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THE WEIERSTRASS–WHITTAKER INTEGRAL TRANSFORM

Dedicated to the memory of
Professor Viktor Kupradze (1903–1985)
on the 110th anniversary of his birthday

Abstract. We introduce a Weierstrass type transform associated with the Whittaker integral transform, which we refer to as *Weierstrass–Whittaker integral transform*. We examine some properties of the transform and show, in particular, that it is helpful in solving of a generalized non-stationary heat equation with an initial condition.

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რეზიუმე. ჩვენ განვმარტავთ ვაიერშტრასის ტიპის გარდაქმნას, რომელიც დაკავშირებულია უაითეკერის ინტეგრალურ გარდაქმნასთან და რომელსაც ჩვენ ვუწოდებთ ვაიერშტრას-უაითეკერის ინტეგრალურ გარდაქმნას. ჩვენ ვსწავლობთ ამ გარდაქმნის ზოგიერთ თვისებას და, კერძოდ, ვაჩვენებთ, რომ ის სასარგებლოა განზოგადოებული არასტაციონარული სითბოგამტარებლობის განტოლების ამოსახსნელად საწყისი პირობით.

1. Introduction

The Whittaker functions $M_{\mu,\nu}$ and $W_{\mu,\nu}$ of first and second order have acquired an increasing significance due to their frequent use in applications of mathematics to physical and technical problems (cf., e.g., [2]). Moreover, they are closely related to the confluent hypergeometric functions which play an important role in various branches of applied mathematics and theoretical physics. For instance, this is the case in fluid mechanics, electromagnetic diffraction theory and atomic structure theory. This justifies a continuous effort in studying properties of these functions and in gathering information about them, as well as the integral equations and transforms generated by them.

For a somehow much more detailed account of several significant results on the Whittaker and Weierstrass type transforms, over the last half-century, we refer to [1, 3–7, 11–14].

Let us consider the integral transform

$$[Wf](\tau) = \int_0^{+\infty} e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau) f(x) e^{-(x+\frac{1}{x})} x^{\alpha}\, dx, \quad \tau > 0, \tag{1.1}$$

where $\alpha > 0$. The main purpose of this work is to define an integral transform associated with the Whittaker integral transform (1.1) – which will be called *Weierstrass–Whittaker transform* – and to study some of its properties and possible applications. We define such integral transform by

$$[\mathcal{W}_t f](x) = \int_0^{+\infty} \mathcal{K}_t(x,y) f(y) e^{-(y+\frac{1}{y})} y^{\alpha}\, dy, \tag{1.2}$$

where $\mathcal{K}_t(x,y)$ is the heat kernel associated with the Whittaker transform (to be also studied later) and which is defined as

$$\mathcal{K}_t(x,y) = \int_0^{+\infty} e^{-4\nu^2\tau t} e^{-\frac{y\tau}{2}} W_{\mu,\nu}(y\tau) e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau) e^{-(\tau+\frac{1}{\tau})} \tau^{\alpha}\, d\tau$$

for $t, x, y > 0$.

The integral transform $\mathcal{W}_t f$ is a variant of the usual Weierstrass transform [9] and solves the heat type problem

$$\begin{cases} \partial_t [\mathcal{W}_t f](x) = -L_x [\mathcal{W}_t f](x), \\ \lim\limits_{t\to 0} [\mathcal{W}_t f](x) = f(x), \end{cases} \qquad t, x > 0,$$

where

$$L_x = 4\tau^3 x^2 \frac{d^2}{dx^2} + 4\tau^4 x^2 \frac{d}{dx} + \tau^3 x^2(\tau^2 - 1) + 4\mu\tau^2 x + \tau.$$

2. The Whittaker Integral Transform

In this section, we study some of the mapping properties of the integral transform (1.1) which may, in fact, be viewed as an operator acting from $L^2(\mathbb{R}^+, e^{-(x+\frac{1}{x})}x^\alpha\, dx)$ into $L^2(\mathbb{R}^+, e^{-(\tau+\frac{1}{\tau})}\tau^\alpha\, d\tau)$.

So, we consider the weighted Hilbert spaces $L^2(\mathbb{R}^+, e^{-(x+\frac{1}{x})}x^\alpha\, dx)$ endowed with the inner product

$$\langle f, g\rangle_{L^2(\mathbb{R}^+, e^{-(x+\frac{1}{x})}x^\alpha\, dx)} = \int_0^{+\infty} f(x)\overline{g(x)} e^{-(x+\frac{1}{x})}x^\alpha\, dx \tag{2.1}$$

which generates the associated norm

$$\|f\|_{L^2(\mathbb{R}^+, e^{-(x+\frac{1}{x})}x^\alpha\, dx)} = \left(\int_0^{+\infty} |f(x)|^2 e^{-(x+\frac{1}{x})}x^\alpha\, dx \right)^{1/2}. \tag{2.2}$$

In order to prove the convergence of the integral transform (1.1), we have the following auxiliary result.

Theorem 2.1. *Let* $f \in L^2(\mathbb{R}^+, e^{-(x+\frac{1}{x})}x^\alpha\, dx)$ *and*

$$\alpha > \max\left\{2|\nu| - 2, 0\right\}.$$

The integral transform (1.1) *is absolutely convergent and the following uniform estimate*

$$\left|[Wf](\tau)\right| \leq C_{\mu,\nu}(\tau)\|f\|_{L^2(\mathbb{R}^+, e^{-(x+\frac{1}{x})}x^\alpha\, dx)}. \tag{2.3}$$

holds.

Proof. Invoking the Cauchy–Schwarz inequality and relation (2.19.24.7) in [8], we have

$$\begin{aligned}
\left|[Wf](\tau)\right| &\leq \int_0^{+\infty} \left| e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau) f(x) e^{-(x+\frac{1}{x})}x^\alpha \right| dx \leq \\
&\leq \left(\int_0^{+\infty} e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau) e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau) e^{-(x+\frac{1}{x})}x^\alpha\, dx \right)^{1/2} \times \\
&\qquad \times \left(\int_0^{+\infty} |f(x)|^2 e^{-(x+\frac{1}{x})}x^\alpha\, dx \right)^{1/2} \leq \\
&\leq \left(\int_0^{+\infty} e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau) e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau) x^\alpha\, dx \right)^{1/2} \|f\|_{L^2(\mathbb{R}^+, e^{-(x+\frac{1}{x})}x^\alpha\, dx)} = \\
&= C_{\mu,\nu}(\tau)\, \|f\|_{L^2(\mathbb{R}^+, e^{-(x+\frac{1}{x})}x^\alpha\, dx)},
\end{aligned} \tag{2.4}$$

where

$$C_{\mu,\nu}(\tau) = \tau^{-\frac{\alpha+1}{2}} \Bigg(\frac{\Gamma(-2\nu)\Gamma(\alpha+2\nu+2)\Gamma(2+\alpha)}{\Gamma(\frac{1}{2}-\mu-\nu)\Gamma(\frac{5}{2}-\mu+\alpha+\nu)} \times$$
$$\times\, 3^F 2\Big(\frac{1}{2}+\mu+\nu, 2+\alpha+2\nu, 2+\alpha; 1+2\nu, \frac{5}{2}+\alpha+\nu-\mu; 1\Big) +$$
$$+ \frac{\Gamma(2\nu)\Gamma(\alpha-2\nu+2)\Gamma(2+\alpha)}{\Gamma(\frac{1}{2}-\mu+\nu)\Gamma(\frac{5}{2}-\mu+\alpha+\nu)} \times$$
$$\times\, 3^F 2\Big(\frac{1}{2}-\mu+\nu, 2+\alpha, 2+\alpha-2\nu; 1-2\nu, \frac{5}{2}+\alpha-\nu-\mu; -1\Big)\Bigg)^{1/2},$$

with $\tau > 0$, and where $3^F 2$ denotes the generalized hypergeometric function. Hence, besides the estimation in question, the convergence of the integral transform (1.1) is also obtained. □

We now concentrate on the image of the integral transform for the elements considered above. Namely, for that elements, in the next result we obtain that $Wf \in L^2(\mathbb{R}^+, e^{-(\tau+\frac{1}{\tau})}\tau^\alpha\, d\tau)$.

Theorem 2.2. *Let* $\alpha > \max\{2|\nu| - 2, 0\}$.

If $f \in L^2(\mathbb{R}^+, e^{-(x+\frac{1}{x})}x^\alpha\, dx)$, *then the Whittaker integral transform* $[Wf](\tau)$ *belongs to the space* $L^2(\mathbb{R}^+, e^{-(\tau+\frac{1}{\tau})}\tau^\alpha\, d\tau)$.

Proof. From the definition of the norm in $L^2(\mathbb{R}^+, e^{-(\tau+\frac{1}{\tau})}\tau^\alpha\, d\tau)$, taking into account that $f \in L^2(\mathbb{R}^+, e^{-(x+\frac{1}{x})}x^\alpha\, dx)$ and using (2.4), we obtain

$$\|Wf\|^2_{L^2(\mathbb{R}^+, e^{-(\tau+\frac{1}{\tau})}\tau^\alpha\, d\tau)} = \int_0^{+\infty} \big|[Wf](\tau)\big|^2 e^{-(\tau+\frac{1}{\tau})}\tau^\alpha\, d\tau \le$$
$$\le \int_0^{+\infty} (C_{\mu,\nu}(\tau))^2 \|f\|^2_{L^2(\mathbb{R}^+, e^{-(x+\frac{1}{x})}x^\alpha)} e^{-(\tau+\frac{1}{\tau})}\tau^\alpha\, d\tau =$$
$$= C^*_{\mu,\nu} \|f\|^2_{L^2(\mathbb{R}^+, e^{-(x+\frac{1}{x})}x^\alpha\, dx)} \int_0^{+\infty} \tau^{-(\alpha+1)} e^{-(\tau+\frac{1}{\tau})}\tau^\alpha\, d\tau \le$$
$$\le \Big(\Gamma(0,1) + \frac{1}{e}\Big) C^*_{\mu,\nu} \|f\|^2_{L^2(\mathbb{R}^+, e^{-(x+\frac{1}{x})}x^\alpha\, dx)}, \tag{2.5}$$

where

$$C^*_{\mu,\nu} = \frac{\Gamma(-2\nu)\Gamma(\alpha+2\nu+2)\Gamma(2+\alpha)}{\Gamma(\frac{1}{2}-\mu-\nu)\Gamma(\frac{5}{2}-\mu+\alpha+\nu)} \times$$
$$\times\, 3^F 2\Big(\frac{1}{2}+\mu+\nu, 2+\alpha+2\nu, 2+\alpha; 1+2\nu, \frac{5}{2}+\alpha+\nu-\mu; 1\Big) +$$
$$+ \frac{\Gamma(2\nu)\Gamma(\alpha-2\nu+2)\Gamma(2+\alpha)}{\Gamma(\frac{1}{2}-\mu+\nu)\Gamma(\frac{5}{2}-\mu+\alpha+\nu)} \times$$

$$\times\, 3^{F}2\Big(\frac{1}{2}-\mu+\nu, 2+\alpha, 2+\alpha-2\nu; 1-2\nu, \frac{5}{2}+\alpha-\nu-\mu; -1\Big), \quad (2.6)$$

and

$$\int_0^{+\infty} \tau^{-(\alpha+1)} e^{-(\tau+\frac{1}{\tau})} \tau^{\alpha}\, d\tau =$$

$$= \int_0^{1} \tau^{-(\alpha+1)} e^{-(\tau+\frac{1}{\tau})} \tau^{\alpha}\, d\tau + \int_1^{+\infty} \tau^{-(\alpha+1)} e^{-(\tau+\frac{1}{\tau})} \tau^{\alpha}\, d\tau \le$$

$$\le \int_0^{1} \tau^{-1} e^{-\frac{1}{\tau}} e^{-\tau}\, d\tau + \int_1^{+\infty} \tau^{-\alpha} e^{-(\tau+\frac{1}{\tau})} \tau^{\alpha}\, d\tau \le$$

$$\le \int_0^{1} \tau^{-1} e^{-\frac{1}{\tau}}\, d\tau + \int_1^{+\infty} e^{-(\tau+\frac{1}{\tau})}\, d\tau \le$$

$$\le \int_0^{1} \tau^{-1} e^{-\frac{1}{\tau}}\, d\tau + \int_1^{+\infty} e^{-\tau}\, d\tau = \Gamma(0,1) + \frac{1}{e}\,, \quad (2.7)$$

with $\Gamma(a, x)$ denoting the incomplete Gamma function. □

3. The Heat Kernel Related to the Whittaker Integral Transform

In order to introduce in a formal way the Weierstrass–Whittaker transform (1.2), we need first to study the heat kernel associated with the Whittaker transform. Therefore, we will introduce in this section the heat kernel associated with the Whittaker integral transform. Moreover, we will define and examine some of its properties.

Let us introduce the Hilbert space $H_K(\mathbb{R}^+)$, defined as the subspace of $L^2(\mathbb{R}^+, e^{-(x+\frac{1}{x})} x^{\alpha}\, dx)$ formed by all functions f such that

$$Wf \in L^2(\mathbb{R}^+, e^{-(\tau+\frac{1}{\tau})} \tau^{\alpha}\, d\tau).$$

$H_K(\mathbb{R}^+)$ is endowed with the inner product

$$\langle f, g\rangle_{H_K} = \int_0^{+\infty} [Wf](\tau)\overline{[Wg](\tau)} e^{-(\tau+\frac{1}{\tau})} \tau^{\alpha}\, d\tau \quad (3.1)$$

and, consequently, the norm of $H_K(\mathbb{R}^+)$ is given by

$$\|f\|_{H_K} = \sqrt{\langle f, f\rangle_{H_K}} = \Big(\int_0^{+\infty} \big|[Wf](\tau)\big|^2 e^{-(\tau+\frac{1}{\tau})} \tau^{\alpha}\, d\tau \Big)^{1/2}. \quad (3.2)$$

Proposition 3.1. *Let $\alpha > \max\{2|\nu| - 2, 0\}$. For $t > 0$, we introduce $\mathcal{K}_t(x,y)$ defined on $]0,+\infty[\times]0,+\infty[$ by*

$$\mathcal{K}_t(x,y) = \int_0^{+\infty} e^{-4\nu^2\tau t} e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau) e^{-\frac{y\tau}{2}} W_{\mu,\nu}(y\tau) e^{-(\tau+\frac{1}{\tau})}\tau^\alpha \, d\tau. \quad (3.3)$$

For all $y \in]0,+\infty[$, the function

$$x \mapsto \mathcal{K}_t(x,y)$$

belongs to $H_K(\mathbb{R}^+)$.

Proof. Invoking the Cauchy–Schwarz inequality and the relation (2.19.24.7) in [8], we will be able to prove first the fact that the kernel belongs to $L^2(\mathbb{R}^+, e^{-(x+\frac{1}{x})}x^\alpha \, dx)$. Indeed,

$$\|\mathcal{K}_t\|^2_{L^2(\mathbb{R}^+, e^{-(x+\frac{1}{x})}x^\alpha \, dx)} = \int_0^{+\infty} |\mathcal{K}_t(x,y)|^2 e^{-(x+\frac{1}{x})} x^\alpha \, dx =$$

$$= \int_0^{+\infty} \left(\int_0^{+\infty} e^{-4\nu^2\tau t} e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau) e^{-\frac{y\tau}{2}} W_{\mu,\nu}(y\tau) e^{-(\tau+\frac{1}{\tau})}\tau^\alpha d\tau \right)^2 e^{-(x+\frac{1}{x})} x^\alpha dx \le$$

$$\le \int_0^{+\infty} \left(\int_0^{+\infty} (e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau))^2 e^{-(\tau+\frac{1}{\tau})}\tau^\alpha \, d\tau \right) \times$$

$$\times \left(\int_0^{+\infty} \left(e^{-\frac{y\tau}{2}} W_{\mu,\nu}(y\tau)\right)^2 e^{-(\tau+\frac{1}{\tau})}\tau^\alpha \, d\tau \right) e^{-(x+\frac{1}{x})} x^\alpha \, dx \le$$

$$\le \int_0^{+\infty} \left(\int_0^{+\infty} (e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau))^2 \tau^\alpha \, d\tau \right) e^{-(x+\frac{1}{x})} x^\alpha \, dx \times$$

$$\times \left(\int_0^{+\infty} \left(e^{-\frac{y\tau}{2}} W_{\mu,\nu}(y\tau)\right)^2 \tau^\alpha \, d\tau \right) =$$

$$= (C^*_{\mu,\nu})^2 y^{-(\alpha+1)} \int_0^{+\infty} x^{-(\alpha+1)} e^{-(x+\frac{1}{x})} x^\alpha \, dx \le$$

$$\le \left(\Gamma(0,1) + \frac{1}{e} \right) (C^*_{\mu,\nu})^2 y^{-(\alpha+1)}, \quad (3.4)$$

where $C^*_{\mu,\nu}$ is given by (2.6).

In order to prove that $\mathcal{K}_t \in H_K(\mathbb{R}^+)$, we still need to prove that $W\mathcal{K}_t \in L^2(\mathbb{R}^+, e^{-(\tau+\frac{1}{\tau})}\tau^\alpha \, d\tau)$.

For $\alpha > \max\{2|\nu| - 2, 0\}$, we obtain the following estimate by using the Cauchy–Schwarz inequality:

$$|W\mathcal{K}_t| = \left| \int_0^{+\infty} e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau)\mathcal{K}_t(x,y) e^{-(x+\frac{1}{x})} x^\alpha \, dx \right| \leq$$

$$\leq \left(\int_0^{+\infty} (e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau))^2 e^{-(x+\frac{1}{x})} x^\alpha \, dx \right)^{1/2} \times$$

$$\times \left(\int_0^{+\infty} |\mathcal{K}_t(x,y)|^2 e^{-(x+\frac{1}{x})} x^\alpha \, dx \right)^{1/2} \leq$$

$$\leq \left(\int_0^{+\infty} \left(e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau)\right)^2 x^\alpha \, dx \right)^{1/2} \|\mathcal{K}_t\|_{L^2(\mathbb{R}^+, e^{-(x+\frac{1}{x})} x^\alpha \, dx)} =$$

$$= (C^*_{\mu,\nu})^{1/2} \tau^{-\frac{\alpha+1}{2}} \|\mathcal{K}_t\|_{L^2(\mathbb{R}^+, e^{-(x+\frac{1}{x})} x^\alpha \, dx)}.$$

Taking into account the previous inequality, we have

$$\|W\mathcal{K}_t\|^2_{L^2(\mathbb{R}^+, e^{-(\tau+\frac{1}{\tau})} \tau^\alpha \, d\tau)} = \int_0^{+\infty} \left|W\mathcal{K}_t(x,y)\right|^2 e^{-(\tau+\frac{1}{\tau})} \tau^\alpha \, d\tau \leq$$

$$\leq C^*_{\mu,\nu} \|\mathcal{K}_t\|^2_{L^2(\mathbb{R}^+, e^{-(x+\frac{1}{x})} x^\alpha \, dx)} \int_0^{+\infty} \tau^{-(\alpha+1)} e^{-(\tau+\frac{1}{\tau})} \tau^\alpha \, d\tau \leq$$

$$\leq \left(\Gamma(0,1) + \frac{1}{e}\right) C^*_{\mu,\nu} \|\mathcal{K}_t\|^2_{L^2(\mathbb{R}^+, e^{-(x+\frac{1}{x})} x^\alpha \, dx)}. \quad (3.5)$$

Therefore, we have just proved that, for $y > 0$, the function $x \mapsto \mathcal{K}_t(x,y)$ belongs to $H_K(\mathbb{R}^+)$. □

In order to obtain some important results related to the heat kernel and the Weierstrass transform, we need to introduce a new Hilbert space which we denote by $H^*_K(\mathbb{R}^+)$. Towards this end, we need first to guarantee the following result (which will ensure that the above-mentioned new space definition will be coherent with our purposes).

Lemma 3.2. *If $f \in H_K(\mathbb{R}^+)$, then*

$$\int_0^{+\infty} [Wf](\tau) e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau) e^{-(\tau+\frac{1}{\tau})} \tau^\alpha \, d\tau \quad (3.6)$$

belongs to $H_K(\mathbb{R}^+)$.

Proof. Having in mind the definition of $H_K(\mathbb{R}^+)$, under the above hypothesis, we realize that we have to prove that both the element in (3.6) and its image under W must belong to $L^2(\mathbb{R}^+, e^{-(x+\frac{1}{x})} x^\alpha \, dx)$.

For start, we will directly prove that for all elements $f \in H_K(\mathbb{R}^+)$ we have

$$\int_0^{+\infty} [Wf](\tau) e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau) e^{-(\tau+\frac{1}{\tau})} \tau^\alpha \, d\tau \in L^2\big(\mathbb{R}^+, e^{-(x+\frac{1}{x})} x^\alpha \, dx\big).$$

Indeed,

$$\int_0^{+\infty} \bigg| \int_0^{+\infty} [Wf](\tau) e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau) e^{-(\tau+\frac{1}{\tau})} \tau^\alpha \, d\tau \bigg|^2 e^{-(x+\frac{1}{x})} x^\alpha \, dx \le$$

$$\le \int_0^{+\infty} \bigg(\int_0^{+\infty} \big([Wf](\tau)\big)^2 e^{-(\tau+\frac{1}{\tau})} \tau^\alpha \, d\tau \bigg) \times$$

$$\times \bigg(\int_0^{+\infty} \big(e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau)\big)^2 e^{-(\tau+\frac{1}{\tau})} \tau^\alpha \, d\tau \bigg) e^{-(x+\frac{1}{x})} x^\alpha \, dx \le$$

$$\le \int_0^{+\infty} \bigg(\int_0^{+\infty} \big([Wf](\tau)\big)^2 e^{-(\tau+\frac{1}{\tau})} \tau^\alpha \, d\tau \bigg) \times$$

$$\times \bigg(\int_0^{+\infty} \big(e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau)\big)^2 \tau^\alpha \, d\tau \bigg) e^{-(x+\frac{1}{x})} x^\alpha \, dx \le$$

$$\le C^*_{\mu,\nu} \|Wf\|_{L^2(\mathbb{R}^+, e^{-(\tau+\frac{1}{\tau})}\tau^\alpha \, d\tau)} \int_0^{+\infty} x^{-\alpha-1} e^{-(x+\frac{1}{x})} x^\alpha \, dx \le$$

$$\le C^*_{\mu,\nu} \|Wf\|_{L^2(\mathbb{R}^+, e^{-(\tau+\frac{1}{\tau})}\tau^\alpha \, d\tau)} \int_0^{+\infty} x^{-\alpha-1} e^{-x} x^\alpha \, dx \le$$

$$\le C^*_{\mu,\nu} \Big(\Gamma(0,1) + \frac{1}{e}\Big) \|Wf\|_{L^2(\mathbb{R}^+, e^{-(\tau+\frac{1}{\tau})}\tau^\alpha \, d\tau)}. \tag{3.7}$$

From the previous inequality, taking into account the definition of the Whittaker integral transform (1.1), we have the following inequality related with the Whittaker transform:

$$\bigg| W \bigg[\int_0^{+\infty} [Wf](\tau) e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau) e^{-(\tau+\frac{1}{\tau})} \tau^\alpha \, d\tau \bigg] \bigg|^2 =$$

$$= \bigg| \int_0^{+\infty} e^{-\frac{x\tau'}{2}} W_{\mu,\nu}(x\tau') \bigg(\int_0^{+\infty} [Wf](\tau) e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau) \times$$

$$\times e^{-(\tau+\frac{1}{\tau})} \tau^\alpha \, d\tau \bigg) e^{-(x+\frac{1}{x})} x^\alpha \, dx \bigg|^2 \le$$

$$\leq \int_0^{+\infty} \left(e^{-\frac{x\tau'}{2}} W_{\mu,\nu}(x\tau') e^{-(x+\frac{1}{x})} x^{\alpha} \right)^2 \times$$

$$\times \Bigg(\int_0^{+\infty} [Wf](\tau) e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau) e^{-(\tau+\frac{1}{\tau})} \tau^{\alpha}\, d\tau \Bigg)^2 dx \leq$$

$$\leq C^*_{\mu,\nu} \|Wf\|_{L^2(\mathbb{R}^+, e^{-(\tau+\frac{1}{\tau})}\tau^{\alpha}\, d\tau)} \int_0^{+\infty} \big(e^{-\frac{x\tau'}{2}} W_{\mu,\nu}(x\tau') \big)^2 x^{2\alpha} x^{-\alpha-1}\, dx \leq$$

$$\leq (C^*_{\mu,\nu})^2 (\tau')^{-\alpha} \|Wf\|_{L^2(\mathbb{R}^+, e^{-(\tau+\frac{1}{\tau})}\tau^{\alpha}\, d\tau)}. \tag{3.8}$$

Therefore, for $f \in H_K$, we have

$$W\Bigg(\int_0^{+\infty} [Wf](\tau) e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau) e^{-(\tau+\frac{1}{\tau})} \tau^{\alpha}\, d\tau \Bigg) \in L^2\big(\mathbb{R}^+, e^{-(\tau'+\frac{1}{\tau'})} (\tau')^{\alpha}\, d\tau'\big)$$

i.e.,

$$\int_0^{+\infty} e^{-\frac{x\tau'}{2}} W_{\mu,\nu}(x\tau') \Bigg(\int_0^{+\infty} [Wf](\tau) e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau) e^{-(\tau+\frac{1}{\tau})} \tau^{\alpha}\, d\tau \Bigg) e^{-(x+\frac{1}{x})} x^{\alpha} dx$$

$$\in L^2\big(\mathbb{R}^+, e^{-(\tau'+\frac{1}{\tau'})} (\tau')^{\alpha}\, d\tau'\big).$$

Indeed, from (3.8), we get

$$\int_0^{+\infty} \Bigg| \Bigg[W\Bigg(\int_0^{+\infty} [Wf](\tau) e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau) e^{-(\tau+\frac{1}{\tau})} \tau^{\alpha}\, d\tau \Bigg) \Bigg] (\tau') \Bigg|^2 \times$$

$$\times\, e^{-(\tau'+\frac{1}{\tau'})} (\tau')^{\alpha}\, d\tau' \leq$$

$$\leq (C^*_{\mu,\nu})^2 \|Wf\|_{L^2(\mathbb{R}^+, e^{-(\tau+\frac{1}{\tau})}\tau^{\alpha}\, d\tau)} \int_0^{+\infty} e^{-(\tau'+\frac{1}{\tau'})} (\tau')^{\alpha} (\tau')^{-\alpha}\, d\tau' \leq$$

$$\leq (C^*_{\mu,\nu})^2 \|Wf\|_{L^2(\mathbb{R}^+, e^{-(\tau+\frac{1}{\tau})}\tau^{\alpha}\, d\tau)}.$$

□

Having in mind Lemma 3.2, we are now in a position to define $H^*_K(\mathbb{R}^+)$ as the space of elements $f \in H_K(\mathbb{R}^+)$ which admit the integral representation

$$f(x) = \int_0^{+\infty} [Wf](\tau) e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau) e^{-(\tau+\frac{1}{\tau})} \tau^{\alpha}\, d\tau. \tag{3.9}$$

We will now exhibit a significative result based on the representation of the elements of the space $H^*_K(\mathbb{R}^+)$ and the definition of the heat kernel.

Lemma 3.3. *Let $\mathcal{K}_t \in H_K^*(\mathbb{R}^+)$. Then, the Whittaker type transform* (1.1) *of the heat kernel is given by*

$$[W\mathcal{K}_t](\tau, x) = e^{-4\nu^2\tau t} e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau). \tag{3.10}$$

Proof. From Proposition 3.1, we find that $\mathcal{K}_t \in H_K(\mathbb{R}^+)$. Taking into account the definition of heat kernel (3.3) and since $\mathcal{K}_t \in H_K^*(\mathbb{R}^+)$, we get $[W\mathcal{K}_t](\tau, x) = e^{-4\nu^2\tau t} e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau)$. □

4. Properties of the Weierstrass–Whittaker Transform

In this section, we shall define the above-mentioned Weierstrass–Whittaker transform in a formal way, and derive some of its properties.

Definition 4.1. The Weierstrass transform associated with the Whittaker integral transform and called *Weierstrass–Whittaker transform*, is defined in $L^2(\mathbb{R}^+, e^{-(y+\frac{1}{y})} y^\alpha \, dy)$ by

$$[\mathcal{W}_t f](x) = \int\limits_0^{+\infty} \mathcal{K}_t(x, y) f(y) e^{-(y+\frac{1}{y})} y^\alpha \, dy. \tag{4.1}$$

For the classical Weierstrass transform, one can see [9].

Proposition 4.2. *Let $\alpha > \max\{0, 2\nu - 2\}$. For all $t > 0$, the Weierstrass type transform $\mathcal{W}_t f$ is a bounded operator from $L^2(\mathbb{R}^+, e^{-(y+\frac{1}{y})} y^\alpha \, dy)$ into $L^2(\mathbb{R}^+, e^{-(x+\frac{1}{x})} x^\alpha \, dx)$ and, for all $f \in L^2(\mathbb{R}^+, e^{-(y+\frac{1}{y})} y^\alpha \, dy)$, we have*

$$\|\mathcal{W}_t f\|^2_{L^2(\mathbb{R}^+, e^{-(x+\frac{1}{x})} x^\alpha \, dx)} \leq$$
$$\leq (C^*_{\mu,\nu})^2 \Big(\Gamma(0,1) + \frac{1}{e}\Big)^2 \|f\|^2_{L^2(\mathbb{R}^+, e^{-(y+\frac{1}{y})} y^\alpha \, dy)}. \tag{4.2}$$

Proof. The absolutely convergence of the integral (4.1) follows from the Cauchy–Schwarz inequality and Proposition 3.1. Indeed,

$$\big|[\mathcal{W}_t f](x)\big| \leq \int\limits_0^{+\infty} |\mathcal{K}_t(x, y)| \, |f(y)| e^{-(y+\frac{1}{y})} y^\alpha \, dy \leq$$
$$\leq \Bigg(\int\limits_0^{+\infty} |\mathcal{K}_t(x, y)|^2 e^{-(y+\frac{1}{y})} y^\alpha \, dy \Bigg)^{1/2} \Bigg(\int\limits_0^{+\infty} |f(y)|^2 e^{-(y+\frac{1}{y})} y^\alpha \, dy \Bigg)^{1/2} \leq$$
$$\leq \Bigg(\int\limits_0^{+\infty} (C^*_{\mu,\nu})^2 x^{-(\alpha+1)} y^{-(\alpha+1)} e^{-(y+\frac{1}{y})} y^\alpha \, dy \Bigg)^{1/2} \|f\|_{L^2(\mathbb{R}^+, e^{-(y+\frac{1}{y})} y^\alpha \, dy)} \leq$$
$$\leq C^*_{\mu,\nu} \Big(\Gamma(0,1) + \frac{1}{e}\Big)^{\frac{1}{2}} x^{-\frac{\alpha+1}{2}} \|f\|_{L^2(\mathbb{R}^+, e^{-(y+\frac{1}{y})} y^\alpha \, dy)}. \tag{4.3}$$

Then, for all $f \in L^2(\mathbb{R}^+, e^{-(y+\frac{1}{y})}y^\alpha\, dy)$ and using the relation (4.3), we have

$$\|\mathcal{W}_t f\|^2_{L^2(\mathbb{R}^+, e^{-(x+\frac{1}{x})}x^\alpha\, dx)} = \int_0^{+\infty} |[\mathcal{W}_t f](x)|^2 e^{-(x+\frac{1}{x})} x^\alpha\, dx \leq$$

$$\leq (C^*_{\mu,\nu})^2 \Big(\Gamma(0,1) + \frac{1}{e}\Big) \|f\|^2_{L^2(\mathbb{R}^+, e^{-(y+\frac{1}{y})}y^\alpha\, dy)} \int_0^{+\infty} x^{-(\alpha+1)} e^{-(x+\frac{1}{x})} x^\alpha\, dx \leq$$

$$\leq (C^*_{\mu,\nu})^2 \Big(\Gamma(0,1) + \frac{1}{e}\Big)^2 \|f\|^2_{L^2(\mathbb{R}^+, e^{-(y+\frac{1}{y})}y^\alpha\, dy)}. \quad \square$$

Proposition 4.3. *Let* $\alpha > \max\{0, 2\nu - 2\}$*. For all* $t > 0$*, the Weierstrass–Whittaker transform* $\mathcal{W}_t f$ *belongs to the space* $H_K(\mathbb{R}^+)$*.*

Proof. From the previous proposition we have

$$\mathcal{W}_t f \in L^2\big(\mathbb{R}^+, e^{-(x+\frac{1}{x})} x^\alpha\, dx\big).$$

Now, in order to prove that $\mathcal{W}_t f$ belongs to the space $H_K(\mathbb{R}^+)$, we need to show that $W[\mathcal{W}_t f] \in L^2(\mathbb{R}^+, e^{-(\tau+\frac{1}{\tau})}\tau^\alpha\, d\tau)$.

From the definition of the Whittaker type transform, we obtain

$$\big|\big[W[\mathcal{W}_t f]\big](\tau)\big| \leq \int_0^{+\infty} e^{-\frac{x\tau}{2}} |W_{\mu,\nu}(x\tau)|\, |\mathcal{W}_t f(x)| e^{-(x+\frac{1}{x})} x^\alpha\, dx$$

and by using (4.3) and taking into account the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
\big|[W\big[\mathcal{W}_t f\big]](\tau)\big| &\leq \Big(\Gamma(0,1) + \frac{1}{e}\Big)^{\frac{1}{2}} C^*_{\mu,\nu} \|f\|_{L^2(\mathbb{R}^+, e^{-(y+\frac{1}{y})}y^\alpha\, dy)} \times \\
&\quad \times \int_0^{+\infty} e^{-\frac{x\tau}{2}} |W_{\mu,\nu}(x\tau)| x^{-\frac{\alpha+1}{2}} e^{-(x+\frac{1}{x})} x^\alpha\, dx \leq \\
&\leq \Big(\Gamma(0,1) + \frac{1}{e}\Big)^{\frac{1}{2}} C^*_{\mu,\nu} \|f\|_{L^2(\mathbb{R}^+, e^{-(y+\frac{1}{y})}y^\alpha\, dy)} \times \\
&\quad \times \Big(\int_0^{+\infty} \big(e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau)\big)^2 e^{-(x+\frac{1}{x})} x^\alpha\, dx \Big)^{1/2} \times \\
&\quad \times \Big(\int_0^{+\infty} x^{-(\alpha+1)} e^{-(x+\frac{1}{x})} x^\alpha\, dx \Big)^{1/2} \leq \\
&\leq \tau^{-\frac{\alpha+1}{2}} \Big(\Gamma(0,1) + \frac{1}{e}\Big) (C^*_{\mu,\nu})^{\frac{3}{2}} \|f\|_{L^2(\mathbb{R}^+, e^{-(y+\frac{1}{y})}y^\alpha\, dy)}.
\end{aligned}$$

Having in mind the previous inequality, we obtain the following estimate:

$$\big\|W[\mathcal{W}_t f]\big\|^2_{L^2(\mathbb{R}^+, e^{-(\tau+\frac{1}{\tau})}\tau^\alpha\, d\tau)} = \int_0^{+\infty} \big|W[\mathcal{W}_t f](\tau)\big|^2 e^{-(\tau+\frac{1}{\tau})}\tau^\alpha\, d\tau \leq$$

$$\leq \Big(\Gamma(0,1)+\frac{1}{e}\Big)^2 (C^*_{\mu,\nu})^3 \|f\|^2_{L^2(\mathbb{R}^+, e^{-(y+\frac{1}{y})}y^\alpha\, dy)} \int_0^{+\infty} \tau^{-(\alpha+1)} e^{-(\tau+\frac{1}{\tau})}\tau^\alpha\, d\tau \leq$$

$$\leq \Big(\Gamma(0,1)+\frac{1}{e}\Big)^3 (C^*_{\mu,\nu})^3 \|f\|^2_{L^2(\mathbb{R}^+, e^{-(y+\frac{1}{y})}y^\alpha\, dy)}. \tag{4.4}$$

Hence, it follows that the composition of the Whittaker type transform (1.1) with the Weierstrass–Whittaker transform (4.1) belongs to the space $L^2(\mathbb{R}^+, e^{-(\tau+\frac{1}{\tau})}\tau^\alpha\, d\tau)$ and therefore $\mathcal{W}_t f \in H_K(\mathbb{R}^+)$. □

The just used composition of integral transformations can be described in an even more detailed way if we invoke the representation of the elements of the space $H^*_K(\mathbb{R}^+)$ and the definition of the Weierstrass–Whittaker transform, as we shall see in the next result.

Lemma 4.4. *Let $\mathcal{W}_t f \in H^*_K(\mathbb{R}^+)$. For all $t > 0$, we have*

$$\big[W[\mathcal{W}_t f]\big](\tau) = e^{-4\nu^2\tau t}[Wf](\tau). \tag{4.5}$$

Proof. From the definition of Weierstrass–Whittaker transform, the definition of inner product in $H_K(\mathbb{R}^+)$, Proposition 3.1, Proposition 4.3 and Lemma 3.3, we deduce

$$\begin{aligned}
[\mathcal{W}_t f](x) &= \int_0^{+\infty} \mathcal{K}_t(x,y) f(y) e^{-(y+\frac{1}{y})} y^\alpha\, dy = \\
&= \int_0^{+\infty} [W\mathcal{K}_t](\tau) W[f](\tau) e^{-(\tau+\frac{1}{\tau})}\tau^\alpha\, d\tau = \\
&= \int_0^{+\infty} e^{-4\nu^2\tau t} e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau)[Wf](\tau) e^{-(\tau+\frac{1}{\tau})}\tau^\alpha\, d\tau.
\end{aligned}$$

Since $\mathcal{W}_t f \in H^*_K(\mathbb{R}^+)$, invoking (3.9), we find

$$\big[W[\mathcal{W}_t f]\big](\tau) = e^{-4\nu^2\tau t}[Wf](\tau). \tag{4.6}$$

□

5. The Weierstrass–Whittaker Transform as a Solution of a Heat Type Equation

In this last section we will show that the Weierstrass–Whittaker transform $\mathcal{W}_t f$ solves a non-stationary heat type equation (cf. (5.2)). To this end, first of all, we need to prove that the kernel $\mathcal{K}_t(x,y)$ is a solution of a variant of the heat equation.

We start by recalling that the Whittaker function is an eigenfunction of a second order differential operator. More precisely,

$$A_z W_{\mu,\nu}(z) = 4\nu^2 W_{\mu,\nu}(z),$$

where

$$A_z = 4z^2 \frac{d^2}{dz^2} - z^2 + 4\mu z + 1. \tag{5.1}$$

From the differential properties of the Whittaker function, the absolute and uniform convergence of the integral (1.3) and its derivatives with respect to t and x, we directly arrive at the following result.

Corollary 5.1. *The kernel $\mathcal{K}_t(x,y)$ satisfies the non-stationary heat type equation*

$$\partial_t u(t,x,y) = -L_x u(t,x,y), \quad t,x,y>0, \tag{5.2}$$

where

$$L_x = 4\tau^3 x^2 \frac{d^2}{dx^2} + 4\tau^4 x^2 \frac{d}{dx} + \tau^3 x^2(\tau^2-1) + 4\mu\tau^2 x + \tau. \tag{5.3}$$

is a second order differential operator which satisfies

$$L_x\big(e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau)\big) = 4\nu^2 \tau e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau). \tag{5.4}$$

Furthermore, the kernel $\mathcal{K}_t(x,y)$ is also a solution of the non-stationary heat type equation

$$\partial_t u(t,x,y) = -L_y u(t,x,y), \quad t,x,y>0, \tag{5.5}$$

where

$$L_y = 4\tau^3 y^2 \frac{d^2}{dy^2} + 4\tau^4 y^2 \frac{d}{dy} + \tau^3 y^2(\tau^2-1) + 4\mu\tau^2 y + \tau \tag{5.6}$$

is a second order differential operator which satisfies

$$L_y\big(e^{-\frac{y\tau}{2}} W_{\mu,\nu}(y\tau)\big) = 4\nu^2 \tau e^{-\frac{y\tau}{2}} W_{\mu,\nu}(y\tau). \tag{5.7}$$

Theorem 5.2. *Let $f \in H_K(\mathbb{R}^+)$. For all $t>0$ and for all $\mathcal{W}_t f \in H_K^*(\mathbb{R}^+)$, the function $\mathcal{W}_t f$ solves the generalized heat equation* (5.2), *with the initial condition $\lim_{t\to 0}[\mathcal{W}_t f](x) = f(x)$ in $H_K(\mathbb{R}^+)$.*

Proof. Propositions 3.1 and 4.2 guarantee the necessary differential properties of $\mathcal{W}_t f$, and from the differential properties of the Whittaker function we deduce that the function $\mathcal{W}_t f$ is a solution of (5.2).

We will now prove the initial condition. From the definition of the norm of $H_K(\mathbb{R}^+)$ (cf. (3.2)) and using Lemma 4.4, we have

$$\|\mathcal{W}_t f - f\|^2_{L^2(\mathbb{R}^+, e^{-(x+\frac{1}{x})}x^\alpha\, dx)} =$$

$$= \int_0^{+\infty} \Big|\big[W[\mathcal{W}_t f]\big](\tau) - [Wf](\tau)\Big|^2 e^{-(\tau+\frac{1}{\tau})}\tau^\alpha\, d\tau =$$

$$= \int_0^{+\infty} \big|e^{-4\nu^2\tau t} - 1\big|^2 \big|[Wf](\tau)\big|^2 e^{-(\tau+\frac{1}{\tau})}\tau^\alpha\, d\tau. \tag{5.8}$$

Since $4\nu^2\tau t > 0$, we realize that the right-hand side of (5.8) is estimated by $\int\limits_0^{+\infty} |[Wf](\tau)|^2 e^{-(\tau+\frac{1}{\tau})}\tau^\alpha\, d\tau$. Then, we can pass to the limit $\to 0$ through equation (5.8) and the desired result is obtained. □

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