**Memoirs on Differential Equations and Mathematical Physics**
Volume 54, 2011, 51–68

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# CONDITIONAL WELL-POSEDNESS OF NONLOCAL PROBLEMS FOR FOURTH ORDER LINEAR HYPERBOLIC EQUATIONS WITH SINGULARITIES

*Dedicated to the blessed memory of Professor T. Chanturia*

**Abstract.** Unimprovable in a sense sufficient conditions of well-posedness of nonlocal problems are established for fourth order linear hyperbolic equations with singular coefficient.

**2010 Mathematics Subject Classification.** 34B05, 34B10.
**Key words and phrases.** Linear, singular differential equation, nonlocal problem.

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## 1. Formulation of the Main Results

1.1. **Statement of the problem.** In the rectangle $\Omega = [0,a] \times [0,b]$ consider the linear hyperbolic equation

$$u^{(2,2)} = \sum_{i=1}^{2}\sum_{k=1}^{2} h_{ik}(x,y)u^{(i-1,k-1)} + h(x,y) \tag{1.1}$$

with the nonlocal boundary conditions

$$\begin{aligned} &\int_0^a u(s,y)\,d\alpha_i(s) = 0 \ \text{ for } \ 0 \le y \le b \ \ (i=1,2), \\ &\int_0^b u(x,t)\,d\beta_k(t) = 0 \ \text{ for } \ 0 \le x \le a \ \ (k=1,2). \end{aligned} \tag{1.2}$$

Here

$$u^{(i,k)}(x,y) = \frac{\partial^{i+k}u(x,y)}{\partial x^i \partial y^k} \ \ (i,\,k=0,1,2),$$

$h_{ik} : \Omega \to \mathbb{R}$ $(i,\,k=1,2)$ are measurable functions, $h \in L(\Omega)$, and $\alpha_i : [0,a] \to \mathbb{R}$ and $\beta_i : [0,b] \to \mathbb{R}$ $(i=1,2)$ are functions of bounded variation.

We will use the following notation.

$L(\Omega)$ is the Banach space of Lebesgue integrable functions $v : \Omega \to \mathbb{R}$ with the norm

$$\|v\|_L = \int_0^a\int_0^b |v(x,y)|\,dx\,dy.$$

$C^{1,1}(\Omega)$ is the space of functions $u : \Omega \to \mathbb{R}$, continuous together with $u^{(i-1,k-1)}$ $(i,\,k=1,2)$, with the norm

$$\|u\|_{C^{1,1}} = \max\bigg\{\sum_{i=1}^{2}\sum_{k=1}^{2}\big|u^{(i-1,k-1)}(x,y)\big| : \ (x,y) \in \Omega\bigg\}.$$

$\widetilde{C}^{1,1}(\Omega)$ is the space of functions $u \in C^{1,1}(\Omega)$ for which $u^{(1,1)}$ is absolutely continuous (see, e.g., [1,4]).

The function $u \in \widetilde{C}^{1,1}(\Omega)$ is said to be a *solution of equation* (1.1) if it satisfies that equation almost everywhere on $\Omega$.

A solution of equation (1.1) satisfying boundary conditions (1.2) is called a *solution of problem* (1.1), (1.2).

Along with the equation (1.1) consider the corresponding homogeneous and perturbed equations

$$u^{(2,2)} = \sum_{i=1}^{2}\sum_{k=1}^{2} h_{ik}(x,y)u^{(i-1,k-1)}, \tag{1.1$_0$}$$

$$u^{(2,2)} = \sum_{i=1}^{2}\sum_{k=1}^{2} h_{ik}(x,y)u^{(i-1,k-1)} + \widetilde{h}(x,y), \tag{1.1$'$}$$

with the nonhomogeneous boundary conditions

$$\begin{gathered}\int\limits_0^a u(s,y)\,d\alpha_i(s)=\int\limits_0^a c(s,y)\,d\alpha_i(s) \ \text{ for } \ 0\le y\le b \ \ (i=1,2),\\ \int\limits_0^b u(x,t)\,d\beta_k(t)=\int\limits_0^b c(x,t)\,d\beta_k(t) \ \text{ for } \ 0\le x\le a \ \ (k=1,2).\end{gathered}\tag{1.2'}$$

Following [2], introduce the definitions.

**Definition 1.1.** Problem (1.1), (1.2) is said to be *well-posed* if for arbitrary $\widetilde{h}\in L(\Omega)$ and $c\in\widetilde{C}^{1,1}(\Omega)$ problem (1.1′), (1.2′) is uniquely solvable, and there exists a positive constant $r$ independent of $\widetilde{h}$ and $c$ such that

$$\|\widetilde{u}-u\|_{C^{1,1}}\le r\big(\|c\|_{C^{1,1}}+\|\widetilde{h}-h\|_L\big),$$

where $u$ and $\widetilde{u}$, respectively, are solutions of problems (1.1), (1.2) and (1.1′), (1.2′).

**Definition 1.2.** Problem (1.1), (1.2) is said to be *conditionally well-posed* if for an arbitrary $\widetilde{h}\in L(\Omega)$ problem (1.1′), (1.2) is uniquely solvable, and the exists a positive constant $r$ independent of $\widetilde{h}$ such that

$$\|\widetilde{u}-u\|_{C^{1,1}}\le r\|\widetilde{h}-h\|_L,$$

where $u$ and $\widetilde{u}$, respectively, are solutions of problems (1.1), (1.2) and (1.1′), (1.2).

In the case where the coefficients of equation (1.1) are continuous functions sufficient conditions of well-posedness of problems of type (1.1), (1.2) are established in [3–7]. We are interested in the singular case, where some of the coefficients $h_{ik}$ $(i,\,k=1,2)$ are nonintegrable on $\Omega$. Until recently, for singular equations only the Dirichlet problem has been studied [8].

General theorems on conditional well-posedness of nonlocal problems for higher order linear hyperbolic equations with singular coefficients are proved in [2]. In the present paper effective and unimprovable in a sense conditions, guaranteeing conditional well-posedness of the singular problem (1.1), (1.2), are established on the basis of those results.

The following boundary conditions are the particular cases of (1.2):

$$\begin{gathered}u(0,y)=0,\ \ u(a,y)=0 \ \text{ for } \ 0\le y\le b,\\ u(x,0)=0,\ \ u(x,b)=0 \ \text{ for } \ 0\le x\le a\end{gathered}\tag{1.2$_1$}$$

and

$$\begin{gathered}u(0,y)=0,\ \ \int\limits_0^a u(s,y)\,d\alpha(s)=0 \ \text{ for } \ 0\le y\le b,\\ u(x,0)=0,\ \ \int\limits_0^b u(x,t)\,d\beta(t)=0 \ \text{ for } \ 0\le x\le a,\end{gathered}\tag{1.2$_2$}$$

where $\alpha : [0,a] \to \mathbb{R}$ and $\beta : [0,b] \to \mathbb{R}$ are functions of bounded variation.

The theorems proved below imply new sufficient conditions of conditional well-posedness of problems $(1.1), (1.2_1)$ and $(1.1), (1.2_2)$.

1.2. **Theorems on the Conditional Well-Posedness of Problem (1.1), (1.2).** Let

$$
\begin{aligned}
\Delta_1(x) &= \alpha_2(a) \int\limits_x^a \alpha_1(s)\,ds - \alpha_1(a) \int\limits_x^a \alpha_2(s)\,ds, \\
\Delta_2(y) &= \beta_2(b) \int\limits_y^b \beta_1(t)\,dt - \beta_1(b) \int\limits_y^b \beta_2(t)\,dt.
\end{aligned} \tag{1.3}
$$

We study problem $(1.1), (1.2)$ in the case, where

$$
\alpha_i(0) = 0, \ \ \beta_i(0) = 0, \ \ \Delta_i(0) \neq 0 \quad (i = 1, 2). \tag{1.4}
$$

Introduce the functions

$$
\chi(t,s) = \begin{cases} 1 & \text{for } \ s \le t, \\ 0 & \text{for } \ s > t, \end{cases} \tag{1.5}
$$

$$
\begin{aligned}
g_1(x,s) = \frac{1}{\Delta_1(0)} \biggl[ &\int\limits_0^a \alpha_1(\tau)\,d\tau \int\limits_s^a \alpha_2(\tau)\,d\tau - \int\limits_s^a \alpha_1(\tau)\,d\tau \int\limits_0^a \alpha_2(\tau)\,d\tau + \\
&+ (s-a)\Delta_1(0) + (a-x)\Delta_1(s) \biggr] + \chi(x,s)(x-s) \ \text{ for } \ 0 \le x, \ s \le a,
\end{aligned} \tag{1.6}
$$

$$
\begin{aligned}
g_2(y,t) = \frac{1}{\Delta_2(0)} \biggl[ &\int\limits_0^b \beta_1(\tau)\,d\tau \int\limits_t^b \beta_2(\tau)\,d\tau - \int\limits_t^b \beta_1(\tau)\,d\tau \int\limits_0^b \beta_2(\tau)\,d\tau + \\
&+ (t-b)\Delta_2(0) + (b-y)\Delta_1(t) \biggr] + \chi(y,t)(y-t) \ \text{ for } \ 0 \le y, \ t \le b,
\end{aligned} \tag{1.7}
$$

$$
\begin{aligned}
\varphi_{11}(x) &= \max\bigl\{ |g_1(x,s)| : \ 0 \le s \le a \bigr\}, \\
\varphi_{12}(x) &= \sup\Bigl\{ \bigl|g_1^{(1,0)}(x,s)\bigr| : \ 0 \le s \le a, \ s \neq x \Bigr\},
\end{aligned} \tag{1.8}
$$

$$
\begin{aligned}
\varphi_{21}(y) &= \max\bigl\{ |g_2(y,t)| : \ 0 \le t \le b \bigr\}, \\
\varphi_{22}(y) &= \sup\Bigl\{ \bigl|g_2^{(1,0)}(y,t)\bigr| : \ 0 \le t \le b, \ t \neq y \Bigr\}.
\end{aligned} \tag{1.9}
$$

**Theorem 1.1.** *If along with* $(1.4)$ *the inequalities*

$$
\int\limits_0^b \int\limits_0^a \varphi_{1i}(x)\varphi_{2k}(y)|h_{ik}(x,y)|\,dx\,dy < +\infty \ \ (i, \, k = 1, 2) \tag{1.10}
$$

*hold, then problem* $(1.1), (1.2)$ *is conditionally well-posed if and only if the corresponding homogeneous problem* $(1.1_0), (1.2)$ *has only the trivial solution.*

**Theorem 1.2.** *If along with* (1.4) *the condition*

$$\sum_{i=1}^{2}\sum_{k=1}^{2}\int_0^b\int_0^a \varphi_i(x)\psi_k(y)|h_{ik}(x,y)|\,dx\,dy<1 \tag{1.11}$$

*holds, then problem* (1.1), (1.2) *is conditionally well-posed. Moreover, if*

$$h_{ik}\in L(\Omega)\ \ (i,k=1,2), \tag{1.12}$$

*then problem* (1.1), (1.2) *is well-posed.*

**Theorem 1.3.** *If conditions* (1.4) *and* (1.11) *hold, and*

$$\int_0^b\int_0^a |h_{11}(x,y)|\,dx\,dy=+\infty, \tag{1.13}$$

*then problem* (1.1), (1.2) *is conditionally well-posed but not well-posed.*

1.3. **Corollaries for problem (1.1), ($1.2_1$).**

**Corollary 1.1.** *If*

$$\int_0^b\int_0^a\Big[x\Big(1-\frac{x}{a}\Big)\Big]^{2-i}\Big[y\Big(1-\frac{y}{b}\Big)\Big]^{2-k}|h_{ik}(x,y)|\,dx\,dy<+\infty\ \ (i,\,k=1,2) \tag{1.14}$$

*hold, then problem* (1.1), ($1.2_1$) *is conditionally well-posed if and only if the corresponding homogeneous problem* ($1.1_0$), ($1.2_1$) *has only the trivial solution.*

**Corollary 1.2.** *Let either*

$$\sum_{i=1}^{2}\sum_{k=1}^{2}\int_0^b\int_0^a\Big[x\Big(1-\frac{x}{a}\Big)\Big]^{2-i}\Big[y\Big(1-\frac{y}{b}\Big)\Big]^{2-k}|h_{ik}(x,y)|\,dx\,dy<1, \tag{1.15}$$

*or*

$$\operatorname{ess\,sup}\bigg\{\sum_{i=1}^{2}\sum_{k=1}^{2}\Big[x\Big(1-\frac{x}{a}\Big)\Big]^{2-i}\Big[y\Big(1-\frac{y}{b}\Big)\Big]^{2-k}|h_{ik}(x,y)|:(x,y)\in\Omega\bigg\}< \\ <\frac{4}{ab}\,. \tag{1.16}$$

*Then problem* (1.1), ($1.2_1$) *is conditionally well-posed. Moreover, if along with* (1.15) (*along with* (1.16)) *condition* (1.12) *holds, then problem* (1.1), ($1.2_1$) *is well-posed.*

**Corollary 1.3.** *Let along with condition* (1.13) *either of conditions* (1.15) *and* (1.16) *hold. Then problem* (1.1), ($1.2_1$) *is conditionally well-posed but not well-posed.*

1.4. **Corollaries for problem (1.1), (1.2₂).** We study problem (1.1), $(1.2_2)$ in the case, where

$$\begin{aligned} &\alpha(0)=0, \ \ \alpha(x)\le\alpha(a) \ \text{ a.e. on } [0,a], \ \ \int_0^a \alpha(x)\,dx < a\alpha(a),\\ &\beta(0)=0, \ \ \beta(y)\le\beta(b) \ \text{ a.e. on } [0,b], \ \ \int_0^b \beta(y)\,dy < b\beta(b). \end{aligned} \tag{1.17}$$

**Corollary 1.4.** *If along with* (1.17) *the condition*

$$\int_0^b\int_0^a x^{2-i}y^{2-k}|h_{ik}(x,y)|\,dx\,dy < +\infty \ \ (i,\,k=1,2) \tag{1.18}$$

*holds, then problem* $(1.1), (1.2_2)$ *is conditionally well-posed if and only if the corresponding homogeneous problem* $(1.1_0), (1.2_2)$ *has only the trivial solution.*

**Corollary 1.5.** *If along with* (1.17) *the condition*

$$\sum_{i=1}^2\sum_{k=1}^2\int_0^b\int_0^a x^{2-i}y^{2-k}|h_{ik}(x,y)|\,dx\,dy < 1 \tag{1.19}$$

*holds, then problem* $(1.1), (1.2_2)$ *is conditionally well-posed. Moreover, if along with* (1.17) *and* (1.19) *condition* (1.12) *holds, then problem* $(1.1), (1.2_2)$ *is well-posed.*

**Corollary 1.6.** *If along with* (1.17) *and* (1.19) *condition* (1.13) *holds, then problem* $(1.1), (1.2_2)$ *is conditionally well-posed but not well-posed.*

1.5. **Examples.** The examples below demonstrate that in Theorem 1.2 (in Corollary 1.2) condition (1.11) (condition (1.15), as well as condition (1.16)) is unimprovable in a sense.

**Example 1.1.** Let $\varepsilon$ be an arbitrary positive number and $\gamma > 1$ be sufficiently large number such that

$$\Big(\frac{\gamma+1}{\gamma-1}\Big)^2 < 1+\varepsilon. \tag{1.20}$$

Set

$$h_0(t) = \begin{cases} (\gamma+1)t^{\gamma-2} - t^{2\gamma-2} & \text{for } \ 0\le t\le 1,\\ (\gamma+1)(2-t)^{\gamma-2} - (2-t)^{2\gamma-2} & \text{for } \ 1<t\le 2, \end{cases} \tag{1.21}$$

$$w_0(t) = \begin{cases} t\exp\Big(-\dfrac{t^\gamma}{\gamma}\Big) & \text{for } \ 0\le t\le 1,\\ (2-t)\exp\Big(-\dfrac{(2-t)^\gamma}{\gamma}\Big) & \text{for } \ 1<t\le 2, \end{cases}$$

and consider the differential equation (1.1), where $h \in L(\Omega)$ and

$$h_{11}(x,y) = \frac{16}{a^2b^2} h_0\Big(\frac{2x}{a}\Big) h_0\Big(\frac{2y}{b}\Big), \quad h_{ik}(x,y) = 0 \quad \text{for } (x,y) \in \Omega, \;\; i+k \neq 2. \tag{1.22}$$

Then problem $(1.1), (1.2_1)$ is *not* conditionally well-posed since the corresponding homogeneous problem $(1.1_0), (1.2_1)$ has the nontrivial solution

$$u(x,y) = w_0\Big(\frac{2x}{a}\Big) w_0\Big(\frac{2y}{b}\Big).$$

On the other hand, according to (1.21) and (1.22) we have

$$\sum_{i=1}^{2}\sum_{k=1}^{2}\int_0^b\int_0^a \bigg[x\Big(1-\frac{x}{a}\Big)\bigg]^{2-i}\bigg[y\Big(1-\frac{y}{b}\Big)\bigg]^{2-k} |h_{ik}(x,y)|\,dx\,dy =$$

$$= \frac{16}{a^2b^2}\int_0^a x\Big(1-\frac{x}{a}\Big) h_0\Big(\frac{2x}{a}\Big)\,dx \int_0^b y\Big(1-\frac{y}{b}\Big) h_0\Big(\frac{2y}{b}\Big)\,dy \leq$$

$$\leq \frac{1}{ab}\int_0^a h_0\Big(\frac{2x}{a}\Big)\,dx \int_0^b h_0\Big(\frac{2y}{b}\Big)\,dy = \frac{1}{4}\int_0^2 h_0(t)\,dt =$$

$$= \bigg(\int_0^1 h_0(t)\,dt\bigg)^2 < \bigg(\frac{\gamma+1}{\gamma-1}\bigg)^2.$$

Hence, by (1.20) it follows that

$$\sum_{i=1}^{2}\sum_{k=1}^{2}\int_0^b\int_0^a \bigg[x\Big(1-\frac{x}{a}\Big)\bigg]^{2-i}\bigg[y\Big(1-\frac{y}{b}\Big)\bigg]^{2-k} |h_{ik}(x,y)|\,dx\,dy < 1+\varepsilon. \tag{1.23}$$

Consequently, in Corollary 1.2 condition (1.15) cannot be replaced by the condition

$$\sum_{i=1}^{2}\sum_{k=1}^{2}\int_0^b\int_0^a \varphi_{1i}(x)\varphi_{2k}(y)|h_{ik}(x,y)|\,dx\,dy < 1+\varepsilon$$

no matter how small $\varepsilon > 0$ might be.

**Example 1.2.** Let

$$h_{11}(x,y) = \frac{4}{x(a-x)y(b-y)}, \quad h_{ik}(x,y) = 0 \;\text{ for }\; (x,y) \in \Omega, \;\; i+k > 2.$$

Then

$$\operatorname{ess\,sup}\left\{\sum_{i=1}^{2}\sum_{k=1}^{2}\Big[x\Big(1-\frac{x}{a}\Big)\Big]^{2-i}\Big[y\Big(1-\frac{y}{b}\Big)\Big]^{2-k}|h_{ik}(x,y)| : (x,y)\in\Omega\right\} = \\ = \frac{4}{ab}\,. \tag{1.24}$$

On the other hand, problem $(1.1),(1.2_1)$ is *not* conditionally well-posed, since its corresponding homogeneous problem $(1.1_0),1.2_1()$ has the nontrivial solution

$$u(x,y)=x(x-a)y(y-b).$$

Consequently, in Corollary 1.2 inequality (1.16) cannot be replaced by equality (1.24).

## 2. Auxiliary Statements

By $L([0,T])$ we denote the space of Lebesgue integrable functions $v : [0,T]\to\mathbb{R}$ endowed with the norm

$$\|v\|_L=\int_0^T|v(t)|\,dt,$$

and by $\widetilde{C}^1([0,T])$ we denote the space of continuously differentiable functions $u:[0,T]\to\mathbb{R}$ for which $u'$ is absolutely continuous.

Also, we will need to consider the second order ordinary differential equation

$$u''=q(t) \tag{2.1}$$

with the nonlocal boundary conditions

$$\int_0^T u(t)\,d\gamma_i(t)=0 \quad (i=1,2), \tag{2.2}$$

where $q\in L([0,T])$, and $\gamma_i:[0,T]\to\mathbb{R}$ $(i=1,2)$ are functions of bounded variation such that

$$\gamma_i(0)=0 \quad (i=1,2). \tag{2.3}$$

A solution of problem $(2.1),(2.2)$ will be sought in the space $\widetilde{C}^1([0,T])$.

2.1. **Lemmas on estimates of solutions to problems of type (2.1), (2.2).** Let

$$\Delta(t)=\gamma_2(T)\int_t^T\gamma_1(s)\,ds-\gamma_1(T)\int_t^T\gamma_2(s)\,ds \quad\text{for}\quad 0\le t\le T. \tag{2.4}$$

If $\Delta(0)\neq 0$, then set

$$g(t,s)=\frac{1}{\Delta(0)}\Big[\int\limits_0^T\gamma_1(\tau)\,d\tau\int\limits_s^T\gamma_2(\tau)\,d\tau-\int\limits_s^T\gamma_1(\tau)\,d\tau\int\limits_0^T\gamma_2(\tau)\,d\tau\Big]+$$
$$+\frac{1}{\Delta(0)}\big[(s-T)\Delta(0)+(T-t)\Delta(s)\big]+\chi(t,s)(t-s)\ \text{ for }\ 0\le t,s\le T, \tag{2.5}$$

where $\chi$ is the function given by equality (1.5).

**Lemma 2.1.** *Problem* (2.1) *is uniquely solvable if and only if*

$$\Delta(0)\neq 0. \tag{2.6}$$

*Moreover, is condition* (2.6) *holds, then the function $g$ given equality* (2.5) *is the Green's function of the boundary value problem*

$$u''=0;\quad \int\limits_0^T u(t)\,d\gamma_i(t)=0\quad (i=1,2), \tag{2.7}$$

*and a solution $u$ of problem* (2.1), (2.2) *admits the estimates*

$$|u^{(i-1)}(t)|\le\varphi_i(t)\|h\|_L\ \ \textit{for}\ \ 0\le t\le T (i=1,2), \tag{2.8}$$

*where*

$$\begin{aligned}&\varphi_1(t)=\max\big\{|g(t,s)|:\ 0\le s\le T\big\},\\ &\varphi_2(t)=\sup\Big\{|g^{(1,0)}(t,s)|:\ 0\le s\le T,\ \ s\neq t\Big\}.\end{aligned} \tag{2.9}$$

*Proof.* An arbitrary solution of equation (2.1) admits the representation

$$u(t)=c_1+c_2t+\int\limits_0^t(t-s)q(s)\,ds\ \text{ for }\ 0\le t\le T. \tag{2.10}$$

In view of (2.3) the function $u$ is a solution of problem (2.1), (2.2) if and only if $(c_1,c_2)$ is a solution of the system of linear algebraic equation

$$\gamma_i(T)c_1+\Big(\int\limits_0^T\tau\,d\gamma_i(\tau)\Big)c_2=\int\limits_0^T\Big(\int\limits_0^s(\tau-s)q(\tau)\,d\tau\Big)d\gamma_i(s)\ \ (i=1,2). \tag{2.11}$$

However,

$$\int\limits_0^T\tau\,d\gamma_i(\tau)=T\gamma_i(T)-\int\limits_0^T\gamma_i(\tau)\,d\tau\ \ (i=1,2),$$
$$\int\limits_0^T\Big(\int\limits_0^s(\tau-s)q(\tau)\,d\tau\Big)d\gamma_i(s)=$$

$$= \gamma_i(T)\int_0^T (s-T)q(s)\,ds + \int_0^T \Big(\int_0^s q(\tau)\,d\tau\Big)\gamma_i(s)\,ds =$$

$$= \gamma_i(T)\int_0^T (s-T)q(s)\,ds + \int_0^T \Big(\int_s^T \gamma_i(\tau)\,d\tau\Big)q(s)\,ds =$$

$$= \int_0^T \Big(\int_s^T \gamma_i(\tau)\,d\tau - \gamma_i(T)(T-s)\Big)q(s)\,ds \quad (i=1,2).$$

Therefore system (2.11) is equivalent to system

$$\gamma_i(T)c_1 + \Big(T\gamma_i(T) - \int_0^T \gamma_i(\tau)\,d\tau\Big)c_2 =$$

$$= \int_0^T \Big(\int_s^T \gamma_i(\tau)\,d\tau - \gamma_i(T)(T-s)\Big)q(s)\,ds \quad (i=1,2).$$

In view of notation (2.4) the latter system is uniquely solvable if and only if inequality (2.6) holds. Besides, if this inequality holds, then

$$c_1 = \frac{1}{\Delta(0)}\int_0^T \Big[\int_0^T \gamma_1(\tau)\,d\tau \int_s^T \gamma_2(\tau)\,d\tau - \int_s^T \gamma_1(\tau)\,d\tau \int_0^T \gamma_2(\tau)\,d\tau\Big]q(s)\,ds +$$

$$+\frac{1}{\Delta(0)}\int_0^T [T\Delta(s) + (s-T)\Delta(0)]q(s)\,ds, \quad c_2 = -\int_0^T \frac{\Delta(s)}{\Delta(0)}q(s)\,ds.$$

Substituting $c_1$ and $c_2$ in (2.10) and taking into account (2.5), we get

$$u(t) = \int_0^T g(t,s)q(s)\,ds \ \text{ for } \ 0 \le t \le T.$$

Consequently $g$ is the Green's function of problem (2.7). On the other hand, the obtained representation of a solution of problem (2.1), (2.2) implies estimates (2.8), where $\varphi_i$ $(i=1,2)$ are the functions given by equalities (2.9). $\square$

**Lemma 2.2.** *If inequality* (2.6) *holds, then the functions* $\varphi_1$ *and* $\varphi_2$*, given by equalities* (2.9)*, are continuous on* $[0,T]$*. Moreover,* $\varphi_1$ *has at most two zeros, and* $\varphi_2$ *is positive in* $[0,T]$*.*

*Proof.* According to equalities (2.4) and (2.5) the function $g : [0,T] \times [0,T] \to \mathbb{R}$ is continuous, that guarantees continuity of function $\varphi_1$. On the other hand

$$g^{(1,0)}(t,s)=\begin{cases}1-\dfrac{\Delta(s)}{\Delta(0)} & \text{for } 0\le s<t\le T,\\ -\dfrac{\Delta(s)}{\Delta(0)} & \text{for } 0\le t<s\le T.\end{cases} \tag{2.12}$$

Therefore

$$\varphi_2(t)=\frac{1}{2}(\varphi_{21}(t)+\varphi_{22}(t)+|\varphi_{22}(t)-\varphi_{21}(t)|) \ \text{ for } \ 0\le t\le T,$$

where

$$\varphi_{21}(t)=\max\left\{\left|1-\frac{\Delta(s)}{\Delta(0)}\right|:\ 0\le s\le t\right\},\ \ \varphi_{22}(t)=\max\left\{\left|\frac{\Delta(s)}{\Delta(0)}\right|:\ t\le s\le T\right\}.$$

Consequently, in view of continuity if the function $\Delta$, the functions $\varphi_{21}$, $\varphi_{22}$ and $\varphi_2$ are continuous. Besides,

$$\varphi_2(t)\ge\frac{1}{2}(\varphi_{21}(t)+\varphi_{22}(t))\ge\frac{1}{2}\left(\left|1-\frac{\Delta(t)}{\Delta(0)}\right|+\left|\frac{\Delta(t)}{\Delta(0)}\right|\right)\ge\frac{1}{2} \ \text{ for } \ 0\le t\le T.$$

To complete the proof it remains to show that the function $\varphi_1$ has at most two zeros in $[0,T]$. Assume the contrary that $\varphi_1$ has at least three zeros $t_1$, $t_2$ and $t_3$, where $0\le t_1<t_2<t_3\le T$. Let $s_0\in(t_1,t_2)$ be arbitrarily fixed and set

$$v(t)=g(t,s_0) \ \text{ for } \ 0\le t\le T.$$

Then, in view of the equalities $\varphi_1(t_i)=0$ $(i=1,2,3)$, we have $v(t_i)=0$ $(i=1,2,3)$. Hence, in view of equality (2.12), it follows that $v'(t)=1-\frac{\Delta(s_0)}{\Delta(0)}=0$ for $t_2\le t\le t_3$. Consequently,

$$v'(t)=\begin{cases}-1 & \text{for } t_1\le t<s_0,\\ 0 & \text{for } s_0<t\le t_2.\end{cases}$$

But this is impossible since $v(t_1)=v(t_2)=0$. The obtained contradiction proves the lemma. □

If

$$\gamma_1(t)=\begin{cases}0 & \text{for } t=0\\ 1 & \text{for } 0<t\le T\end{cases},\quad \gamma_2(t)=\gamma(t) \ \text{ for } \ 0\le t\le T, \tag{2.13}$$

where $\gamma:[0,T]\to\mathbb{R}$ is a function of bounded variation, then boundary condition (2.2) receives the form

$$u(0)=0,\quad \int_0^T u(s)\,d\gamma(s)=0. \tag{2.14}$$

**Lemma 2.3.** *If*

$$\gamma(0)=0,\ \ \gamma(t)\le\gamma(T)\ \text{ a.e. on }\ [0,T],\quad \int\limits_0^T\gamma(s)\,ds<T\gamma(T), \tag{2.15}$$

*then problem* (2.1), (2.14) *is uniquely solvable and the Green's function of the problem*

$$u''=0;\quad u(0)=0,\quad \int\limits_0^T u(s)\,d\gamma(s)=0$$

*admits the estimates*

$$\begin{gathered}\max\big\{|g(t,s)|:\ 0\le s\le T\big\}\le t,\\ \sup\Big\{|g^{(1,0)}(t,s)|:\ 0\le s\le T,\ s\ne t\Big\}\le 1\ \ \textit{for}\ \ 0\le t\le T.\end{gathered} \tag{2.16}$$

*Proof.* According to conditions (2.13) and (2.15) from inequalities (2.4) and (2.5) we find

$$\Delta(0)=T\gamma(T)-\int\limits_0^T\gamma(s)\,ds>0, \tag{2.17}$$

$$0\le\Delta(t)=(T-t)\gamma(T)-\int\limits_t^T\gamma(s)\,ds\le\Delta(0)\ \text{ for }\ 0\le t\le T, \tag{2.18}$$

$$g(t,s)=-\frac{\Delta(s)}{\Delta(0)}t+\chi(t,s)(t-s)\ \text{ for }\ 0\le t,s,\le T. \tag{2.19}$$

By Lemma 2.1, inequality (2.17) guarantees unique solvability of problem (2.1), (2.14). On the other hand, by virtue of inequalities (2.17) and (2.18), estimates (2.16) follow from representation (2.19). □

In conclusion of this subsection consider equation (2.1) with the Dirichlet boundary conditions

$$u(0)=0,\quad u(T)=0. \tag{2.20}$$

**Lemma 2.4.** *Problem* (2.1), (2.20) *is uniquely solvable and the Green's function of the problem*

$$u''=0;\quad u(0)=0,\ \ u(T)=0$$

*admits the estimates*

$$\begin{gathered}\max\big\{|g(t,s)|:\ 0\le s\le T\big\}\le t\Big(1-\frac{t}{T}\Big),\\ \sup\Big\{\big|g^{(1,0)}(t,s)\big|:\ 0\le s\le T,\ s\ne t\Big\}\le 1\ \ \textit{for}\ \ 0\le t\le T,\end{gathered} \tag{2.21}$$

$$\int_0^T \left|g^{(i-1,0)}(t,s)\right| ds \le \frac{T}{2}\Big[t\Big(1-\frac{t}{T}\Big)\Big]^{2-i} \ \textit{for}\ 0 \le t \le T \ \ (i=1,2). \quad (2.22)$$

*Proof.* Boundary condition (2.20) follow from conditions (2.2) in the case where

$$\gamma_1(t) = \begin{cases} 0 & \text{for } t=0 \\ 1 & \text{for } 0<t\le T \end{cases}, \quad \gamma_2(t) = \begin{cases} 0 & \text{for } 0 \le t < T \\ 1 & \text{for } t=T \end{cases}. \quad (2.23)$$

Therefore equalities (2.4) and (2.5) imply

$$\Delta(t) = T - t \ \text{ for } \ 0 \le t \le T, \ \ \Delta(0) = T > 0$$

and

$$g(t,s) = \begin{cases} s\Big(\dfrac{t}{T}-1\Big) & \text{for } 0 \le s \le t \le T \\ t\Big(\dfrac{s}{T}-1\Big) & \text{for } 0 \le t < s \le T \end{cases}. \quad (2.24)$$

By Lemma 2.1 problem (2.1), (2.2) is uniquely solvable. On the other hand, estimates (2.21) and (2.22) immediately follow from representation (2.24).

2.2. **Lemma on estimates of functions satisfying conditions** $(1.2_1)$**.**

**Lemma 2.5.** *Let* $u \in \widetilde{C}^{1,1}(\Omega)$ *be a function satisfying boundary conditions* $(1.2_1)$*. Then*

$$|u^{(i-1,k-1)}(x,y)| \le$$
$$\le \|u^{(2,2)}\|_L \Big[x\Big(1-\frac{x}{a}\Big)\Big]^{2-i}\Big[y\Big(1-\frac{y}{b}\Big)\Big]^{2-k} \ \textit{for}\ (x,y)\in\Omega \ (i,k=1,2). \quad (2.25)$$

*Moreover, if*

$$\rho = \operatorname{ess\,sup}\{|u^{(2,2)}(x,y)| : \ (x,y)\in\Omega\} < +\infty, \quad (2.26)$$

*then*

$$|u^{(i-1,k-1)}(x,y)| \le$$
$$\le \frac{ab}{4}\Big[x\Big(1-\frac{x}{a}\Big)\Big]^{2-i}\Big[y\Big(1-\frac{y}{b}\Big)\Big]^{2-k}\rho \ \textit{for}\ (x,y)\in\Omega \ \ (i,k=1,2). \quad (2.27)$$

*Proof.* By Lemma 2.6 from [2], the function $u$ satisfies inequality (2.25) and admits the representation

$$u(x,y) = \int_0^b \int_0^a g_2(y,t) g_1(x,s) u^{(2,2)}(s,t)\, ds\, dt \ \text{ for } \ (x,y)\in\Omega, \quad (2.28)$$

where $g_1 : [0,a]\times[0,a] \to \mathbb{R}$ and $g_2 : [0,b]\times[0,b] \to \mathbb{R}$, respectively, are the Green's functions of the boundary value problems

$$v'' = 0; \quad v(0)=0, \quad v(a)=0 \quad (2.29)$$

and

$$w'' = 0; \quad w(0) = 0, \quad w(b) = 0. \tag{2.30}$$

On the other hand, according to Lemma 2.4, the functions $g_1$ and $g_2$ admit the estimates

$$\big|g_1^{(i-1,0)}(x,s)\big| \le \Big[x\Big(1-\frac{x}{a}\Big)\Big]^{2-i} \quad \text{for } 0\le x,s\le a, \ \ x\ne s \ \ (i=1,2), \tag{2.31}$$

$$\big|g_2^{(0,k-1)}(y,t)\big| \le \Big[y\Big(1-\frac{y}{b}\Big)\Big]^{2-k} \quad \text{for } 0\le y,t\le b, \ \ y\ne t \ \ (k=1,2), \tag{2.32}$$

and

$$\int_0^a \big|g_1^{(i-1,0)}(x,s)\big|\,ds \le \frac{a}{2}\Big[x\Big(1-\frac{x}{a}\Big)\Big]^{2-i} \quad \text{for } 0 \le x \le a \ \ (i=1,2), \tag{2.33}$$

$$\int_0^b \big|g_2^{(0,k-1)}(y,t)\big|\,dt \le \frac{b}{2}\Big[y\Big(1-\frac{y}{b}\Big)\Big]^{2-k} \quad \text{for } 0 \le y \le b \ \ (k=1,2). \tag{2.34}$$

In view of estimates (2.31) and (2.32), estimates (2.25) follow from (2.28).

Now assume that the function $u$ satisfies condition (2.26). Then representation (2.28) yields

$$\big|u^{(i-1,k-1)}(x,y)\big| \le \Big(\int_0^a |g_1^{(i-1,0)}(x,s)|\,ds\Big)\Big(\int_0^b |g_2^{(0,k-1)}(y,t)|\,dt\Big)\rho$$
$$\text{for } (x,y)\in\Omega \ \ (i,k=1,2),$$

whence, by inequalities (2.33) and (2.34), estimates (2.27) follow. □

2.3. **Lemmas on conditional well-posedness of problem (1.1), (1.2).** Let there exist continuous functions $\psi_{1i} : [0,a] \to [0,\infty)$, $\psi_{2i} : [0,b] \to [0,+\infty)$ $(i=1,2)$ such that

$$\psi_{1i}(x) > 0 \ \text{ a.e. on } [0,a], \quad \psi_{2i}(y) > 0 \ \text{ a.e. on } [0,b], \tag{2.35}$$

and arbitrary functions $v \in \widetilde{C}^1([0,a])$ and $w \in \widetilde{C}^1([0,b])$, satisfying the boundary conditions

$$\int_0^a v(x)\,d\alpha_i(x) = 0, \quad \int_0^b w(y)\,d\beta_i(y) = 0 \ \ (i=1,2) \tag{2.36}$$

admit the estimates

$$\begin{aligned} &|v^{(i-1)}(x)| \le \psi_{1i}(x)\|v''\|_L \ \text{ for } 0 \le x \le a \ \ (i=1,2),\\ &|w^{(i-1)}(y)| \le \psi_{2i}(y)\|w''\|_L \ \text{ for } 0 \le y \le b \ \ (i=1,2). \end{aligned} \tag{2.37}$$

Then Theorems 1.4, 1.5 and 1.10 from [2] imply the following lemmas.

**Lemma 2.6.** *If*

$$\int_0^b \int_0^a \psi_{1i}(x)\psi_{2k}(y)|h_{ik}(x,y)|\,dx\,dy < +\infty \quad (i,k=1,2), \tag{2.38}$$

*then problem* (1.1), (1.2) *is conditionally well-posed if and only if the homogeneous problem* $(1.1_0)$, (1.2) *has only the trivial solution.*

**Lemma 2.7.** *If*

$$\sum_{i=1}^{2}\sum_{k=1}^{2}\int_0^b \int_0^a \psi_{1i}(x)\psi_{2k}(y)|h_{ik}(x,y)|\,dx\,dy < 1, \tag{2.39}$$

*then problem* (1.1), (1.2) *is conditionally well-posed. Moreover, if along with* (2.39) *condition* (1.12) *holds, then problem* (1.1), (1.2) *is well-posed.*

**Lemma 2.8.** *If conditions* (1.13) *and* (2.39) *hold, then problem* (1.1), (1.2) *is conditionally well-posed but not well-posed.*

## 3. Proofs of the Main Results

*Proof of Theorem* 1.1. Set

$$\psi_{1i}(x)=\varphi_{1i}(x) \ \text{ for } \ 0\le x\le a, \quad \psi_{2i}(y)=\varphi_{2i}(y) \ \text{ for } \ 0\le y\le b \ \ (i=1,2).$$

Then by conditions (1.4), (1.10) and Lemma 2.2, the functions $\psi_{1i}$ and $\psi_{2i}$ $(i=1,2)$ are continuous and satisfy conditions (2.35) and (2.38). On the other hand, according to Lemma 2.1, functions $v \in \widetilde{C}^1([0,a])$ and $w \in \widetilde{C}^1([0,b])$ satisfying boundary conditions (2.36) admit estimates (2.37). Therefore Theorem 1.1 immediately follows from Lemma 2.6. □

Theorem 1.2 follows from Lemmas 2.1, 2.2 and 2.7, while Theorem 1.3 follows from Lemmas 2.1, 2.2 and 2.8.

*Proof of Corollary* 1.1. Boundary conditions $(1.2_1)$ follow from the conditions (1.2), where

$$\alpha_1(x)=\begin{cases} 0 & \text{for } \ x=0 \\ 1 & \text{for } \ 0<x\le a \end{cases}, \quad \alpha_2(x)=\begin{cases} 0 & \text{for } \ 0\le x<a \\ 1 & \text{for } \ x=a \end{cases},$$

$$\beta_1(y)=\begin{cases} 0 & \text{for } \ y=0 \\ 1 & \text{for } \ 0<y\le b \end{cases}, \quad \beta_2(y)=\begin{cases} 0 & \text{for } \ 0\le y<b \\ 1 & \text{for } \ y=b \end{cases}.$$

In this case, by Lemmas 2.1 and 2.4, the functions $g_1$ and $g_2$, given by equalities (1.6) and (1.7), are Green's functions of problems (2.29) and (2.30), respectively, and the functions $\varphi_{ik}$ $(i,k=1,2)$, given by equalities (1.8) and (1.9), admit the estimates

$$\varphi_{1i}(x) \le \Big[x\Big(1-\frac{x}{a}\Big)\Big]^{2-i} \ \text{ for } \ 0\le x\le a \ \ (i=1,2),$$

$$\varphi_{2i}(y) \leq \Big[y\Big(1-\frac{y}{b}\Big)\Big]^{2-k} \text{ for } 0 \leq y \leq b \ \ (k=1,2).$$

According to those estimates, inequalities (1.10) follow from inequalities (1.14). Now applying Theorem 1.1, the validity of Corollary 1.1 becomes evident. □

*Proof of Corollary* 1.2. In view of Corollary 1.1, in order to prove Corollary 1.2 it is sufficient to show that problem $(1.1_0), (1.2_1)$ has only the trivial solution provided that inequality (1.15) (inequality (1.16)) holds.

Let $u$ be an arbitrary solution of problem $(1.1_0), (1.2)$. Then, in view of Lemma 2.5, estimates (2.25) are valid. Therefore from $(1.1_0)$ we deduce

$$\|u^{(2,2)}\|_L \leq \bigg(\sum_{i=1}^{2}\sum_{k=1}^{2}\int_0^b\int_0^a \Big[x\Big(1-\frac{x}{a}\Big)\Big]^{2-i}\Big[y\Big(1-\frac{y}{b}\Big)\Big]^{2-k} \times \\ \times |h_{ik}(x,y)|\,dx\,dy\bigg)\|u^{(2,2)}\|_L. \quad (3.1)$$

If inequality (1.15) holds, then (3.1) and (2.25) imply that $\|u^{(2,2)}\|_L = 0$ and $u(x,y) \equiv 0$.

To complete the proof it remains to consider the case, where inequality (1.16) holds. In that case according to estimates (2.25) we have

$$\rho = \operatorname{ess\,sup}\{|u^{(2,2)}(x,y)| : \ (x,y) \in \Omega\} \leq l\|u^{(2,2)}\|_L < +\infty, \quad (3.2)$$

where

$$l = \operatorname{ess\,sup}\bigg\{\sum_{i=1}^{2}\sum_{k=1}^{2}\Big[x\Big(1-\frac{x}{a}\Big)\Big]^{2-i}\Big[y\Big(1-\frac{y}{b}\Big)\Big]^{2-k} \times \\ \times |h_{ik}(x,y)| : \ (x,y) \in \Omega\bigg\} < \frac{4}{ab}\,. \quad (3.3)$$

But, by Lemma 2.5, condition (3.2) guarantees the validity of estimates (2.27). Taking in account those estimates from $(1.1_0)$ we obtain

$$\rho \leq \frac{ab}{4}\,l\,\rho. \quad (3.4)$$

In view of inequality (3.3), (3.4) and (2.27) imply that $\rho = 0$ and $u(x,y) \equiv 0$. □

Corollary 1.3 follows from Theorem 1.3 and Lemmas 2.1 and 2.4.

Corollaries 1.4 and 1.5 can be proved in the same manner as Corollaries 1.1 and 1.2. The only difference between the proofs is that instead of Lemma 2.4 one should use Lemma 2.3.

Corollary 1.6 follows from Theorem 1.3 and Lemmas 2.1 and 2.3.

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(Received 16.08.2011)

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