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# VARIATION FORMULAS OF SOLUTION FOR NEUTRAL FUNCTIONAL-DIFFERENTIAL EQUATIONS WITH REGARD FOR THE DELAY FUNCTION PERTURBATION AND THE CONTINUOUS INITIAL CONDITION

**Abstract.** Variation formulas of solution are obtained for linear with respect to prehistory of the phase velocity (quasi-linear) neutral functional-differential equations with variable delays. In the variation formulas, the effect of perturbation of the delay function appearing in the phase coordinates is stated.

**რეზიუმე.** ფაზური სიჩქარის წინაისტორიის მიმართ წრფივი ნეიტრალური ფუნქციონალურ-დიფერენციალური განტოლებებისთვის ცვლადი დაგვიანებებით მიღებულია ამონახსნის ვარიაციის ფორმულები. ვარიაციის ფორმულებში გამოვლენილია ფაზურ კოორდინატებში შემავალი დაგვიანების ფუნქციის შეშფოთების ეფექტი.

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Let $I = [a, b]$ be a finite interval and $\mathbb{R}^n$ be the $n$-dimensional vector space of points $x = (x^1, \dots, x^n)^T$, where $T$ is the sign of transposition. Suppose that $O \subset \mathbb{R}^n$ is an open set, and $E_f$ is the set of functions $f : I \times O^2 \to \mathbb{R}^n$ satisfying the following conditions: the function $f(t, \cdot) : O^2 \to \mathbb{R}^n$ is continuously differentiable for almost all $t \in I$; the functions $f(t, x, y)$, $f_x(t, x, y)$ and $f_y(t, x, y)$ are measurable on $I$ for any $(x, y) \in O^2$; for each $f \in E_f$ and compact set $K \subset O$, there exists a function $m_{f,K}(t) \in L(I, \mathbb{R}_+)$, $\mathbb{R}_+ = [0, \infty)$, such that

$$|f(t, x, y)| + |f_x(t, x, y)| + |f_y(t, x, y)| \le m_{f,K}(t)$$

for all $(x, y) \in K^2$ and almost all $t \in I$.

Further, let $D$ be the set of continuous differentiable scalar functions (delay functions) $\tau(t), t \in I$, satisfying the conditions:

$$\tau(t) < t, \quad \dot{\tau}(t) > 0, \quad \inf\{\tau(a) : \ \tau \in D\} := \widehat{\tau} > -\infty.$$

Let $\Phi$ be the set of continuously differentiable initial functions $\varphi(t) \in O$, $t \in I_1 = [\widehat{\tau}, b]$.

To each element $\mu = (t_0, \tau, \varphi, f) \in \Lambda = [a, b) \times D \times \Phi \times E_f$ we assign the quasi-linear neutral functional-differential equation

$$\dot{x}(t) = A(t)\dot{x}(\sigma(t)) + f\big(t, x(t), x(\tau(t))\big) \tag{1}$$

with the continuous initial condition

$$x(t) = \varphi(t), \quad t \in [\widehat{\tau}, t_0], \tag{2}$$

where $A(t)$ is a given continuous matrix function of dimension $n \times n$; $\sigma \in D$ is a fixed delay function.

**Definition 1.** Let $\mu = (t_0, \tau, \varphi, f) \in \Lambda$. A function $x(t) = x(t; \mu) \in O$, $t \in [\widehat{\tau}, t_1]$, $t_1 \in (t_0, b]$, is said to be a solution of equation (1) with the initial condition (2), or a solution corresponding to the element $\mu$ and defined on the interval $[\widehat{\tau}, t_1]$, if $x(t)$ satisfies condition (2) and is absolutely continuous on the interval $[t_0, t_1]$ and satisfies equation (1) almost everywhere on $[t_0, t_1]$.

Let $\mu_0 = (t_{00}, \tau_0, \varphi_0, f_0) \in \Lambda$ be the given element and $x_0(t)$ be a solution corresponding to $\mu_0$ and defined on $[\widehat{\tau}, t_{10}]$, with $a < t_{00} < t_{10} < b$.

Let us introduce the set of variations

$$V = \Big\{\delta\mu = (\delta t_0, \delta\tau, \delta\varphi, \delta f) : \ |\delta t_0| \le \alpha, \ \|\delta\tau\| \le \alpha,$$
$$\delta\varphi = \sum_{i=1}^{k} \lambda_i \delta\varphi_i, \ \delta f = \sum_{i=1}^{k} \lambda_i \delta f_i, \ |\lambda_i| \le \alpha, \ i = \overline{1,k}\Big\}.$$

Here

$$\delta t_0 \in \mathbb{R}, \quad \delta\tau \in D - \tau_0, \quad \|\delta\tau\| = \sup\big\{|\delta\tau(t)| : \ t \in I\big\}$$

and

$$\delta\varphi_i \in \Phi - \varphi_0, \quad \delta f_i \in E_f - f_0, \quad i = \overline{1,k},$$

are the fixed functions and $\alpha > 0$ is a fixed number.

There exist the numbers $\delta_1 > 0$ and $\varepsilon_1 > 0$ such that for arbitrary $(\varepsilon, \delta\mu) \in (0, \varepsilon_1] \times V$ the element $\mu_0 + \varepsilon\delta\mu \in \Lambda$ and there corresponds the solution $x(t; \mu_0 + \varepsilon\delta\mu)$ defined on the interval $[\widehat{\tau}, t_{10} + \delta_1] \subset I_1$ ( [1, Theorem 2]).

Due to the uniqueness, the solution $x(t; \mu_0)$ is a continuation of the solution $x_0(t)$ on the interval $[\widehat{\tau}, t_{10} + \delta_1]$. Therefore, the solution $x_0(t)$ is assumed to be defined on the interval $[\widehat{\tau}, t_{10} + \delta_1]$.

Let us define the increment of the solution

$$x_0(t) = x(t; \mu_0) : \ \Delta x(t; \varepsilon\delta\mu) = x(t; \mu_0 + \varepsilon\delta\mu) - x_0(t),$$
$$\forall\, (t, \varepsilon, \delta\mu) \in [\widehat{\tau}, t_{10} + \delta_1] \times (0, \varepsilon_1] \times V.$$

**Theorem 1.** *Let the following conditions hold:*

1) *the function $f_0(t, x, y), (t, x, y) \in I \times O^2$ is bounded;*

2) *there exists the limit*

$$\lim_{z \to z_0} f_0(z) = f_0^-, \quad z = (t, x, y) \in (a, t_{00}] \times O^2,$$

*where* $z_0 = (t_{00}, \varphi_0(t_{00}), \varphi_0(\tau_0(t_{00})))$.

*Then there exist the numbers* $\varepsilon_2 \in (0, \varepsilon_1)$ *and* $\delta_2 \in (0, \delta_1)$ *such that*

$$\Delta x(t; \varepsilon\delta\mu) = \varepsilon\delta x(t; \delta\mu) + o(t; \varepsilon\delta\mu) \tag{3}$$

*for arbitrary* $(t, \varepsilon, \delta\mu) \in [t_{00}, t_{10} + \delta_2] \times (0, \varepsilon_2] \times V^-$, *where* $V^- = \{\delta\mu \in V : \delta t_0 \le 0\}$ *and*

$$\begin{aligned} \delta x(t; \delta\mu) = {} & Y(t_{00}-; t)\Big[\dot{\varphi}_0(t_{00}) - A(t_{00})\dot{\varphi}_0(\sigma(t_{00})) - f_0^-\Big]\delta t_0 + \\ & + \beta(t; \delta\mu), \end{aligned} \tag{4}$$

$$\begin{aligned} \beta(t; \delta\mu) = {} & \Psi(t_{00}; t)\delta\varphi(t_{00}) + \\ & + \int\limits_{\tau_0(t_{00})}^{t_{00}} Y(\gamma_0(s); t) f_{0y}[\gamma_0(s)]\dot{\gamma}_0(s)\delta\varphi(s)\, ds + \\ & + \int\limits_{\sigma(t_{00})}^{t_{00}} Y(\varrho(s); t) A(\varrho(s))\dot{\varrho}(s)\dot{\delta}\varphi(s)\, ds + \\ & + \int\limits_{t_{00}}^{t} Y(s; t) f_{0y}[s]\dot{x}_0(\tau_0(s))\delta\tau(s)\, ds + \\ & + \int\limits_{t_{00}}^{t} Y(s; t)\delta f[s]\, ds, \end{aligned} \tag{5}$$

$$\lim_{\varepsilon\to 0} \frac{o(t; \varepsilon\delta\mu)}{\varepsilon} = 0 \ \ \textit{uniformly for} \ \ (t, \delta\mu) \in [t_{00}, t_{10} + \delta_2] \times V^-,$$

$Y(s; t)$ *and* $\Psi(s; t)$ *are the* $n \times n$*-matrix functions satisfying the system*

$$\begin{cases} \Psi_s(s; t) = -Y(s; t) f_{0x}[t] - Y(\gamma_0(s); t) f_{0y}[\gamma_0(s)]\dot{\gamma}_0(s), \\ Y(s; t) = \Psi(s; t) + Y(\varrho(s); t) A(\varrho(s))\dot{\varrho}(s), \quad s \in [t_{00}, t], \end{cases}$$

*and the condition*

$$\Psi(s; t) = Y(s; t) = \begin{cases} H, & s = t, \\ \Theta, & s > t; \end{cases}$$

$$f_{0y}[s] = f_{0y}\big(s, x_0(s), x_0(\tau_0(s))\big), \quad \delta f[s] = \delta f\big(s, x_0(s), x_0(\tau_0(s))\big);$$

$\gamma_0(s)$ *is the function, inverse to* $\tau_0(t)$, $\varrho(s)$ *is the function, inverse to* $\sigma(t)$, $H$ *is the identity matrix and* $\Theta$ *is the zero matrix.*

**Some comments.** The function $\delta x(t; \delta\mu)$ is called the variation of the solution $x_0(t), t \in [t_{00}, t_{10} + \delta_2]$, and the expression (4) is called the variation formula.

The addend

$$\int\limits_{t_{00}}^{t} Y(s;t) f_{0y}[s]\dot{x}_0(\tau_0(s))\delta\tau(s)\,ds$$

in formula (5) is the effect of perturbation of the delay function $\tau_0(t)$.

The expression

$$Y(t_{00}-;t)\Big[\dot{\varphi}_0(t_{00}) - A(t_{00})\dot{\varphi}_0(\sigma(t_{00})) - f_0^-\Big]\delta t_0$$

is the effect of the continuous initial condition (2) and perturbation of the initial moment $t_{00}$.

The expression

$$\Psi(t_{00};t)\delta\varphi(t_{00}) + \int\limits_{\tau_0(t_{00})}^{t_{00}} Y(\gamma_0(s);t) f_{0y}[\gamma_0(s)]\dot{\gamma}_0(s)\delta\varphi(s)\,ds +$$
$$+ \int\limits_{\sigma(t_{00})}^{t_{00}} Y(\varrho(s);t)A(\varrho(s))\dot{\varrho}(s)\dot{\delta}\varphi(s)\,ds + \int\limits_{t_{00}}^{t} Y(s;t)\delta f[s]\,ds$$

in formula (5) is the effect of perturbations both of the initial function $\varphi_0(t)$ and of the function $f_0(t,x,y)$.

Variation formulas of solutions for various classes of neutral functional-differential equations without perturbation of delay function can be found in [2–4]. The variation formula of solution plays the basic role in proving the necessary conditions of optimality and under sensitivity analysis of mathematical models [5–8]. Finally, it should be noted that the variation formula allows one to get an approximate solution of the perturbed equation

$$\dot{x}(t) = A(t)\dot{x}(\sigma(t)) +$$
$$+ f_0\big(t, x(t), x(\tau_0(t) + \varepsilon\delta\tau(t))\big) + \varepsilon\delta f\big(t, x(t), x(\tau_0(t) + \varepsilon\delta\tau(t))\big)$$

with the perturbed initial condition

$$x(t) = \varphi_0(t) + \varepsilon\delta\varphi(t), \quad t \in [\widehat{\tau}, t_{00} + \varepsilon\delta t_0].$$

In fact, for a sufficiently small $\varepsilon \in (0, \varepsilon_2]$ it follows from (3) that

$$x(t;\mu_0 + \varepsilon\delta\mu) = x_0(t) + \varepsilon\delta x(t;\delta\mu).$$

**Theorem 2.** *Let the following conditions hold:*

1) *the function $f_0(t,x,y), (t,x,y) \in I \times O^2$ is bounded;*

2) *there exists the limit*

$$\lim_{z\to z_0} f_0(z) = f_0^+, \quad z \in [t_{00}, b) \times O^2.$$

*Then for each* $\widehat{t_0} \in (t_{00}, t_{10})$ *there exist the numbers* $\varepsilon_2 \in (0, \varepsilon_1)$ *and* $\delta_2 \in (0, \delta_1)$ *such that for arbitrary* $(t, \varepsilon, \delta\mu) \in [\widehat{t_0}, t_{10} + \delta_2] \times (0, \varepsilon_2] \times V^+$, *where* $V^+ = \{\delta\mu \in V : \ \delta t_0 \geq 0\}$, *formula* (3) *holds, where*

$$\delta x(t; \delta\mu) = Y(t_{00}+; t)(\dot{\varphi}(t_{00}) - A(t_{00})\dot{\varphi}_0(\sigma(t_{00})) - f_0^+)\delta t_0 + \beta(t; \delta\mu).$$

The following assertion is a corollary to Theorems 1 and 2.

**Theorem 3.** *Let the assumptions of Theorems* 1 *and* 2 *be fulfilled. Moreover,* $f_0^- = f_0^+ := \widehat{f}_0$ *and* $t_{00} \notin \{\sigma(t_{10}), \sigma^2(t_{10})), \dots\}$. *Then there exist the numbers* $\varepsilon_2 \in (0, \varepsilon_1)$ *and* $\delta_2 \in (0, \delta_1)$ *such that for arbitrary* $(t, \varepsilon, \delta\mu) \in [t_{10} - \delta_2, t_{10} + \delta_2] \times (0, \varepsilon_2] \times V$ *formula* (3) *holds, where*

$$\delta x(t; \delta\mu) = Y(t_{00}; t)(\dot{\varphi}(t_{00}) - A(t_{00})\dot{\varphi}_0(\sigma(t_{00})) - \widehat{f}_0)\delta t_0 + \beta(t; \delta\mu).$$

All assumptions of Theorem 3 are satisfied if the function $f_0(t, x, y)$ is continuous and bounded. Clearly, in this case

$$\widehat{f}_0 = f_0\big(t_{00}, \varphi_0(t_{00}), \varphi_0(\tau_0(t_{00}))\big).$$


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