{12}w_0\le a_{32}u_0\Big\}\\ \quad\text{if } \mu_2\beta_2\le a_{21}\beta_1+a_{23}\beta_3\le\mu_1\beta_2,\ a_{12}\beta_3\le a_{32}\beta_1,\\ \left\{\begin{array}{l}(u_0,v_0,w_0)\in\mathbb{R}^3:\\ \frac{1}{\mu_2}(a_{21}u_0+a_{23}w_0)\le v_0\le\frac{1}{\mu_1}(a_{21}u_0+a_{23}w_0),\ a_{32}u_0\le a_{12}w_0\end{array}\right\}\\ \quad\text{if } \frac{1}{\mu_2}(a_{21}\beta_1+a_{23}\beta_3)\le\beta_2\le\frac{1}{\mu_1}(a_{21}\beta_1+a_{23}\beta_3),\ a_{32}\beta_1\le a_{12}\beta_3,\\ \left\{\begin{array}{l}(u_0,v_0,w_0)\in\mathbb{R}^3:\\ \frac{1}{\mu_2}(a_{21}u_0+a_{23}w_0)\le v_0\le\frac{1}{\mu_1}(a_{21}u_0+a_{23}w_0),\ a_{32}u_0\ge a_{12}w_0\end{array}\right\}\\ \quad\text{if } \frac{1}{\mu_2}(a_{21}\beta_1+a_{23}\beta_3)\le\beta_2\le\frac{1}{\mu_1}(a_{21}\beta_1+a_{23}\beta_3),\ a_{32}\beta_1\ge a_{12}\beta_3,\end{cases}$$

where

$$\mu_1=\underline{a}-\lambda_1>0>\mu_2=\underline{a}-\lambda_2.$$

Since we use the same methods to treat all the cases, we will tackle only with the first one. We suppose that the reaction terms f, g and h are continuously differentiable, polynomially bounded on Σ,

$$\Big(f(r_1,r_2,r_3),g(r_1,r_2,r_3),h(r_1,r_2,r_3)\Big)$$

is in Σ for all (r_1,r_2,r_3) in $\partial\Sigma$ (we say that (f,g,h) points into Σ on $\partial\Sigma$), i.e.,

$$a_{21}f(r_1,r_2,r_3)+a_{23}h(r_1,r_2,r_3)\le\mu_1 g(r_1,r_2,r_3) \tag{1.6}$$

for all r_1, r_2 and r_3 such that $\mu_2 r_2\le a_{21}r_1+a_{23}r_3=\mu_1 r_2$ and $a_{32}r_1\le a_{12}r_3$, and

$$\mu_2 g(r_1,r_2,r_3)\le a_{21}f(r_1,r_2,r_3)+a_{23}h(r_1,r_2,r_3) \tag{1.6a}$$

for all r_1, r_2 and r_3 such that $\mu_2 r_2=a_{21}r_1+a_{23}r_3\le\mu_1 r_2$ and $a_{32}r_1\le a_{12}r_3$, and

$$a_{32}f(r_1,r_2,r_3)\le a_{12}h(r_1,r_2,r_3) \tag{1.6b}$$

for all r_1, r_2 and r_3 such that $\mu_2 r_2\le a_{21}r_1+a_{23}r_3\le\mu_1 r_2$ and $a_{32}r_1=a_{12}r_3$, and for positive constants E and D, we have

$$(Ef+Dg+h)(u,v,w)\le C_1(u+v+w+1), \tag{1.7}$$

for all (u,v,w) in Σ, where C_1 is a positive constant.

In the two-component case, where $a_{12}=0$, Kouachi and Youkana [13] generalized the method of Haraux and Youkana [4] with the reaction terms $f(u,v)=-\lambda F(u,v)$ and $g(u,v)=\mu F(u,v)$ with $F(u,v)\ge0$, requiring the condition

$$\lim_{s\to+\infty}\left[\frac{\ln(1+F(r,s))}{s}\right]<\alpha^*\ \text{ for any }\ r\ge0,$$

with

$$\alpha^* = \frac{2a_{11}a_{22}}{n\,(a_{11}-a_{22})^2\,\|u_0\|_\infty}\ \min\left\{\frac{\lambda}{\mu}, \frac{a_{11}-a_{22}}{a_{21}}\right\},$$

where the positive diffusion coefficients a_{11}, a_{22} satisfy $a_{11} > a_{22}$, and a_{21}, λ, μ are positive constants. This condition reflects the weak exponential growth of the reaction term F. Kanel and Kirane [6] proved the global existence in the case where $g(u,v) = -f(u,v) = uv^n$ and n is an odd integer, under the embarrassing condition

$$|a_{12} - a_{21}| < C_p,$$

where C_p contains a constant from Solonnikov's estimate [18]. Later they improved their results in [7] to obtain the global existence under the restrictions

H$_1$. $a_{22} < a_{11} + a_{21}$,

H$_2$. $a_{12} < \varepsilon_0 = \dfrac{a_{11}a_{22}\,(a_{11}+a_{21}-a_{22})}{a_{11}a_{22}+a_{21}\,(a_{11}+a_{21}-a_{22})}$ if $a_{11} \le a_{22} < a_{11} + a_{21}$,

H$_3$. $a_{12} < \min\left\{\dfrac{1}{2}(a_{11}+a_{21}), \varepsilon_0\right\}$ if $a_{22} < a_{11}$,

and

$$|F(v)| \le C_F\left(1 + |v|^{1-\varepsilon}\right), \quad vF(v) \ge 0 \quad \text{for all} \quad v \in \mathbb{R},$$

where ε and C_F are positive constants with $\varepsilon < 1$ and

$$g(u,v) = -f(u,v) = uF(v).$$

Kouachi [12] has proved global existence for solutions of two-component reaction-diffusion systems with a general full matrix of diffusion coefficients and nonhomgeneous boundary conditions.

Many chemical and biological operations are described by reaction-diffusion systems with a tridiagonal matrix of diffusion coefficients. The components $u(t,x)$, $v(t,x)$ and $w(t,x)$ can represent either chemical concentrations or biological population densities (see, e.g., Cussler [1] and [2]).

We note that the case of strongly coupled systems which are not triangular in the diffusion part is more difficult. As a consequence of the blow-up of the solutions found in [16], we can indeed prove that there is a blow-up of the solutions in finite time for such nontriangular systems even though the initial data are regular, the solutions are positive and the nonlinear terms are negative, a structure that ensured the global existence in the diagonal case. For this purpose, we construct invariant domains in which we can demonstrate that for any initial data in these domains, the problem (1.1)–(1.5) is equivalent to the problem for which the global existence follows from the usual techniques based on Lyapunov functionals (see Kirane and Kouachi [8], Kouachi and Youkana [13] and Kouachi [12]).

2. Local Existence and Invariant Regions

This section is devoted to proving that if (f,g,h) points into Σ on $\partial\Sigma$, then Σ is an invariant domain for the problem (1.1)–(1.5), i.e., the solution remains in Σ for any initial data in Σ. Once the invariant domains are constructed, both problems of the local and global existence become easier to be established. For the first problem we demonstrate that the system (1.1)–(1.3) with the boundary conditions (1.4) and the initial data in Σ is equivalent to a problem for which the local existence throughout the time interval $[0,T^*[$ can be obtained by the known procedure, and for the second one we need invariant domains as explained in the preceeding section.

The main result of this section is

Proposition 1. *Suppose that (f,g,h) points into Σ on $\partial\Sigma$. Then for any (u_0,v_0,w_0) in Σ the solution (u,v,w) of the problem* (1.1)–(1.5) *remains in Σ for all t's in* $[0,T^*[$.

Proof. Let $(x_{i1},x_{i2},x_{i3})^t$, $i=1,2,3$, be the eigenvectors of the matrix A^t associated with its eigenvalues λ_i, $i=1,2,3$ $(\lambda_1<\lambda_3<\lambda_2)$. Multiplying the equations (1.1), (1.2) and (1.3) of the given reaction-diffusion system by x_{i1}, x_{i2} and x_{i3}, respectively, and summing the resulting equations, we get

$$\frac{\partial}{\partial t}z_1-\lambda_1\Delta z_1=F_1(z_1,z_2,z_3)\quad\text{in}\quad]0,T^*[\times\Omega, \tag{2.1}$$

$$\frac{\partial}{\partial t}z_2-\lambda_2\Delta z_2=F_2(z_1,z_2,z_3)\quad\text{in}\quad]0,T^*[\times\Omega, \tag{2.2}$$

$$\frac{\partial}{\partial t}z_3-\lambda_3\Delta z_3=F_3(z_1,z_2,z_3)\quad\text{in}\quad]0,T^*[\times\Omega, \tag{2.3}$$

with the boundary conditions

$$\lambda z_i+(1-\lambda)\frac{\partial z_i}{\partial\eta}=\rho_i,\quad i=1,2,3,\quad\text{on}\quad]0,T^*[\times\partial\Omega, \tag{2.4}$$

and the initial data

$$z_i(0,x)=z_i^0(x),\quad i=1,2,3,\quad\text{in}\quad\Omega, \tag{2.5}$$

where

$$z_i=x_{i1}u+x_{i2}v+x_{i3}w,\quad i=1,2,3,\quad\text{in}\quad]0,T^*[\times\Omega, \tag{2.6}$$
$$\rho_i=x_{i1}\beta_1+x_{i2}\beta_2+x_{i3}\beta_3,\quad i=1,2,3,$$

and

$$F_i(z_1,z_2,z_3)=x_{i1}f+x_{i2}g+x_{i3}h,\quad i=1,2,3, \tag{2.7}$$

for all (u,v,w) in Σ.

First, as has been mentioned above, note that the condition of the parabolicity of the system (1.1)–(1.3) implies the parabolicity of the system

(2.1)–(2.3) since

$$(a_{12}+a_{21})^2+(a_{23}+a_{32})^2<4a_{11}a_{22}\Longrightarrow$$
$$\Longrightarrow(\det A>0 \ \text{ and } \ a_{11}a_{22}-a_{23}a_{32}>0).$$

Since λ_1, λ_2 and λ_3 $(\lambda_1<\lambda_3<\lambda_2)$ are the eigenvalues of the matrix A^t, the problem (1.1)–(1.5) is equivalent to the problem (2.1)–(2.5) and to prove that Σ is an invariant domain for the system (1.1)–(1.3) it suffices to prove that the domain

$$\left\{\left(z_1^0,z_2^0,z_3^0\right)\in\mathbb{R}^3:\ z_i^0\geq 0,\ i=1,2,3\right\}=\left(\mathbb{R}^+\right)^3 \tag{2.8}$$

is invariant for the system (2.1)–(2.3) and that

$$\Sigma=\left\{(u_0,v_0,w_0)\in\mathbb{R}^3:\ z_i^0=x_{i1}u_0+x_{i2}v_0+x_{i3}w_0\geq 0,\ i=1,2,3\right\}. \tag{2.9}$$

Since $(x_{i1},x_{i2},x_{i3})^t$ is an eigenvector of the matrix A^t associated to the eigenvalue λ_i, $i=1,2,3$, we have

$$\begin{aligned}(a_{11}-\lambda_i)\,x_{i1}+a_{21}x_{i2}&=0,\\ a_{12}x_{i1}+(a_{22}-\lambda_i)\,x_{i2}+a_{32}x_{i3}&=0\ , \quad i=1,2,3,\\ a_{23}x_{i2}+(a_{11}-\lambda_i)\,x_{i3}&=0.\end{aligned}$$

If we assume, without loss of generality, that $a_{11}\leq a_{22}$ and choose $x_{12}=\mu_1$, $x_{22}=-\mu_2$ and $x_{33}=a_{12}$, then we have

$$x_{i1}u_0+x_{i2}v_0+x_{i3}w_0\geq 0,\quad i=1,2,3\Longleftrightarrow\begin{cases}-a_{21}u_0+\mu_1v_0-a_{23}w_0\geq 0,\\ a_{21}u_0-\mu_2v_0+a_{23}w_0\geq 0,\\ -a_{32}u_0+a_{12}w_0\geq 0.\end{cases}\Longleftrightarrow$$
$$\Longleftrightarrow\mu_2v_0\leq a_{21}u_0+a_{23}w_0\leq\mu_1v_0,\quad a_{32}u_0\leq a_{12}w_0.$$

Thus (2.9) is proved and (2.6) can be written as

$$\begin{cases}z_1=-a_{21}u+\mu_1v-a_{23}w,\\ z_2=a_{21}u-\mu_2v+a_{23}w,\\ z_3=-a_{32}u+a_{12}w.\end{cases} \tag{2.6a}$$

Now, to prove that the domain $(\mathbb{R}^+)^3$ is invariant for the system (2.1)–(2.3), it suffices to show that $F_i\,(z_1,z_2,z_3)\geq 0$ for all (z_1,z_2,z_3) such that $z_i=0$ and $z_j\geq 0$, $j=1,2,3$ $(j\neq i)$, $i=1,2,3$, thanks to the invariant domain method (see Smoller [17]). Using the expressions (2.7), we get

$$\begin{cases}F_1=-a_{21}f+\mu_1g-a_{23}h,\\ F_2=a_{21}f-\mu_2g+a_{23}h,\\ F_3=-a_{32}f+a_{12}h\end{cases} \tag{2.7a}$$

for all (u,v,w) in Σ. Since from (1.6), (1.6a) and (1.6b) we have $F_i\,(z_1,z_2,z_3)\geq 0$ for all (z_1,z_2,z_3) such that $z_i=0$ and $z_j\geq 0$, $j=1,2,3$ $(j\neq i)$, $i=1,2,3$, we obtain $z_i\,(t,x)\geq 0$, $i=1,2,3$, for all $(t,x)\in[0,T^*[\times\Omega$. Then Σ is an invariant domain for the system (1.1)–(1.3). $\square$

In addition, the system (1.1)–(1.3) with the boundary conditions (1.4) and initial data in Σ is equivalent to the system (2.1)–(2.3) with the boundary conditions (2.4) and positive initial data (2.5). As has been mentioned at the beginning of this section and since ρ_i, $i = 1, 2, 3$, given by

$$\begin{cases} \rho_1 = -a_{21}\beta_1 + \mu_1\beta_2 - a_{23}\beta_3, \\ \rho_2 = a_{21}\beta_1 - \mu_2\beta_2 + a_{23}\beta_3, \\ \rho_3 = -a_{32}\beta_1 + a_{12}\beta_3, \end{cases}$$

are positive, we have for any initial data in $\mathbb{C}\left(\overline{\Omega}\right)$ or $\mathbb{L}^p(\Omega)$, $p \in \left]1, +\infty\right]$, the local existence and uniqueness of solutions to the initial value problem (2.1)–(2.5) and consequently those of the problem (1.1)–(1.5) follow from the basic existence theory for abstract semilinear differential equations (see Friedman [3], Henry [5] and Pazy [15]). These solutions are classical on $[0, T^*[\times \Omega$, where T^* denotes the eventual blow up time in $\mathbb{L}^\infty(\Omega)$. A local solution is continued globally by a priori estimates.

Once invariant domains are constructed, one can apply the Lyapunov technique and establish the global existence of unique solutions for (1.1)–(1.5).

3. Global Existence

As the determinant of the linear algebraic system (2.6), with respect to the variables u, v and w, is different from zero, to prove the global existence of solutions of the problem (1.1)–(1.5) one needs to prove it for the problem (2.1)–(2.5). To this end, it suffices (see Henry [5]) to derive a uniform estimate of $\left\|F_i(z_1, z_2, z_3)\right\|_p$, $i = 1, 2, 3$ on $[0, T]$, $T < T^*$, for some $p > N/2$, where $\|\cdot\|_p$ denotes the usual norms in spaces $\mathbb{L}^p(\Omega)$ defined by

$$\|u\|_p^p = \frac{1}{|\Omega|}\int_\Omega |u(x)|^p \, dx, \quad 1 \le p < \infty, \quad \text{and} \quad \|u\|_\infty = \operatorname*{esssup}_{x\in\Omega} |u(x)|.$$

Let θ and σ be two positive constants such that

$$\theta > A_{12}, \tag{3.1}$$

$$\left(\theta^2 - A_{12}^2\right)\left(\sigma^2 - A_{23}^2\right) > \left(A_{13} - A_{12}A_{23}\right)^2, \tag{3.2}$$

where

$$A_{ij} = \frac{\lambda_i + \lambda_j}{2\sqrt{\lambda_i\lambda_j}}, \quad i, j = 1, 2, 3 \quad (i < j),$$

and let

$$\theta_q = \theta^{(p-q+1)^2} \quad \text{and} \quad \sigma_p = \sigma^{p^2}, \quad \text{for} \quad q = 0, 1, \ldots, p \quad \text{and} \quad p = 0, 1, \ldots, n, \tag{3.3}$$

where n is a positive integer. The main result of this section is

Theorem 1. *Let* $(z_1(t,\cdot), z_2(t,\cdot), z_3(t,\cdot))$ *be any positive solution of* (2.1)–(2.5). *Introduce the functional*

$$t \longmapsto L(t) = \int_\Omega H_n\big(z_1(t,x), z_2(t,x), z_3(t,x)\big)\,dx, \tag{3.4}$$

where

$$H_n(z_1,z_2,z_3) = \sum_{p=0}^{n}\sum_{q=0}^{p} C_n^p C_p^q \theta_q \sigma_p z_1^q z_2^{p-q} z_3^{n-p}, \tag{3.5}$$

with n *being a positive integer and* $C_n^p = \frac{n!}{(n-p)!p!}$.

Then the functional L *is uniformly bounded on the interval* $[0,T]$, $T < T^*$.

For the proof of Theorem 1 we need some preparatory Lemmas.

Lemma 1. *Let* H_n *be the homogeneous polynomial defined by* (3.5). *Then*

$$\frac{\partial H_n}{\partial z_1} = n\sum_{p=0}^{n-1}\sum_{q=0}^{p} C_{n-1}^p C_p^q \theta_{q+1}\sigma_{p+1} z_1^q z_2^{p-q} z_3^{(n-1)-p}, \tag{3.6}$$

$$\frac{\partial H_n}{\partial z_2} = n\sum_{p=0}^{n-1}\sum_{q=0}^{p} C_{n-1}^p C_p^q \theta_{q}\sigma_{p+1} z_1^q z_2^{p-q} z_3^{(n-1)-p}, \tag{3.7}$$

$$\frac{\partial H_n}{\partial z_3} = n\sum_{p=0}^{n-1}\sum_{q=0}^{p} C_{n-1}^p C_p^q \theta_{q}\sigma_{p} z_1^q z_2^{p-q} z_3^{(n-1)-p}. \tag{3.8}$$

Proof. Differentiating H_n with respect to z_1 and using the fact that

$$qC_p^q = pC_{p-1}^{q-1} \text{ and } pC_n^p = nC_{n-1}^{p-1} \tag{3.9}$$

for $q = 1,2,\dots,p$, $p = 1,2,\dots,n$, we get

$$\frac{\partial H_n}{\partial z_1} = n\sum_{p=1}^{n}\sum_{q=1}^{p} C_{n-1}^{p-1} C_{p-1}^{q-1} \theta_{q}\sigma_{p} z_1^{q-1} z_2^{p-q} z_3^{n-p}.$$

Replacing in the sums the indexes $q-1$ by q and $p-1$ by p, we deduce (3.6). For the formula (3.7), differentiating H_n with respect to z_2, taking into account

$$C_p^q = C_p^{p-q},\ \ q = 0,1,\dots,p-1 \text{ and } p = 1,2,\dots,n, \tag{3.10}$$

using (3.9) and replacing the index $p-1$ by p, we get (3.7).

Finally, we have

$$\frac{\partial H_n}{\partial z_3} = \sum_{p=0}^{n-1}\sum_{q=0}^{p} (n-p)\, C_n^p C_p^q \theta_q\sigma_p z_1^q z_2^{p-q} z_3^{n-p-1}.$$

Since $(n-p)\,C_n^p = (n-p)\,C_n^{n-p} = nC_{n-1}^{n-p-1} = nC_{n-1}^p$, we get (3.8). $\square$

Lemma 2. *The second partial derivatives of H_n are given by*

$$\frac{\partial^2 H_n}{\partial z_1^2} = n\,(n-1)\sum_{p=0}^{n-2}\sum_{q=0}^{p} C_{n-2}^{p} C_{p}^{q} \theta_{q+2}\sigma_{p+2} z_1^{q} z_2^{p-q} z_3^{(n-2)-p}, \tag{3.11}$$

$$\frac{\partial^2 H_n}{\partial z_1 \partial z_2} = n\,(n-1)\sum_{p=0}^{n-2}\sum_{q=0}^{p} C_{n-2}^{p} C_{p}^{q} \theta_{q+1}\sigma_{p+2} z_1^{q} z_2^{p-q} z_3^{(n-2)-p}, \tag{3.12}$$

$$\frac{\partial^2 H_n}{\partial z_1 \partial z_3} = n\,(n-1)\sum_{p=0}^{n-2}\sum_{q=0}^{p} C_{n-2}^{p} C_{p}^{q} \theta_{q+1}\sigma_{p+1} z_1^{q} z_2^{p-q} z_3^{(n-2)-p}, \tag{3.13}$$

$$\frac{\partial^2 H_n}{\partial z_2^2} = n\,(n-1)\sum_{p=0}^{n-2}\sum_{q=0}^{p} C_{n-2}^{p} C_{p}^{q} \theta_{q}\sigma_{p+2} z_1^{q} z_2^{p-q} z_3^{(n-2)-p}, \tag{3.14}$$

$$\frac{\partial^2 H_n}{\partial z_2 \partial z_3} = n\,(n-1)\sum_{p=0}^{n-2}\sum_{q=0}^{p} C_{n-2}^{p} C_{p}^{q} \theta_{q}\sigma_{p+1} z_1^{q} z_2^{p-q} z_3^{(n-2)-p}, \tag{3.15}$$

$$\frac{\partial^2 H_n}{\partial z_3^2} = n\,(n-1)\sum_{p=0}^{n-2}\sum_{q=0}^{p} C_{n-2}^{p} C_{p}^{q} \theta_{q}\sigma_{p} z_1^{q} z_2^{p-q} z_3^{(n-2)-p}. \tag{3.16}$$

Proof. Differentiating $\dfrac{\partial H_n}{\partial z_1}$ given by (3.6) with respect to z_1 yields

$$\frac{\partial^2 H_n}{\partial z_1^2} = n\sum_{p=1}^{n-1}\sum_{q=1}^{p} q C_{n-1}^{p} C_{p}^{q} \theta_{q+1}\sigma_{q+1} z_1^{q-1} z_2^{p-q} z_3^{(n-1)-p}.$$

Using (3.9), we get (3.11).

$$\frac{\partial^2 H_n}{\partial z_1 \partial z_2} = n\sum_{p=1}^{n-1}\sum_{q=0}^{p-1} (p-q)\, C_{n-1}^{p} C_{p}^{q} \theta_{q+1}\sigma_{p+1} z_1^{q} z_2^{p-q-1} z_3^{(n-1)-p}.$$

Applying (3.10) and then (3.9), we get (3.12).

$$\frac{\partial^2 H_n}{\partial z_1 \partial z_3} = n\sum_{p=0}^{n-2}\sum_{q=0}^{p} ((n-1)-p)\, C_{n-1}^{p} C_{p}^{q} \theta_{q+1}\sigma_{p+1} z_1^{q} z_2^{p-q} z_3^{(n-2)-p}.$$

Applying successively (3.10), (3.9) and (3.10) for the second time, we deduce (3.13).

$$\frac{\partial^2 H_n}{\partial z_2^2} = n\sum_{p=1}^{n-1}\sum_{q=0}^{p-1} (p-q)\, C_{n-1}^{p} C_{p}^{q} \theta_{q}\sigma_{p+1} z_1^{q} z_2^{p-q-1} z_3^{(n-1)-p}.$$

The application of (3.10) and then of (3.9) yields (3.14).

$$\frac{\partial^2 H_n}{\partial z_2 \partial z_3} = n\sum_{p=0}^{n-2}\sum_{q=0}^{p} ((n-1)-p)\, C_{n-1}^{p} C_{p}^{q} \theta_{q}\sigma_{p} z_1^{q} z_2^{p-q} z_3^{(n-2)-p}.$$

Applying (3.10) and then (3.9) yields (3.15). Finally we get (3.16) by differentiating $\frac{\partial H_n}{\partial z_3}$ with respect to z_3 and applying successively (3.10), (3.9) and (3.10) for the second time. $\square$

Proof of Theorem 1. Differentiating L with respect to t yields

$$\begin{aligned}
L'(t) &= \int_\Omega \left(\frac{\partial H_n}{\partial z_1} \frac{\partial z_1}{\partial t} + \frac{\partial H_n}{\partial z_2} \frac{\partial z_2}{\partial t} + \frac{\partial H_n}{\partial z_3} \frac{\partial z_3}{\partial t} \right) dx = \\
&= \int_\Omega \left(\lambda_1 \frac{\partial H_n}{\partial z_1} \Delta z_1 + \lambda_2 \frac{\partial H_n}{\partial z_2} \Delta z_2 + \lambda_3 \frac{\partial H_n}{\partial z_3} \Delta z_3 \right) dx + \\
&\quad + \int_\Omega \left(\frac{\partial H_n}{\partial z_1} F_1 + \frac{\partial H_n}{\partial z_2} F_2 + \frac{\partial H_n}{\partial z_3} F_3 \right) dx = \\
&=: I + J.
\end{aligned}$$

Using Green's formula in I, we get $I = I_1 + I_2$, where

$$I_1 = \int_{\partial\Omega} \left(\lambda_1 \frac{\partial H_n}{\partial z_1} \frac{\partial z_1}{\partial \eta} + \lambda_2 \frac{\partial H_n}{\partial z_2} \frac{\partial z_2}{\partial \eta} + \lambda_3 \frac{\partial H_n}{\partial z_3} \frac{\partial z_3}{\partial \eta} \right) ds,$$

where ds denotes the $(n-1)$-dimensional surface element, and

$$\begin{aligned}
I_2 = -\int_\Omega \Bigg[&\lambda_1 \frac{\partial^2 H_n}{\partial z_1^2} |\nabla z_1|^2 + (\lambda_1 + \lambda_2) \frac{\partial^2 H_n}{\partial z_1 \partial z_2} \nabla z_1 \nabla z_2 \\
&+ (\lambda_1 + \lambda_3) \frac{\partial^2 H_n}{\partial z_1 \partial z_3} \nabla z_1 \nabla z_3 + \lambda_2 \frac{\partial^2 H_n}{\partial z_2^2} |\nabla z_2|^2 \\
&+ (\lambda_2 + \lambda_3) \frac{\partial^2 H_n}{\partial z_2 \partial z_3} \nabla z_2 \nabla z_3 + \lambda_3 \frac{\partial^2 H_n}{\partial z_3^2} |\nabla z_3|^2 \Bigg] dx.
\end{aligned}$$

We prove that there exists a positive constant C_2 independent of $t \in [0, T^*[$ such that

$$I_1 \le C_2 \quad \text{for all} \quad t \in [0, T^*[\,, \tag{3.17}$$

and that

$$I_2 \le 0. \tag{3.18}$$

To see this, we follow the same reasoning as in [11].

(i) If $0 < \lambda < 1$, using the boundary conditions (2.4) we get

$$I_1 = \int_{\partial\Omega} \left(\lambda_1 \frac{\partial H_n}{\partial z_1} (\gamma_1 - \alpha z_1) + \lambda_2 \frac{\partial H_n}{\partial z_2} (\gamma_2 - \alpha z_2) + \lambda_3 \frac{\partial H_n}{\partial z_3} (\gamma_3 - \alpha z_3) \right) ds,$$

where $\alpha = \frac{\lambda}{1-\lambda}$ and $\gamma_i = \frac{\rho_i}{1-\lambda}$, $i = 1, 2, 3$. Since

$$\begin{aligned}
H(z_1, z_2, z_3) &= \lambda_1 \frac{\partial H_n}{\partial z_1} (\gamma_1 - \alpha z_1) + \lambda_2 \frac{\partial H_n}{\partial z_2} (\gamma_2 - \alpha z_2) + \lambda_3 \frac{\partial H_n}{\partial z_3} (\gamma_3 - \alpha z_3) \\
&= P_{n-1}(z_1, z_2, z_3) - Q_n(z_1, z_2, z_3),
\end{aligned}$$

where P_{n-1} and Q_n are polynomials with positive coefficients and respective degrees $n-1$ and n, and since the solution is positive, we obtain

$$\limsup_{(|z_1|+|z_2|+|z_3|)\to+\infty} H(z_1,z_2,z_3)=-\infty, \tag{3.19}$$

which proves that H is uniformly bounded on $(\mathbb{R}^+)^3$, and consequently (3.17).

(ii) If $\lambda=0$, then $I_1=0$ on $[0,T^*[$.

(iii) The case of the homogeneous Dirichlet conditions is trivial since the positivity of the solution on $[0,T^*[\times\Omega$ implies $\frac{\partial z_1}{\partial\eta}\le 0$, $\frac{\partial z_2}{\partial\eta}\le 0$ and $\frac{\partial z_3}{\partial\eta}\le 0$ on $[0,T^*[\times\partial\Omega$. Consequently, one again gets (3.17) with $C_2=0$.

Now, we prove (3.18). Applying Lemma 1 and Lemma 2, we get

$$I_2=-n(n-1)\int_\Omega\sum_{p=0}^{n-2}\sum_{q=0}^{p}C_{n-2}^{p}C_p^q\left[(B_{pq}z)\cdot z\right]dx,$$

where

$$B_{pq}=\begin{pmatrix}\lambda_1\theta_{q+2}\sigma_{p+2} & \frac{\lambda_1+\lambda_2}{2}\theta_{q+1}\sigma_{p+2} & \frac{\lambda_1+\lambda_3}{2}\theta_{q+1}\sigma_{p+1}\\ \frac{\lambda_1+\lambda_2}{2}\theta_{q+1}\sigma_{p+2} & \lambda_2\theta_q\sigma_{p+2} & \frac{\lambda_2+\lambda_3}{2}\theta_q\sigma_{p+1}\\ \frac{\lambda_1+\lambda_3}{2}\theta_{q+1}\sigma_{p+1} & \frac{\lambda_2+\lambda_3}{2}\theta_q\sigma_{p+1} & \lambda_3\theta_q\sigma_p\end{pmatrix},$$

for $q=0,1,\ldots,p$, $p=0,1,\ldots,n-2$ and $z=(\nabla z_1,\nabla z_2,\nabla z_3)^t$.

The quadratic forms (with respect to $\nabla z_1,\nabla z_2$ and ∇z_3) associated with the matrices B_{pq}, $q=0,1,\ldots,p$, $p=0,1,\ldots,n-2$, are positive since their main determinants Δ_1, Δ_2 and Δ_3 are positive too, according to the Sylvester criterion. To see this, we have

1. $\Delta_1=\lambda_1\theta_{q+2}\sigma_{p+2}>0$ for $q=0,1,\ldots,p$ and $p=0,1,\ldots,n-2$.
2. $$\Delta_2=\begin{vmatrix}\lambda_1\theta_{q+2}\sigma_{p+2} & \frac{\lambda_1+\lambda_2}{2}\theta_{q+1}\sigma_{p+2}\\ \frac{\lambda_1+\lambda_2}{2}\theta_{q+1}\sigma_{p+2} & \lambda_2\theta_q\sigma_{p+2}\end{vmatrix}$$
   $$=\lambda_1\lambda_2\theta_{q+1}^2\sigma_{p+2}^2\left(\theta^2-A_{12}^2\right),$$
   for $q=0,1,\ldots,p$ and $p=0,1,\ldots,n-2$.

   Using (3.1), we get $\Delta_2>0$.
3. $$\Delta_3=\begin{vmatrix}\lambda_1\theta_{q+2}\sigma_{p+2} & \frac{\lambda_1+\lambda_2}{2}\theta_{q+1}\sigma_{p+2} & \frac{\lambda_1+\lambda_3}{2}\theta_{q+1}\sigma_{p+1}\\ \frac{\lambda_1+\lambda_2}{2}\theta_{q+1}\sigma_{p+2} & \lambda_2\theta_q\sigma_{p+2} & \frac{\lambda_2+\lambda_3}{2}\theta_q\sigma_{p+1}\\ \frac{\lambda_1+\lambda_3}{2}\theta_{q+1}\sigma_{p+1} & \frac{\lambda_2+\lambda_3}{2}\theta_q\sigma_{p+1} & \lambda_3\theta_q\sigma_p\end{vmatrix}$$
   $$=\lambda_1\lambda_2\lambda_3\theta_{q+1}^2\theta_q\sigma_{p+2}\sigma_{p+1}^2\left[(\theta^2-A_{12}^2)(\sigma^2-A_{23}^2)-(A_{13}-A_{12}A_{23})^2\right],$$
   for $q=0,1,\ldots,p$ and $p=0,1,\ldots,n-2$.

Using (3.2), we get $\Delta_3 > 0$. Consequently we have (3.18).

Substitution of the expressions of the partial derivatives given by Lemma 1 in the second integral yields

$$J = \int\limits_{\Omega} \left[n \sum_{p=0}^{n-1} \sum_{q=0}^{p} C_{n-1}^{p} C_{p}^{q} z_1^q z_2^{p-q} z_3^{(n-1)-p} \right] \times$$
$$\times \left(\theta_{q+1}\sigma_{p+1}F_1 + \theta_q\sigma_{p+1}F_2 + \theta_q\sigma_p F_3 \right) dx.$$

Using the expressions (2.7a), we get

$$\begin{aligned}
&\theta_{q+1}\sigma_{p+1}F_1 + \theta_q\sigma_{p+1}F_2 + \theta_q\sigma_p F_3 = \\
&= \big(-\theta_{q+1}\sigma_{p+1}a_{21} + a_{21}\theta_q\sigma_{p+1} - a_{32}\theta_q\sigma_p \big) f + \left(\theta_{q+1}\sigma_{p+1}\mu_1 - \mu_2\theta_q\sigma_{p+1} \right) g + \\
&\qquad + \big(-\theta_{q+1}\sigma_{p+1}a_{23} + a_{23}\theta_q\sigma_{p+1} + a_{12}\theta_q\sigma_p \big) h = \\
&= \Big(a_{23}(\theta_q\sigma_{p+1} - \theta_{q+1}\sigma_{p+1}) + a_{12}\theta_q\sigma_p \Big) \left(\frac{a_{21}\left(\theta_q\sigma_{p+1} - \theta_{q+1}\sigma_{p+1}\right) - a_{32}\theta_q\sigma_p}{a_{23}(\theta_q\sigma_{p+1} - \theta_{q+1}\sigma_{p+1}) + a_{12}\theta_q\sigma_p} f + \right. \\
&\qquad \left. + \frac{\theta_{q+1}\sigma_{p+1}\mu_1 - \mu_2\theta_q\sigma_{p+1}}{a_{23}\left(\theta_q\sigma_{p+1} - \theta_{q+1}\sigma_{p+1}\right) + a_{12}\theta_q\sigma_p} g + h \right) = \\
&\qquad = \theta_{q+1}\sigma_p \left(a_{23}\frac{\sigma_{p+1}}{\sigma_p}\left(\frac{\theta_q}{\theta_{q+1}} - 1 \right) + a_{12}\frac{\theta_q}{\theta_{q+1}} \right) \times \\
&\times \left(\frac{a_{21}\frac{\sigma_{p+1}}{\sigma_p}\left(\frac{\theta_q}{\theta_{q+1}} - 1\right) - a_{32}\frac{\theta_q}{\theta_{q+1}}}{a_{23}\frac{\sigma_{p+1}}{\sigma_p}\left(\frac{\theta_q}{\theta_{q+1}} - 1\right) + a_{12}\frac{\theta_q}{\theta_{q+1}}} f + \frac{\mu_1\frac{\sigma_{p+1}}{\sigma_p} - \mu_2\frac{\theta_q}{\theta_{q+1}}\frac{\sigma_{p+1}}{\sigma_p}}{a_{23}\frac{\sigma_{p+1}}{\sigma_p}\left(\frac{\theta_q}{\theta_{q+1}} - 1\right) + a_{12}\frac{\theta_q}{\theta_{q+1}}} g + h \right).
\end{aligned}$$

Since $\frac{\theta_q}{\theta_{q+1}}$ and $\frac{\sigma_{p+1}}{\sigma_p}$ are sufficiently large if we choose θ and σ sufficiently large, using the condition (1.7) and the relation (2.6a) successively we get, for an appropriate constant C_3,

$$J \le C_3 \int\limits_{\Omega} \left[\sum_{p=0}^{n-1} \sum_{q=0}^{p} \left(z_1 + z_2 + z_3 + 1 \right) C_{n-1}^{p} C_{p}^{q} z_1^q z_2^{p-q} z_3^{(n-1)-p} \right] dx.$$

To prove that the functional L is uniformly bounded on the interval $[0, T]$, we first write

$$\sum_{p=0}^{n-1} \sum_{q=0}^{p} \left(z_1 + z_2 + z_3 + 1 \right) C_{n-1}^{p} C_{p}^{q} z_1^q z_2^{p-q} z_3^{(n-1)-p} =$$
$$= R_n\left(z_1, z_2, z_3\right) + S_{n-1}\left(z_1, z_2, z_3\right),$$

where $R_n\left(z_1, z_2, z_3\right)$ and $S_{n-1}\left(z_1, z_2, z_3\right)$ are two homogeneous polynomials of degrees n and $n-1$, respectively. First, since the polynomials H_n and R_n are of degree n, there exists a positive constant C_4 such that

$$\int\limits_{\Omega} R_n\left(z_1, z_2, z_3\right) dx \le C_4 \int\limits_{\Omega} H_n\left(z_1, z_2, z_3\right) dx.$$

Applying Hölder's inequality to the integral $\int_\Omega S_{n-1}(z_1, z_2, z_3)\, dx$, one gets

$$\int\limits_\Omega S_{n-1}(z_1, z_2, z_3)\, dx \le (meas\,\Omega)^{\frac{1}{n}} \left(\int\limits_\Omega (S_{n-1}(z_1, z_2, z_3))^{\frac{n}{n-1}}\, dx \right)^{\frac{n-1}{n}}.$$

Since for all $z_1 \ge 0$ and $z_2, z_3 > 0$

$$\frac{(S_{n-1}(z_1, z_2, z_3))^{\frac{n}{n-1}}}{H_n(z_1, z_2, z_3)} = \frac{(S_{n-1}(\xi_1, \xi_2, 1))^{\frac{n}{n-1}}}{H_n(\xi_1, \xi_2, 1)},$$

where $\xi_1 = \frac{z_1}{z_2}$, $\xi_2 = \frac{z_2}{z_3}$ and

$$\lim_{\substack{\xi_1 \to +\infty \\ \xi_2 \to +\infty}} \frac{(S_{n-1}(\xi_1, \xi_2, 1))^{\frac{n}{n-1}}}{H_n(\xi_1, \xi_2, 1)} < +\infty,$$

one asserts that there exists a positive constant C_5 such that

$$\frac{(S_{n-1}(z_1, z_2, z_3))^{\frac{n}{n-1}}}{H_n(z_1, z_2, z_3)} \le C_5 \ \text{ for all } \ z_1, z_2, z_3 \ge 0.$$

Hence the functional L satisfies the differential inequality

$$L'(t) \le C_6 L(t) + C_7 L^{\frac{n-1}{n}}(t),$$

which for $Z = L^{\frac{1}{n}}$ can be written as

$$nZ' \le C_6 Z + C_7.$$

A simple integration gives a uniform bound of the functional L on the interval $[0, T]$. This completes the proof of Theorem 1. □

Corollary 1. *Suppose that the functions $f(r_1, r_2, r_3)$, $g(r_1, r_2, r_3)$ and $h(r_1, r_2, r_3)$ are continuously differentiable on Σ, point into Σ on $\partial\Sigma$ and satisfy the condition* (1.7). *Then all uniformly bounded on Ω solutions of* (1.1)–(1.5) *with the initial data in Σ are in $L^\infty(0, T; L^p(\Omega))$ for all $p \ge 1$.*

Proof. The proof of this Corollary is an immediate consequence of Theorem 1, the trivial inequality

$$\int\limits_\Omega (z_1 + z_2 + z_3)^p\, dx \le L(t) \quad \text{on} \quad [0, T^*[,$$

and (2.6a). □

Proposition 2. *Under the hypothesis of Corollary* 1, *if $f(r_1, r_2, r_3)$, $g(r_1, r_2, r_3)$ and $h(r_1, r_2, r_3)$ are polynomially bounded, then all uniformly bounded on Ω solutions of* (1.1)–(1.4) *with the initial data in Σ are global in time.*

Proof. As has been mentioned above, it suffices to derive a uniform estimate of $\|F_1(z_1,z_2,z_3)\|_p$, $\|F_2(z_1,z_2,z_3)\|_p$ and $\|F_3(z_1,z_2,z_3)\|_p$ on $[0,T]$, $T<T^*$ for some $p>\frac{N}{2}$. Since the reactions $f\,(u,v,w)$, $g\,(u,v,w)$ and $h\,(u,v,w)$ are polynomially bounded on Σ, by using relations (2.6a) and (2.7a) we get that so are $F_1(z_1,z_2,z_3)$, $F_2(z_1,z_2,z_3)$ and $F_3(z_1,z_2,z_3)$, and the proof becomes an immediate consequence of Corollary 1. □

Acknowledgments

The authors would like to thank the anonymous referee for his/her valuable comments and suggestions that helped to improve the presentation of this paper.

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(Received 22.01.2008; revised 23.09.2009)

Authors' addresses:

Said Kouachi
Department of Mathematics
College of Science
Qassim University
P.O.Box 6644, Al-Gassim, Buraydah 51452
Saudi Arabia
E-mail: kouachi.said@caramil.com

Belgacem Rebiai
Department of Mathematics
University of Tebessa, 12002
Algeria
E-mail: brebiai@gmail.com