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ON THE CAUCHY–NICOLETTI WEIGHTED PROBLEM FOR HIGHER ORDER NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

Abstract. The unimprovable in a certain sense conditions are established which, respectively, ensure the solvability and well-posedness of the weighted Cauchy–Nicoletti problem for higher order nonlinear singular differential equations.

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Let $-\infty < a < b < +\infty$, $n \geq 2$ be a natural number and f be an operator defined on some set $D(f) \subset C^{n-1}([a,b])$ and mapping $D(f)$ onto $L([a,b])$. We consider the functional differential equation

$$u^{(n)}(t) = f(u)(t) \tag{1}$$

with the Cauchy–Nicoletti weighted conditions

$$\limsup_{t \to t_i} \Big(\frac{|u^{(i-1)}(t)|}{\rho_i(t)} \Big) < +\infty \quad (i = 1, \dots, n). \tag{2}$$

Here $t_i \in [a,b]$ $(i = 1, \dots, n)$ and $\rho_i : [a,b] \to [0;+\infty[$ $(i = 1, \dots, n)$ are continuous functions such that

$$\rho_n(t_n) = 0, \quad \rho_n(t) > 0 \ \text{ for } \ t \neq t_n, \quad \rho_i(t_i) = 0,$$
$$\Big| \int_{t_i}^{t} \rho_{i+1}(s)\, ds \Big| \leq \rho_i(t) \ \text{ for } \ a \leq t \leq b \ \ (i = 1, \dots, n-1).$$

By $C^{n-1}_{\rho_1,\dots,\rho_n}([a,b])$ we denote a set of functions $u \in C^{n-1}([a,b])$ such that

$$\mu(u) = \max \big\{ \mu_1(u), \dots, \mu_n(u) \big\} < +\infty,$$

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where

$$\mu_i(u) = \sup\left\{\frac{|u^{(i-1)}(t)|}{\rho_i(t)} \ :\ a \le t \le b,\ t \ne t_i\right\}.$$

For an arbitrary $x > 0$, assume

$$C^{n-1}_{\rho_1,\dots,\rho_n;x}([a,b]) = \Big\{u \in C_{\rho_1,\dots,\rho_n}([a,b]) :\ \mu(u) \le x\Big\},$$

$$f^*(\rho_1,\dots,\rho_n;x)(t) = \sup\Big\{|f(u)(t)| :\ u \in C^{n-1}_{\rho_1,\dots,\rho_n;x}([a,b])\Big\}.$$

We investigate the problem (1), (2) in the case, where

$$C^{n-1}_{\rho_1,\dots,\rho_n}([a,b]) \subset D(f) \tag{3}$$

and for any $x > 0$ the conditions

$$f : C^{n-1}_{\rho_1,\dots,\rho_n;x}([a,b]) \longrightarrow L([a,b]) \text{ is continuous} \tag{4}$$

and

$$\int\limits_a^b f^*(\rho_1,\dots,\rho_n;x)(t)\,dt < +\infty$$

are fulfilled.

Of special interest is the case, where

$$D(f) \ne C^{n-1}([a,b]).$$

In this sense the equation (1) is singular one.

In the case, where f is the Nemytski's operator, i.e., when

$$f(u)(t) \equiv f_0\big(t, u(t), \dots, u^{(n-1)}(t)\big),$$

where $f : (]a,b[\,\setminus\{t_1,\dots,t_n\}) \times \mathbb{R}^n \to \mathbb{R}$ is the function satisfying the local Carathéodory conditions, the problems of the type (1), (2) are investigated thoroughly (see [1]–[6] and references therein). The problem (1), (2) is also investigated in the case, where

$$f(u)(t) \equiv f_0\big(t, u(\tau_1(t)), \dots, u^{(n-1)}(\tau_n(t))\big);$$
$$t_1 = \dots = t_n \ \text{ and } \ \rho_{i+1}(t) = \rho_i'(t) \ \ (i = 1,\dots,n)$$

(see [7]–[9]).

However, the problem mentioned above remains still little studied in a general case. Just this case we consider in the present paper.

The function $u \in D(f)$ with an absolutely continuous $(n-1)$th derivative is said to be a solution of the equation (1) if it almost everywhere on $]a,b[$ satisfies this equation.

A solution of the equation (1) satisfying the boundary conditions (2) is called a solution of the problem (1), (2).

Theorem 1. *Let the conditions* (3) *and* (4) *be fulfilled, and there exist constants* $\alpha \in]0,1[$ *and* $x_0 > 0$ *such that*

$$\Big|\int_{t_n}^{t} f^*(\rho_1,\dots,\rho_n;x)(s)\,ds\Big| \le \alpha\rho_n(x) \ \ for\ \ a \le t \le b,\ \ x \ge x_0. \tag{5}$$

Then the problem (1), (2) *has at least one solution.*

Corollary 1. *Let there exist integrable functions* p *and* $q : [a,b] \to [0;+\infty[$ *such that*

$$\sup\bigg\{\Big|\int_{t_n}^{t} p(s)\,ds\Big|/\rho_n(t): \ \ a \le t \le b,\ \ t \ne t_n\bigg\} < 1, \tag{6}$$

$$\sup\bigg\{\Big|\int_{t_n}^{t} q(s)\,ds\Big|/\rho_n(t): \ \ a \le t \le b,\ \ t \ne t_n\bigg\} < +\infty \tag{7}$$

and for any $u \in C^{n-1}_{\rho_1,\dots,\rho_n}([a,b])$ *almost everywhere on* $]a,b[$ *the condition*

$$\big|f(u)(t)\big| \le \rho(t)\mu(u) + q(t)$$

is fulfilled. Then the problem (1), (2) *has at least one solution.*

Along with the problem (1), (2) we consider the perturbed problem

$$v^{(n)}(t) = f(v)(t) + h(t), \tag{8}$$

$$\limsup_{t\to t_i}\Big(\frac{|v^{(i-1)}(t)|}{\rho_i(t)}\Big) < +\infty \ \ (i = 1,\dots,n), \tag{9}$$

where $h :]a,b[\to \mathbb{R}$ is the integrable function such that

$$\mu_0(h) = \sup\bigg\{\Big|\int_{t_n}^{t} h(s)\,ds\Big|/\rho_n(t): \ \ a \le t \le b,\ \ t \ne t_n\bigg\} < +\infty. \tag{10}$$

Definition 1. The problem (1), (2) is said to be well-posed if for any integrable function $h :]a,b[\to \mathbb{R}$ satisfying the condition (10), the problem (8), (9) is uniquely solvable, and there exists an independent of h positive constant r such that

$$\mu(u - v) \le r\mu_0(h),$$

where u and v are, respectively, the solutions of the problems (1), (2) and (8), (9).

Theorem 2. *Let there exist an integrable function* $p : [a,b] \to [0,+\infty[$ *satisfying the inequality* (6) *such that for any* u *and* $v \in C^{n-1}_{\rho_1,\dots,\rho_n}([a,b])$ *almost everywhere on* $]a,b[$ *the condition*

$$\big|f(u)(t) - f(v)(t)\big| \le p(t)\mu(u - v)$$

is fulfilled. If, moreover, the inequality (7), *where* $q(t) \equiv |f(0)(t)|$, *is fulfilled, then the problem* (1), (2) *is well-posed.*

Note that the condition (5) in Theorem 1, where $\alpha \in]0,1[$, is unimprovable and it cannot be replaced by the condition

$$\Big|\int_{t_n}^{t} f^*(\rho_1,\dots,\rho_n;x)(s)\,ds\Big| \le \rho_n(t)x \ \text{ for } \ a \le t \le b, \ \ x \ge x_0.$$

Similarly, in Corollary 1 and in Theorem 2, the strict inequality (6) cannot be replaced by the nonstrict inequality

$$\sup\bigg\{\Big|\int_{t_n}^{t} p(s)\,ds\Big|/\rho_n(t): \ \ a \le t \le b, \ \ t \ne t_n\bigg\} \le 1.$$

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