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A NOTE ON THE EXISTENCE OF SLOWLY GROWING POSITIVE SOLUTIONS TO SECOND ORDER QUASILINEAR ORDINARY DIFFERENTIAL EQUATIONS

Dedicated to Professor Takaŝi Kusano on his 80th birthday

Abstract. In this paper the second order quasilinear ordinary differential equations are considered, and a sufficient condition for the existence of a slowly growing positive solution is given.

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1. Introduction

In this paper we consider the second order quasilinear ordinary differential equation

$$\left(|x'|^{\alpha}\operatorname{sgn} x'\right)' + p(t)|x|^{\beta}\operatorname{sgn} x = 0, \tag{1.1}$$

where α and β are positive constants and $p(t)$ is a positive and continuous function on an interval $[t_0, \infty)$. By a solution of (1.1) we mean a real-valued function $x = x(t)$ such that $x \in C^1[T, \infty)$, $T \geq t_0$, and $|x'|^{\alpha}\operatorname{sgn} x' \in C^1[T, \infty)$ and $x(t)$ satisfies (1.1) at every point of $[T, \infty)$, where T may depend on $x(t)$. A solution $x(t)$ of (1.1) is said to be *oscillatory* if there is a sequence $\{t_i\}_{i=1}^{\infty}$ such that $\lim_{i\to\infty} t_i = \infty$ and $x(t_i) = 0$ $(i = 1, 2, \dots)$. If a solution $x(t)$ of (1.1) is not oscillatory, then it is said to be *nonoscillatory*. In other words, a solution $x(t)$ of (1.1) is called nonoscillatory if $x(t)$ is eventually positive or eventually negative. If $x(t)$ is a solution of (1.1), then so is $-x(t)$. Therefore there is no loss of generality in assuming that a nonoscillatory solution of (1.1) is eventually positive.

It is easily shown (Elbert [2], Elbert and Kusano [3]) that an eventually positive solution $x(t)$ of (1.1) satisfies one and only one of the following three conditions:

$$\lim_{t\to\infty} x(t) \text{ exists and is a positive finite number;} \tag{1.2}$$

$$\lim_{t\to\infty} x(t) = \infty \text{ and } \lim_{t\to\infty} \frac{x(t)}{t} = 0; \tag{1.3}$$

$$\lim_{t\to\infty} \frac{x(t)}{t} \text{ exists and is a positive finite number.} \tag{1.4}$$

A solution $x(t)$ of (1.1) which satisfies (1.2) [*resp.* (1.4)] is asymptotically equal to a positive constant function c [*resp.* a linear function ct] as $t \to \infty$ for some constant $c > 0$. The asymptotic growth of a solution $x(t)$ of (1.1) which satisfies (1.3) is asymptotically bigger than positive constant functions, and is asymptotically smaller than positive unbounded linear functions. In this paper we refer to eventually positive solutions $x(t)$ satisfying (1.3) as *slowly growing positive solutions*. Eventually positive solutions $x(t)$ satisfying (1.2), (1.3) and (1.4) are sometimes called *subdominant solutions*, *intermediate solutions* and *dominant solutions*, respectively ([1]).

It is well known that the following results hold ([2], [3], [7], [8]).

(A) Equation (1.1) has an eventually positive solution $x(t)$ satisfying (1.2) if and only if

$$\int_{t_0}^{\infty} \left[\int_{t}^{\infty} p(s)\, ds \right]^{1/\alpha} dt < \infty. \tag{1.5}$$

(B) Equation (1.1) has an eventually positive solution $x(t)$ satisfying (1.4) if and only if

$$\int_{t_0}^{\infty} t^{\beta} p(t)\, dt < \infty. \tag{1.6}$$

(C) Let $\alpha < \beta$. Equation (1.1) has an eventually positive solution if and only if (1.5) is satisfied.

(D) Let $\alpha > \beta$. Equation (1.1) has an eventually positive solution if and only if (1.6) is satisfied.

Now consider the problem of the existence of an eventually positive solution $x(t)$ satisfying (1.3), namely, a slowly growing positive solution. For the case $\alpha > \beta$ this problem has been solved finally by Naito [9]. The following statement is true:

(E) Let $\alpha > \beta$. Equation (1.1) has a slowly growing positive solution if and only if

$$\int_{t_0}^{\infty} t^{\beta} p(t)\, dt < \infty \quad \text{and} \quad \int_{t_0}^{\infty} \left[\int_{t}^{\infty} p(s)\, ds \right]^{1/\alpha} dt = \infty. \tag{1.7}$$

More precisely, the statement (E) was proved by Kusano and Naito [6] for the case $\alpha = 1 > \beta$. The "if" part of (E) for the general case $\alpha > \beta$ was proved by Elbert and Kusano [3]. Very recently, the "only if" part of (E) for the general case $\alpha > \beta$ has been proved by Naito [9].

A characterization of the existence of slowly growing positive solutions of (1.1) for the case $\alpha < \beta$ seems to be a more difficult problem. For some results related to this case, see Cecchi, Došlá and Marini [1] and the references therein.

In this paper we attempt to discuss the existence of slowly growing positive solutions of (1.1) for the case $\alpha < \beta$. For this purpose, let us first consider the particular equation

$$\left(|x'|^{\alpha} \operatorname{sgn} x'\right)' + \kappa t^{-\mu} |x|^{\beta} \operatorname{sgn} x = 0 \ \ (\alpha < \beta), \tag{1.8}$$

where κ is a positive constant and μ is a real constant. It is easy to see that (1.8) has a slowly growing positive solution of the form ct^{ν} ($c > 0$, $0 < \nu < 1$) if and only if $\alpha + 1 < \mu < \beta + 1$, and that this solution is uniquely determined by

$$x_0(t) = c_0 t^{\nu_0} \tag{1.9}$$

with

$$\nu_0 = \frac{\mu - 1 - \alpha}{\beta - \alpha} \quad \text{and} \quad c_0 = \left[\frac{\alpha(1 - \nu_0)\nu_0^{\alpha}}{\kappa} \right]^{1/(\beta - \alpha)}. \tag{1.10}$$

Observe here that $0 < \nu_0 < 1$ under the conditions $\alpha < \beta$ and $\alpha + 1 < \mu < \beta + 1$. Then we may conjecture that if $p(t)$ is close to the function $\kappa t^{-\mu}$

($\kappa > 0$, $\alpha + 1 < \mu < \beta + 1$) in some sense, then (1.1) has a slowly growing positive solution $x(t)$ satisfying

$$\begin{cases} x(t) = x_0(t)(1 + o(1)) & (t \to \infty), \\ x'(t) = x_0'(t)(1 + o(1)) & (t \to \infty), \end{cases} \tag{1.11}$$

where $x_0(t)$ is defined by (1.9) and (1.10). This conjecture is true to a certain extent. In fact, the following theorem can be proved. For convenience, we write the equation (1.1) in the form

$$\big(|x'|^{\alpha} \mathrm{sgn}\, x'\big)' + \kappa t^{-\mu}(1 + \varepsilon(t))|x|^{\beta} \mathrm{sgn}\, x = 0, \tag{1.12}$$

where $\varepsilon(t)$ is a continuous function on $[t_0, \infty)$, $t_0 > 0$, such that $1+\varepsilon(t) > 0$ for $t \geq t_0$.

Theorem 1.1. *Consider the equation* (1.12) *under the condition*

$$0 < \alpha < \beta, \;\; \alpha + 1 < \mu < \beta + 1, \;\; \kappa > 0. \tag{1.13}$$

Set $x_0(t) = c_0 t^{\nu_0}$, where c_0 and ν_0 are constants given by (1.10)*. Suppose that there exists $\ell > 0$ such that*

$$\ell(\ell - 2\nu_0 + 1) - |1 - \alpha|(1 - \nu_0)\ell - (\beta - \alpha)(1 - \nu_0)\nu_0 > 0 \tag{1.14}$$

and

$$\lim_{t \to \infty} t^{\ell - 2\nu_0 + 1} \int_t^{\infty} s^{2(\nu_0 - 1)} |\varepsilon(s)|\, ds = 0. \tag{1.15}$$

Then the equation (1.12) *has a slowly growing positive solution $x(t)$ with the asymptotic property*

$$\begin{cases} x(t) = x_0(t)\big(1 + O(t^{-\ell})\big) & (t \to \infty), \\ x'(t) = x_0'(t)\big(1 + O(t^{-\ell})\big) & (t \to \infty). \end{cases}$$

The condition (1.14) is satisfied if $\ell > 0$ is taken sufficiently large. Therefore, if

$$\lim_{t \to \infty} t^{m} \int_t^{\infty} s^{2(\nu_0 - 1)} |\varepsilon(s)|\, ds = 0 \;\text{ for all }\; m > 0, \tag{1.16}$$

then there is $\ell_0 > 0$ such that for all $\ell \geq \ell_0$, both of the conditions (1.14) and (1.15) are satisfied. On the other hand, it is easy to see that (1.16) is equivalent to

$$\int_{t_0}^{\infty} s^{n} |\varepsilon(s)|\, ds < \infty \;\text{ for all }\; n > 0. \tag{1.17}$$

Thus we can conclude the following result as a corollary of Theorem 1.1.

Corollary 1.1. *Consider the equation* (1.12) *under the condition* (1.13)*. If* (1.17) *holds, then the equation* (1.12) *has a slowly growing positive solution $x(t)$ with the asymptotic property* (1.11)*.*

We give a simple example illustrating our theorem in the case $\alpha = 1$.

Example 1.1. Consider the equation

$$x'' + \kappa t^{-3}(1+\varepsilon(t))|x|^3 \mathrm{sgn}\, x = 0, \quad \kappa > 0, \tag{1.18}$$

where $\varepsilon(t)$ is a continuous function on $[1,\infty)$ such that $1+\varepsilon(t) > 0$ for $t \geq 1$. For this equation, $\alpha = 1$, $\beta = 3$, $\mu = 3$, $\kappa > 0$; and hence $\nu_0 = 1/2$ and $c_0 = 1/\big[2\sqrt{\kappa}\,\big]$. Consequently, the conditions (1.14) and (1.15) reduce to

$$\ell > \frac{\sqrt{2}}{2} \quad \text{and} \quad \lim_{t\to\infty} t^{\ell} \int_t^{\infty} \frac{|\varepsilon(s)|}{s}\, ds = 0. \tag{1.19}$$

Therefore, by Theorem 1.1, we can conclude that if (1.19) is satisfied for some ℓ, then (1.18) has a slowly growing positive solution $x(t)$ such that

$$\begin{cases} x(t) = \dfrac{1}{2\sqrt{\kappa}}\, t^{1/2}\big(1+O(t^{-\ell})\big) & (t\to\infty), \\ x'(t) = \dfrac{1}{4\sqrt{\kappa}}\, t^{-1/2}\big(1+O(t^{-\ell})\big) & (t\to\infty). \end{cases}$$

In the case $0 < \alpha < \beta$, assuming the existence of slowly growing positive solutions of (1.1), Kamo and Usami [4] have obtained the asymptotic forms as $t \to \infty$ of such solutions under a certain condition. Note, however, that the existence of slowly growing positive solutions of (1.1) is not proved.

In the case $0 < \beta < \alpha$, the asymptotic forms as $t \to \infty$ of slowly growing positive solutions of (1.1) has been discussed by Naito [9]. See also [4], [5].

2. Proof of Theorem 1.1

In this section we give the proof of Theorem 1.1. First notice that if $x(t)$ is a positive solution of (1.1) on an interval $[T,\infty)$, $T \geq t_0$, then $x'(t) > 0$ for $t \geq T$. This fact is easily checked. For the proof of Theorem 1.1, we make use of the following lemma. In this lemma we consider the equations (1.1) and the auxiliary equation

$$\big(|x'|^{\alpha} \mathrm{sgn}\, x'\big)' + p_0(t)|x|^{\beta} \mathrm{sgn}\, x = 0, \tag{2.1}$$

where $p_0(t)$ is a positive continuous function on $[t_0,\infty)$, $t_0 > 0$.

Lemma 2.1. *Let $x_0(t)$ be an eventually positive solution of the auxiliary equation* (2.1)*. If $x(t)$ is an eventually positive solution of* (1.1)*, then*

$$u(t) = \frac{x(t)}{x_0(t)} \quad \text{and} \quad v(t) = x_0(t)^2 \Big(\frac{x(t)}{x_0(t)}\Big)' \tag{2.2}$$

satisfy

$$u(t) > 0 \quad \text{and} \quad \frac{1}{x_0(t)}\, v(t) + x_0'(t) u(t) > 0 \tag{2.3}$$

for all large t, and $(u(t), v(t))$ is a solution of the binary nonlinear system

$$\begin{cases} u' = \dfrac{1}{x_0(t)^2}\, v, \\ v' = \dfrac{1}{\alpha}\Big\{ p_0(t) x_0(t)^{\beta+1} x_0'(t)^{-\alpha+1} u - \\ \qquad\qquad -p(t) x_0(t)^{\beta+1} \Big[\dfrac{1}{x_0(t)}\, v + x_0'(t) u\Big]^{-\alpha+1} u^\beta \Big\} \end{cases} \tag{2.4}$$

for all large t.

Conversely, if $(u(t), v(t))$ is a solution of (2.4) *satisfying* (2.3)*, then $x(t) = x_0(t)u(t)$ is an eventually positive solution of* (1.1)*.*

Proof. Let $x(t)$ be an eventually positive solution of (1.1). By (2.2), we have

$$x'(t) = \frac{1}{x_0(t)}\, v(t) + x_0'(t)u(t).$$

Since $x'(t) > 0$ for all large t, it is obvious that $(u(t), v(t))$ satisfies (2.3) for all large t. Moreover, $x(t)$ satisfies

$$x''(t) + \frac{1}{\alpha}\, p(t) x(t)^\beta x'(t)^{-\alpha+1} = 0$$

for all large t. An analogous equality also holds for $x_0(t)$. Then we easily see that $(u(t), v(t))$ satisfies (2.4) for all large t. This proves the first half of the lemma.

To prove the second half, let $(u(t), v(t))$ be a solution of (2.4) satisfying (2.3). Then, a straightforward computation shows that $x(t) = x_0(t)u(t)$ is an eventually positive solution of (1.1). The details are left to the reader. The proof of Lemma 2.1 is complete. □

Proof of Theorem 1.1. We apply Lemma 2.1 to the case $p_0(t) = \kappa t^{-\mu}$ and $x_0(t) = c_0 t^{\nu_0}$, where c_0 and ν_0 are constants given by (1.10). Then the existence of a solution $x(t)$ of (1.1) which satisfies $\lim_{t\to\infty} [x(t)/x_0(t)] = 1$ is equivalent to the existence of a solution $(u(t), v(t))$ of (2.4) which satisfies

$$\lim_{t\to\infty} u(t) = 1 \tag{2.5}$$

and

$$\frac{1}{x_0(t)}\, v(t) + x_0'(t)u(t) > 0 \tag{2.6}$$

for all large t. Thus it is natural to consider the integral equation of the form

$$\begin{cases} u(t) = 1 - \int_t^\infty \frac{1}{x_0(s)^2}\, v(s)\, ds, \\ v(t) = -\frac{1}{\alpha} \int_t^\infty \Big\{ p_0(s) x_0(s)^{\beta+1} x_0'(s)^{-\alpha+1} u(s) - \\ \qquad -p(s) x_0(s)^{\beta+1} \Big[\frac{1}{x_0(s)}\, v(s) + x_0'(s) u(s) \Big]^{-\alpha+1} u(s)^\beta \Big\}\, ds, \end{cases} \tag{2.7}$$

where $p(t) = p_0(t)(1+\varepsilon(t)) = \kappa t^{-\mu}(1+\varepsilon(t))$.

Denote by X the set of all vector functions $(u(t), v(t)) \in C[T,\infty) \times C[T,\infty)$ such that

$$|u(t) - 1| \le L t^{-\ell} \ \text{ and } \ |v(t)| \le M t^{-\ell+2\nu_0-1} \ \text{ for } \ t \ge T, \tag{2.8}$$

where ℓ is a positive constant satisfying (1.14) and (1.15), and L, M, T are positive constants to be determined later. Note that, because of $\ell > 0$, the condition (1.14) implies $\ell - 2\nu_0 + 1 > 0$. We seek for a solution $(u(t), v(t))$ of (2.7) in the set X.

On account of (1.14), we can take a sufficiently small positive number d such that $0 < d < 1/2$ and

$$\ell(\ell - 2\nu_0 + 1) - |1-\alpha|(1-\nu_0)\ell(1-2d)^{-\alpha}(1+d)^\beta - \\ - (\beta-\alpha)(1-\nu_0)\nu_0(1+d) > 0. \tag{2.9}$$

Let M be an arbitrary positive number, and set $L = M/(\ell c_0^2)$ (>0). Then, by (2.9),

$$\frac{L}{\ell - 2\nu_0 + 1}\, c_0^2(\beta-\alpha)(1-\nu_0)\nu_0(1+d) + \\ + \frac{M}{\ell - 2\nu_0 + 1}\, |1-\alpha|(1-\nu_0)(1-2d)^{-\alpha}(1+d)^\beta < M.$$

For simplicity, let us use the letters C_1 and C_2 to denote, respectively, the first and the second terms in the left-hand side of the above inequality:

$$C_1 = \frac{L}{\ell - 2\nu_0 + 1}\, c_0^2(\beta-\alpha)(1-\nu_0)\nu_0(1+d) \ \ (>0) \tag{2.10}$$

and

$$C_2 = \frac{M}{\ell - 2\nu_0 + 1}\, |1-\alpha|(1-\nu_0)(1-2d)^{-\alpha}(1+d)^\beta \ \ (\ge 0). \tag{2.11}$$

We have $C_1 + C_2 < M$. Further, let

$$C_3 = D c_0^2 (1-\nu_0)\nu_0(1+d)^\beta \ \ (>0), \tag{2.12}$$

where D is the positive constant defined by

$$D = \begin{cases} (1+2d)^{-\alpha+1} & \text{for } 0 < \alpha \le 1, \\ (1-2d)^{-\alpha+1} & \text{for } \alpha > 1. \end{cases} \tag{2.13}$$

Since

$$\lim_{u \to 1} \frac{u - u^{-\alpha+\beta+1}}{u-1} = \alpha - \beta,$$

there is $\delta > 0$ such that

$$|u - u^{-\alpha+\beta+1}| \le (1+d)(\beta-\alpha)|u-1| \quad \text{for} \quad |u-1| \le \delta. \tag{2.14}$$

We take a number T sufficiently large so that the following inequalities hold for $t \ge T$:

$$Lt^{-\ell} \le d, \quad \frac{M}{c_0^2 \nu_0} t^{-\ell} \le d, \quad Lt^{-\ell} \le \delta, \tag{2.15}$$

and

$$C_1 + C_2 + C_3 t^{\ell - 2\nu_0 + 1} \int_t^\infty s^{2(\nu_0 - 1)} |\varepsilon(s)| \, ds \le M. \tag{2.16}$$

Note that the inequality $C_1 + C_2 < M$ and the assumption (1.15) ensure the inequality (2.16).

Let X be the set of all vector functions $(u(t), v(t)) \in C[T,\infty) \times C[T,\infty)$ such that (2.8) holds. Define the operator $\Phi : X \to C[T,\infty) \times C[T,\infty)$ by $\Phi(u,v)(t) = (\Phi_1(u,v)(t), \Phi_2(u,v)(t))$ with

$$\Phi_1(u,v)(t) = 1 - \int_t^\infty \frac{1}{x_0(s)^2} \, v(s) \, ds, \quad t \ge T,$$

and

$$\Phi_2(u,v)(t) = -\frac{1}{\alpha} \int_t^\infty \Bigg\{ p_0(s) x_0(s)^{\beta+1} x_0'(s)^{-\alpha+1} u(s) -$$
$$- p(s) x_0(s)^{\beta+1} \left[\frac{1}{x_0(s)} \, v(s) + x_0'(s) u(s) \right]^{-\alpha+1} u(s)^\beta \Bigg\} ds, \quad t \ge T.$$

It will be shown with the aid of the Schauder–Tychonoff theorem that Φ has a fixed point $(u(t), v(t))$ in X ($\subset C[T,\infty) \times C[T,\infty)$). Here, the space $C[T,\infty) \times C[T,\infty)$ is regarded as the Fréchet space consisting of all continuous vector functions $(u(t), v(t))$ on $[T,\infty)$ with the topology of uniform convergence on compact subintervals of $[T,\infty)$.

(i) *The operator Φ is well defined on X and maps X into X.*

Let $(u(t), v(t)) \in X$. Then, by the first inequality in (2.15), we obtain $|u(t) - 1| \le Lt^{-\ell} \le d$ for $t \ge T$. Therefore,

$$(0 <) \;\; 1 - d \le u(t) \le 1 + d, \quad t \ge T. \tag{2.17}$$

We can show that

$$-\frac{1}{x_0(t)}|v(t)| + x_0'(t)u(t) \geq (1-2d)c_0\nu_0 t^{\nu_0-1}, \quad t \geq T, \tag{2.18}$$

and

$$\frac{1}{x_0(t)}|v(t)| + x_0'(t)u(t) \leq (1+2d)c_0\nu_0 t^{\nu_0-1}, \quad t \geq T. \tag{2.19}$$

In fact, it follows from (2.17) and the second inequality in (2.15) that

$$\begin{aligned}
-\frac{1}{x_0(t)}|v(t)| + x_0'(t)u(t) &\geq -\frac{1}{c_0 t^{\nu_0}} M t^{-\ell+2\nu_0-1} + c_0\nu_0 t^{\nu_0-1}(1-d) = \\
&= (1-d)c_0\nu_0 t^{\nu_0-1}\Big\{1 - \frac{M}{(1-d)c_0^2\nu_0}\, t^{-\ell}\Big\} \geq \\
&\geq (1-d)c_0\nu_0 t^{\nu_0-1}\Big(1 - \frac{d}{1-d}\Big) = \\
&= (1-2d)c_0\nu_0 t^{\nu_0-1}, \quad t \geq T,
\end{aligned}$$

which shows that (2.18) holds. The inequality (2.19) can be shown in a similar way.

Now let us define $y(t)$ by

$$y(t) = \frac{1}{x_0(t)}\, v(t) + x_0'(t)u(t), \quad t \geq T.$$

Then it follows from (2.18) and (2.19) that

$$(1-2d)c_0\nu_0 t^{\nu_0-1} \leq y(t) \leq (1+2d)c_0\nu_0 t^{\nu_0-1}, \quad t \geq T.$$

In particular, we have $y(t) > 0$ for $t \geq T$ and

$$y(t)^{-\alpha+1} \leq D c_0^{-\alpha+1}\nu_0^{-\alpha+1} t^{(\nu_0-1)(-\alpha+1)}, \quad t \geq T, \tag{2.20}$$

where D is the positive constant defined by (2.13).

For brevity, we define $\varphi_1(u,v)(t)$ and $\varphi_2(u,v)(t)$ by

$$\begin{aligned}
\varphi_1(u,v)(t) &= \frac{1}{x_0(t)^2}\, v(t), \\
\varphi_2(u,v)(t) &= p_0(t)x_0(t)^{\beta+1}x_0'(t)^{-\alpha+1}u(t) - \\
&\quad - p(t)x_0(t)^{\beta+1}\left[\frac{1}{x_0(t)}\, v(t) + x_0'(t)u(t)\right]^{-\alpha+1} u(t)^{\beta},
\end{aligned}$$

so that

$$\Phi_1(u,v)(t) = 1 - \int_t^\infty \varphi_1(u,v)(s)\, ds, \quad t \geq T,$$

$$\Phi_2(u,v)(t) = -\frac{1}{\alpha}\int_t^\infty \varphi_2(u,v)(s)\, ds, \quad t \geq T.$$

By (2.8), we obtain

$$|\varphi_1(u,v)(t)| \le \frac{1}{x_0(t)^2}\,|v(t)| \le M(c_0t^{\nu_0})^{-2}\,t^{-\ell+2\nu_0-1} = L\ell t^{-\ell-1} \qquad (2.21)$$

for $t \ge T$. Thus, $\Phi_1(u,v)(t)$ is well defined on X and

$$|\Phi_1(u,v)(t) - 1| \le L\ell \int_t^\infty s^{-\ell-1}\,ds = Lt^{-\ell}, \ \ t \ge T. \qquad (2.22)$$

Since $p(t) = p_0(t)(1+\varepsilon(t))$, the function $\varphi_2(u,v)(t)$ can be estimated as follows:

$$\begin{aligned}
&|\varphi_2(u,v)(t)| \le \\
&\le \Big|p_0(t)x_0(t)^{\beta+1}x_0'(t)^{-\alpha+1}u(t) - p_0(t)x_0(t)^{\beta+1}x_0'(t)^{-\alpha+1}u(t)^{-\alpha+\beta+1}\Big| + \\
&\quad + \Big|p_0(t)x_0(t)^{\beta+1}x_0'(t)^{-\alpha+1}u(t)^{-\alpha+\beta+1} - \\
&\qquad - p_0(t)(1+\varepsilon(t))x_0(t)^{\beta+1}y(t)^{-\alpha+1}u(t)^{\beta}\Big| \le \\
&\le p_0(t)x_0(t)^{\beta+1}x_0'(t)^{-\alpha+1}\big|u(t) - u(t)^{-\alpha+\beta+1}\big| + \\
&\quad + p_0(t)x_0(t)^{\beta+1}\Big|\big[x_0'(t)u(t)\big]^{-\alpha+1} - y(t)^{-\alpha+1}\Big|u(t)^{\beta} + \\
&\quad + p_0(t)|\varepsilon(t)|x_0(t)^{\beta+1}y(t)^{-\alpha+1}u(t)^{\beta}.
\end{aligned}$$

Denote the first, second and third term of the last side in the above inequality by $\psi_1(u,v)(t)$, $\psi_2(u,v)(t)$ and $\psi_3(u,v)(t)$, respectively. Then

$$|\varphi_2(u,v)(t)| \le \psi_1(u,v)(t) + \psi_2(u,v)(t) + \psi_3(u,v)(t), \ \ t \ge T. \qquad (2.23)$$

In view of (2.8) and (2.15), we get $|u(t) - 1| \le Lt^{-\ell} \le \delta$ for $t \ge T$. Therefore, it follows from (2.14) that

$$\big|u(t) - u(t)^{-\alpha+\beta+1}\big| \le L(1+d)(\beta-\alpha)t^{-\ell}, \ \ t \ge T.$$

Then it is easy to see that

$$\begin{aligned}
\psi_1(u,v)(t) &= p_0(t)x_0(t)^{\beta+1}x_0'(t)^{-\alpha+1}|u(t) - u(t)^{-\alpha+\beta+1}| \le \\
&\le \kappa t^{-\mu}(c_0t^{\nu_0})^{\beta+1}\left(c_0\nu_0t^{\nu_0-1}\right)^{-\alpha+1}L(1+d)(\beta-\alpha)t^{-\ell} = \\
&= \alpha(\ell - 2\nu_0 + 1)C_1t^{-\ell+2\nu_0-2}, \ \ t \ge T,
\end{aligned}$$

where C_1 is the constant given by (2.10).

The mean value theorem implies that if $A > 0$ and $A + B > 0$, then the equality

$$A^{-\alpha+1} - (A+B)^{-\alpha+1} = (\alpha-1)(A+\theta B)^{-\alpha}B$$

holds for some θ, $0 < \theta < 1$. Applying the above equality to the cases $A = x_0'(t)u(t) > 0$ and $B = x_0(t)^{-1}v(t)$, and noting that $A + B = y(t) > 0$, we obtain

$$\Big|\big[x_0'(t)u(t)\big]^{-\alpha+1} - y(t)^{-\alpha+1}\Big| =$$
$$= |\alpha-1|\big[x_0'(t)u(t) + \theta x_0(t)^{-1}v(t)\big]^{-\alpha} x_0(t)^{-1}|v(t)| \le$$
$$\le |\alpha-1|\big[x_0'(t)u(t) - x_0(t)^{-1}|v(t)|\big]^{-\alpha} x_0(t)^{-1}|v(t)|$$

for $t \ge T$. Then, by (2.18) and (2.8), we get

$$\Big|\big[x_0'(t)u(t)\big]^{-\alpha+1} - y(t)^{-\alpha+1}\Big| \le$$
$$\le |\alpha-1|\big[(1-2d)c_0\nu_0 t^{\nu_0-1}\big]^{-\alpha}(c_0t^{\nu_0})^{-1}Mt^{-\ell+2\nu_0-1} =$$
$$= |\alpha-1|(1-2d)^{-\alpha}c_0^{-\alpha-1}\nu_0^{-\alpha}Mt^{-\alpha(\nu_0-1)-\ell+\nu_0-1}$$

for $t \ge T$. Then it is easy to see that

$$\psi_2(u,v)(t) = p_0(t)x_0(t)^{\beta+1}\big|\big[x_0'(t)u(t)\big]^{-\alpha+1} - y(t)^{-\alpha+1}\big|u(t)^\beta \le$$
$$\le \kappa t^{-\mu}(c_0t^{\nu_0})^{\beta+1}\,|\alpha-1|(1-2d)^{-\alpha}c_0^{-\alpha-1}\nu_0^{-\alpha}\times$$
$$\times Mt^{-\alpha(\nu_0-1)-\ell+\nu_0-1}(1+d)^\beta =$$
$$= \alpha(\ell-2\nu_0+1)C_2t^{-\ell+2\nu_0-2}, \quad t \ge T,$$

where C_2 is the constant given by (2.11).

By virtue of (2.20) and (2.17), we find that

$$\psi_3(u,v)(t) = p_0(t)|\varepsilon(t)|x_0(t)^{\beta+1}y(t)^{-\alpha+1}u(t)^\beta \le$$
$$\le \kappa t^{-\mu}|\varepsilon(t)|(c_0t^{\nu_0})^{\beta+1}Dc_0^{-\alpha+1}\nu_0^{-\alpha+1}t^{(\nu_0-1)(-\alpha+1)}(1+d)^\beta =$$
$$= \alpha C_3t^{2(\nu_0-1)}|\varepsilon(t)|, \quad t \ge T,$$

where C_3 is the constant given by (2.12). Therefore, by the above estimates for $\psi_1(u,v)(t)$, $\psi_2(u,v)(t)$ and $\psi_3(u,v)(t)$, and by (2.23), we conclude that

$$|\varphi_2(u,v)(t)| \le \alpha(C_1+C_2)(\ell-2\nu_0+1)t^{-\ell+2\nu_0-2} + \alpha C_3t^{2(\nu_0-1)}|\varepsilon(t)| \quad (2.24)$$

for $t \ge T$. Therefore, $\Phi_2(u,v)(t)$ is well defined on X. Moreover, on account of (2.16), we can conclude that

$$|\Phi_2(u,v)(t)| \le \bigg(C_1 + C_2 + C_3t^{\ell-2\nu_0+1}\int_t^\infty s^{2(\nu_0-1)}|\varepsilon(s)|\,ds\bigg)t^{-\ell+2\nu_0-1} \le$$
$$\le Mt^{-\ell+2\nu_0-1}, \quad t \ge T.$$

Thus, the operator $\Phi = (\Phi_1, \Phi_2)$ is well defined on X and maps X into itself. This proves the claim (i).

(ii) *The operator* $\Phi = (\Phi_1, \Phi_2)$ *is continuous on* X.

Assume that $(u_n, v_n) \in X$ $(n = 1, 2, 3, \dots)$, $(u_\infty, v_\infty) \in X$, and that $(u_n, v_n) \to (u_\infty, v_\infty)$ as $n \to \infty$ uniformly on any compact subinterval $[T, S]$ of $[T, \infty)$. The inequality (2.21) implies that, for every $(u_n, v_n) \in X$, the function $|\varphi_1(u_n, v_n)(t)|$ is bounded by the integrable function $L\ell t^{-\ell-1}$ on $[T, \infty)$. Therefore, by the Lebesgue dominated convergence theorem,

$$\Phi_1(u_n, v_n)(t) \to \Phi_1(u_\infty, v_\infty)(t) \ \text{ as } \ n \to \infty$$

uniformly on any compact subinterval $[T, S]$ of $[T, \infty)$. Similarly, using (2.24) and the Lebesgue dominated convergence theorem, we see that

$$\Phi_2(u_n, v_n)(t) \to \Phi_2(u_\infty, v_\infty)(t) \ \text{ as } \ n \to \infty$$

uniformly on any compact subinterval $[T, S]$ of $[T, \infty)$. This proves the claim (ii).

(iii) $\Phi(X)$ *is relatively compact.*

To prove the relative compactness of $\Phi(X)$, it is enough to show that $\Phi(X)$ is uniformly bounded and equicontinuous on any compact subinterval $[T, S]$ of $[T, \infty)$. The former follows from the fact that the inequalities $|\Phi_1(u, v)(t)| \le 1 + Lt^{-\ell}$ $(t \ge T)$, which is a consequence of (2.22), and $|\Phi_2(u, v)(t)| \le Mt^{-\ell+2\nu_0-1}$ $(t \ge T)$ hold for all $(u, v) \in X$. The latter follows from the fact that the inequalities (2.21) and (2.24) hold for all $(u, v) \in X$.

In view of (i)–(iii), the Schauder–Tychonoff theorem shows that Φ has a fixed point (u, v) in X. This fixed point $(u, v) = (u(t), v(t))$ $(\in X)$ is a solution of (2.7) on $[T, \infty)$, and satisfies (2.5) and (2.6). Consequently, $(u(t), v(t))$ $(\in X)$ is a solution of (2.4) which satisfies (2.3). Therefore, by Lemma 2.1, $x(t) = x_0(t)u(t)$ is an eventually positive solution of (1.12). By the previous arguments it is easy to see that

$$\frac{x(t)}{x_0(t)} = u(t) = 1 + O(t^{-\ell}) \ \text{ as } \ t \to \infty$$

and

$$\begin{aligned} \frac{x'(t)}{x_0'(t)} &= u(t) + \frac{1}{x_0(t)x_0'(t)}\, v(t) = \\ &= u(t) + \frac{1}{c_0^2\nu_0}\, t^{-2\nu_0+1} v(t) = 1 + O(t^{-\ell}) \ \text{ as } \ t \to \infty. \end{aligned}$$

This completes the proof of Theorem 1.1. □

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