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MATRIX SPECTRAL FACTORIZATION WITH PERTURBED DATA

Dedicated to Boris Khvedelidze's 100-th birthday anniversary

Abstract. A necessary condition for the existence of spectral factorization is positive definiteness a.e. on the unit circle of a matrix function which is being factorized. Correspondingly, the existing methods of approximate computation of the spectral factor can be applied only in the case where the matrix function is positive definite. However, in many practical situations an empirically constructed matrix spectral densities may lose this property. In the present paper we consider possibilities of approximate spectral factorization of matrix functions by their known perturbation which might not be positive definite on the unit circle.

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რეზიუმე. სპექტრალური ფაქტორიზაციის არსებობისათვის აუცილებელია გასაფაქტორიზირებელი მატრიც ფუნქცია იყოს დადებითად განსაზღვრული თ.ყ. ერთეულოვან წრეწირზე. შესაბამისად, არსებული მეთოდები სპექტრალური თანამამრავლის მიახლოებით პოვნისა შეიძლება გამოყენებულ იქნას მხოლოდ დადებითად განსაზღვრული მატრიცებისათვის, მაშინ როცა მრავალ პრაქტიკულ სიტუაციაში ემპირიულად შედგენილი სპექტრალური სიმკვრივის მატრიცი შეიძლება უკვე აღარ იყოს ამ თვისების მატარებელი. წინამდებარე ნაშრომში განხილულია შესაძლებლობები მატრიც ფუნქციების მიახლოებითი ფაქტორიზაციისა მათი "შეშფოთებული" ვერსიის მიხედვით, რომელნიც შეიძლება აღარ იყვნენ დადებითად განსაზღვრულნი.

1. Introduction

Matrix Spectral Factorization Theorem [9], [5] asserts that if

$$S(t)=\begin{pmatrix} s_{11}(t) & s_{12}(t) & \cdots & s_{1r}(t)\\ s_{21}(t) & s_{22}(t) & \cdots & s_{2r}(t)\\ \vdots & \vdots & \vdots & \vdots\\ s_{r1}(t) & s_{r2}(t) & \cdots & s_{rr}(t)\end{pmatrix}, \tag{1}$$

$|t|=1$, is a positive definite (a.e.) matrix function with integrable entries, $s_{ij}\in L^1(\mathbb{T})$, and if the Paley–Wiener condition

$$\log\det S\in L^1(\mathbb{T}) \tag{2}$$

is satisfied, then (1) admits a (left) spectral factorization

$$S(t)=S^+(t)S^-(t)=S^+(t)(S^+(t))^*, \tag{3}$$

where S^+ is an $r\times r$ outer analytic matrix function with entries from the Hardy space $H^2(\mathbb{D})$, $\mathbb{D}=\{z\in\mathbb{C}:\ |z|<1\}$, and $S^-(z)=(S^+(1/\overline{z}))^*$, $|z|>1$. It is assumed that (3) holds for boundary values a.e. on $\mathbb{T}$. A spectral factor S^+ is unique up to a constant right unitary multiplier (see e.g. [3]).

In the scalar case, $r=1$, a spectral factor of a positive function f can be explicitly written by the formula

$$f^+(z)=\exp\left(\frac{1}{4\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\,\log f(e^{i\theta})\,d\theta\right) \tag{4}$$

and it is well-known that if (1) is a Laurent polynomial matrix

$$S(t)=\sum_{k=-N}^{N}C_kt^k,\quad C_k\in\mathbb{C}^{r\times r}, \tag{5}$$

then the spectral factor $S^+(t)=\sum_{k=0}^{N}A_kt^k$ is a polynomial matrix of the same order (see e.g. [2]).

A challenging practical problem is actual approximate computation of matrix coefficients of analytic function S^+ for a given matrix function (1). Starting with Wiener's original efforts [10], various methods have been developed to approach this problem (see the survey papers [7], [8] and references therein). Recently, a new algorithm of matrix spectral factorization has been proposed in [6]. This algorithm can be applied to any matrix function which satisfies the necessary and sufficient condition (2) for the existence of factorization. (Most of other algorithms impose additional restrictions on (1), such as S to be rational or strictly positive definite on the boundary.) In the present paper we would like to demonstrate that (at least in the polynomial case) the proposed algorithm can be also applied to the so-called "perturbed" data which looses the property of positive definiteness. Namely, we consider and solve the following problem.

In most practical applications of spectral factorization, a power spectral density S is constructed from empirical observations which are always subject to small numerical errors. Thus, instead of theoretically existing matrix spectral density (1), which is always positive definite (a.e.) on $\mathbb{T}$, we have to deal with $\widehat{S}$ which may no longer be even positive semi-definite on $\mathbb{T}$. The classical illustrative example is when $S(t)=\sum_{k=-n}^{n} C_k t^k$ is a Laurent matrix polynomial with $\det S(t_0)=0$ for some $t_0\in\mathbb{T}$ and we disturb the coefficients C_k. The question arises if the above mentioned spectral factorization algorithm can treat $\widehat{S}$ as positive definite in order to correct this "small error" in data and find S^+ approximately. (Most of existing matrix spectral factorization algorithms do not make sense for non positive definite data.) Below we provide a positive answer to this question. To be specific, for polynomial matrix functions, depending on algorithm proposed in [6], we explicitly describe a computational procedure which can be applied to any polynomial data (say, maps $\mathfrak{C}_n:\mathcal{P}_N(m\times m)\to\mathcal{P}_N^+(m\times m)$, $n=1,2,\dots$, see Section 2 for the notation) in such a manner that the following statement is true.

Theorem 1. *Let S be a polynomial matrix function* (5) *which is positive semi-definite on $\mathbb{T}$ and has a spectral factor S^+, and let S_n, $n=1,2,\dots$, be a sequence of arbitrary* (*not necessarily positive semi-definite on $\mathbb{T}$*) *polynomial matrix functions of the same degree N such that*

$$\|S_n-S\|_{L^1}\to 0. \tag{6}$$

Then

$$\|\mathfrak{C}_n(S_n)-S^+\|_{L^2}\to 0. \tag{7}$$

The paper is organized as follows. In the next section, we introduce the notation that will be used throughout the paper. In Section 3, we review the matrix spectral factorization algorithm proposed in [6] and in Section 4 we describe the strategy dealing with non positive definite matrices. In Section 5, we consider the above formulated problem in the scalar case and solve it for polynomial functions. A partial solution of the problem is provided for general spectral densities. The main Theorem 1 along with some auxiliary lemmas are proved in Section 6.

2. Notation

Let $\mathbb{T}=\{z\in\mathbb{C}:\ |z|=1\}$ with the standard Lebesgue measure $d\mu$ on it and $\overline{\mathbb{D}}=\{z\in\mathbb{C}:\ |z|\le 1\}$. As usual, $L^p=L^p(\mathbb{T})$, $0<p<\infty$, denotes the Lebesgue space of p-integrable complex functions defined on $\mathbb{T}$, and $\mathbb{C}^{m\times m}$, $L^p(\mathbb{T})^{m\times m}$, etc., denote the set of $m\times m$ matrices with entries from $\mathbb{C}$, $L^p(\mathbb{T})$, etc. If $S\in\mathbb{C}^{r\times r}$ is a matrix (function) and $m\le r$, then $S_{[m]}$ stands for the upper-left $m\times m$ submatrix of S ($S_{[0]}$ is assumed to be 1). For a matrix (function) M, its Hermitian conjugate matrix (function) is denoted by $M^* = \overline{M}^T$. Finally, I_m is the $m \times m$ unit matrix.

The kth Fourier coefficient of an integrable (matrix) function $f \in L^1(\mathbb{T})$ ($f \in L^1(\mathbb{T})^{m\times m}$) is denoted by $c_k\{f\}$ ($C_k\{f\} \in \mathbb{C}^{m\times m}$). For $p \geq 1$, $L^p_+(\mathbb{T}) := \{f \in L_p(\mathbb{T}) : \ c_k\{f\} = 0$ whenever $k < 0\}$, and, for $n \geq 0$, $L^p_{n-}(\mathbb{T}) := \{f \in L_p(\mathbb{T}) : \ c_k\{f\} = 0$ whenever $k < -n\}$. Moreover, $\mathcal{P}_N := \{\sum_{k=-N}^{N} c_k z^k, \ c_k \in \mathbb{C}\}$ is the set of trigonometric polynomials of degree at most N and $\mathcal{P}^+_N := \{\sum_{k=0}^{N} c_k z^k, \ c_k \in \mathbb{C}\}$. Also, $\mathcal{P} = \cup \mathcal{P}_N$ and $\mathcal{P}^+ = \cup \mathcal{P}^+_N$, while $\mathbb{Q}[z] := \{p/q : \ p, q \in \mathcal{P}^+\}$ stands for the set of rational functions.

The Hardy space of analytic functions in $\mathbb{D}$, $H^p = H^p(\mathbb{D})$ is identified with $L^p_+(\mathbb{T})$ for $p \geq 1$, and $H^p_O = H^p_O(\mathbb{D})$ is the set of outer analytic functions from H_p. A square matrix function is called outer if its determinant is an outer function.

For a real function f, let ${}_\delta f$ be the truncated function

$$ {}_\delta f(t) = \begin{cases} f(t) & \text{if } f(t) > \delta, \\ \delta & \text{if } f(t) \leq \delta \end{cases} \tag{8}$$

(we usually use the argument "t" for functions defined on $\mathbb{T}$). Also, let $f^{(+)} = \max(0, f)$ and $f^{(-)} = \max(0, -f)$.

The notation $f_n \rightrightarrows f$ means that f_n converges to f in measure. Observe that

$$ f_n \rightrightarrows f \Longrightarrow f_n^{(+)} \rightrightarrows f^{(+)}. \tag{9}$$

We will also use the following implication (see, e.g. [4, Corollary 1]):

$$ \|f_n - f\|_{L^2} \to 0, \ \ |u_n| \leq 1, \ \ u_n \rightrightarrows u \Longrightarrow \|f_n u_n - f u\|_{L^2} \to 0. \tag{10}$$

3. Overview of the Matrix Spectral Factorization Algorithm

The first step of the matrix spectral factorization algorithm proposed in [6] is the triangular factorization

$$ S(t) = M_S(t) M_S^*(t), \tag{11}$$

where $M_S(t)$ is the lower triangular matrix

$$ M_S(t) = \begin{pmatrix} f_1^+(t) & 0 & \cdots & 0 & 0 \\ \xi_{21}(t) & f_2^+(t) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \xi_{r-1,1}(t) & \xi_{r-1,2}(t) & \cdots & f_{r-1}^+(t) & 0 \\ \xi_{r1}(t) & \xi_{r2}(t) & \cdots & \xi_{r,r-1}(t) & f_r^+(t) \end{pmatrix}, \tag{12}$$

$\xi_{ij} \in L^2(\mathbb{T})$, $f_i^+ \in H^2_O$. Then M_S is post multiplied by the unitary matrix functions of the special form $\mathbf{U}_2, \mathbf{U}_3, \ldots, \mathbf{U}_r$, so that to make the left-upper

$m\times m$ submatrices of M_S analytic step-by-step, $m=2,3,\dots,r$. As a result, we get (see [4, formula (47)])

$$S^+(t)=M_S(t)\mathbf{U}_2(t)\mathbf{U}_3(t)\cdots\mathbf{U}_r(t), \tag{13}$$

where each $\mathbf{U}_m$ has a block matrix form

$$\mathbf{U}_m(t)=\begin{pmatrix}U_m(t) & 0\\ 0 & I_{r-m}\end{pmatrix},$$

and $U_m(t)$ is the special unitary matrix function

$$U(t)=\begin{pmatrix}u_{11}(t) & u_{12}(t) & \cdots & u_{1,m-1}(t) & u_{1m}(t)\\ u_{21}(t) & u_{22}(t) & \cdots & u_{2,m-1}(t) & u_{2m}(t)\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ u_{m-1,1}(t) & u_{m-1,2}(t) & \cdots & u_{m-1,m-1}(t) & u_{m-1,m}(t)\\ \overline{u_{m1}(t)} & \overline{u_{m2}(t)} & \cdots & \overline{u_{m,m-1}(t)} & \overline{u_{mm}(t)}\end{pmatrix}, \tag{14}$$

$$u_{ij}\in L^\infty_+,\quad \det U(t)=1\ \text{ a.e.}, \tag{15}$$

while, for each $m\le r$, the left-upper $m\times m$ submatrix of $M_S\mathbf{U}_2\mathbf{U}_3\dots\mathbf{U}_m$ is a spectral factor of the left-upper $m\times m$ submatrix of S, i.e.,

$$\big(M_S(t)\mathbf{U}_2(t)\mathbf{U}_3(t)\cdots\mathbf{U}_m(t)\big)_{[m]}=S^+_{[m]}. \tag{16}$$

An explicit description of the representation (13) and its approximate computation are discussed in [6], [4]. In particular, when the left-upper $(m-1)\times(m-1)$ submatrix of (12) has already been made analytic, the matrix function $\big(M_S\mathbf{U}_2\mathbf{U}_3\dots\mathbf{U}_{m-1}\big)_{[m]}$ has the form

$$\begin{aligned}&\big(M_S\mathbf{U}_2\cdots\mathbf{U}_{m-1}\big)_{[m]}\\ &=\begin{pmatrix} & & & & 0\\ & S^+_{[m-1]} & & & \vdots\\ & & & & 0\\ \zeta_1 & \zeta_2 & \dots & \zeta_{m-1} & f^+_m\end{pmatrix}=\begin{pmatrix} & & & & 0\\ & S^+_{[m-1]} & & & \vdots\\ & & & & 0\\ 0 & 0 & \dots & 0 & 1\end{pmatrix}F,\end{aligned} \tag{17}$$

where F is the matrix function

$$F(t)=\begin{pmatrix}1 & 0 & 0 & \cdots & 0 & 0\\ 0 & 1 & 0 & \cdots & 0 & 0\\ 0 & 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 1 & 0\\ \zeta_1(t) & \zeta_2(t) & \zeta_3(t) & \cdots & \zeta_{m-1}(t) & f^+_m(t)\end{pmatrix}. \tag{18}$$

Remark 1. Note that matrix function (17) multiplied by its Hermitian conjugate gives $S_{[m]}$. Therefore, the following equation

$$S^{+}_{[m-1]} \begin{pmatrix} \overline{\zeta_1} \\ \overline{\zeta_2} \\ \vdots \\ \overline{\zeta_{m-1}} \end{pmatrix} = \begin{pmatrix} s_{1m} \\ s_{2m} \\ \vdots \\ s_{m-1,m} \end{pmatrix} \tag{19}$$

holds.

The analyticity of the m-th row in (17) is achieved by application of the following

Theorem 2 (see [4, Lemma 4]). *For any matrix function F of the form* (18)*, where*

$$\zeta_i \in L^2(\mathbb{T}), \;\; i = 1, 2, \dots, m-1, \;\; f^+ \in H^2_O, \tag{20}$$

there exists a unitary matrix function U of the form (14), (15)*, such that*

$$FU \in L^2_+(\mathbb{T})^{m\times m}.$$

Remark 2. Note that under the above circumstances,

$$S^{+}_{[m]} = \begin{pmatrix} & & & & 0 \\ & & S^{+}_{[m-1]} & & \vdots \\ & & & & 0 \\ \zeta_1 & \zeta_2 & \dots & \zeta_{m-1} & f^+_m \end{pmatrix} U. \tag{21}$$

In order to compute (14) approximately, (18) is approximated by its Fourier series. More specifically, let F_n be the matrix function (18) in which the last row is replaced by

$$\big(\zeta^n_1(t), \zeta^n_2(t), \dots, \zeta^n_{m-1}(t), f^+_n(t)\big), \;\; \zeta^n_i \in L^2(\mathbb{T}), \tag{22}$$

where $\zeta^n_i(t) = \sum\limits_{k=-n}^{\infty} c_k\{\zeta_i\} t^k$, $f^+_n = f^+_m$. Then the following result is invoked:

Theorem 3 (see [6, Theorem 1]). *Let F_n be a matrix function of the form* (18), (22)*, where*

$$\zeta^n_i \in L^2_{n-}(\mathbb{T}), \;\; i = 1, 2, \dots, \;\; \textit{and} \;\; f^+_n(0) \neq 0. \tag{23}$$

Then there exists a unique unitary matrix function U_n of the form (14)*, where $u_{ij} \in \mathcal{P}^+_n$,* $\det U_n(t) = 1$ *(on $\mathbb{T}$), and $U_n(1) = I_m$, such that*

$$F_n U_n \in L^2_+(\mathbb{T})^{m\times m}.$$

Note that [6] in fact provides an explicit construction of U_n.

In order to justify the approximating properties of the algorithm, the following convergence theorem is proved in [4].

Theorem 4 (cf. [4, Theorem 2]). *Let F be a matrix function of the form* (18), (20), *and let F_n, $n = 1, 2, \dots$, be a sequence of matrix functions of the form* (18) *with the last row replaced by* (22). *Let also*

$$\zeta_i^n \to \zeta_i, \quad f_n^+ \to f^+ \ \text{ in } L^2 \ \text{ and } \ f_n^+ \in H_O^2. \tag{24}$$

If U_n, $n = 1, 2, \dots$, are the corresponding unitary matrix functions defined according to Theorem 2, then U_n converges in measure:

$$U_n \rightrightarrows U,$$

and F_nU_n converges in L^2 to the spectral factor of FF^*.

4. Treatment of Nonpositive Definite Matrix Functions

The main argument which helps to deal with the matrix functions $\widehat{S}$ which are close to S, but are not necessarily positive semi-definite (a.e. on $\mathbb{T}$) is the observation that Theorem 4 remains valid if in (24) we replace the condition $f_n^+ \in H_O^2$ by a weaker requirement $f_n^+(0) \neq 0$, as in (23). Theorem 3 guarantees that the corresponding U_n exists in this case as well. Because of the importance of this fact for our goals, we formulate this result separately.

Theorem 5. *Let F be a matrix function of the form* (18), (20), *and let F_n, $n = 1, 2, \dots$, be a sequence of matrix function of the form* (18), (22), (23) *such that*

$$\zeta_i^n \to \zeta_i \ \text{ and } \ f_n^+ \to f^+ \ \text{ in } L^2.$$

If U_n, $n = 1, 2, \dots$, are the corresponding unitary matrix functions defined according to Theorem 3, then U_n converges in measure:

$$U_n \rightrightarrows U,$$

and F_nU_n converges in L^2 to the spectral factor of FF^*.

Remark 3. It should be observed that under the above circumstances F_nU_n might not be the canonical spectral factor of $F_nF_n^*$ (which was the case in the situation of Theorem 4), since $\det(F_nU_n) = \det(F_n) = f_n^+$ might have zeros inside the unit circle. Thus the phrase in the end of the first column on page 2320 in [6] contains a small inaccuracy.

Although Theorems 4 and 5 look alike, there is a significant difference between their meaning, as it is explained in the above remark. Nevertheless, the proof of Theorem 5 does not require any additional efforts, and the proof of Theorem 4 given in [4] goes through without any changes. Therefore we do not provide the proof of Theorem 5 here.

For a positive definite (a.e. on $\mathbb{T}$) matrix function (1) the triangular factorization (11) can be performed by the recurrent formulas which are similar to Cholesky factorization formulas for constant positive definite matrices.

Namely, for the entries of matrix function (12), we can write (see [6], formulas (56)–(58)):

$$f_m^+ = \sqrt{\det S_{[m]}/\det S_{[m-1]}} = \det S_{[m]}^+/\det S_{[m-1]}^+\,, \tag{25}$$

where $\sqrt{f}$ is the scalar spectral factor of f defined by (4),

$$\xi_{i1} = s_{i1}/\overline{f_1^+},\ \ i = 2, 3, \dots, r, \tag{26}$$

$$\xi_{ij} = \Big(s_{ij} - \sum_{k=1}^{j-1} \xi_{ik}\overline{\xi_{jk}}\Big)\Big/\overline{f_j^+},\ \ j = 2, 3, \dots, r-1,\ \ i = j+1, j+2, \dots, r. \tag{27}$$

If now $\widehat{S}$ is not necessarily positive definite, then $\det \widehat{S}_{[m]}$ might become negative on a set of positive measure and we would not be able to define the scalar spectral factor of $\det S_{[m]}/\det S_{[m-1]}$. However, we could still define $M_{\widehat{S}}$ using formulas (25)–(27) if we were able to determine the $\sqrt{f}$ for not necessarily positive function f. In the following section, we define a "scalar spectral factor" of not necessarily positive function for specific cases. If we continue the computational procedures described in Section 3 for $M_{\widehat{S}}$ in place of M_S, we get $M_{\widehat{S}}\mathbf{U}_2\mathbf{U}_3 \cdots \mathbf{U}_r$ which is similar to expression (13) and therefore we would expect its closeness to S^+. For polynomial matrix functions, we perform these procedures in an explicit way.

5. The Scalar Case

If $0 \le f \in L^1(\mathbb{T})$ and $\log f \in L^1(\mathbb{T})$, then the spectral factor f^+ can be written by the formula (4). However, if we only know that $\widehat{f}$ is close to f in L^1 norm, then $\widehat{f}$ might even not be non-negative a.e. (we discard the imaginary part of $\widehat{f}$ if it exists, so we assume here that $\widehat{f}$ is a real function). Even if $\widehat{f}$ were positive, $\log \widehat{f}$ should also be close to $\log f$ in order to claim the closeness of $\widehat{f}^+$ to f^+ (see [4], [1]). Therefore, we consider

$${}_\delta f^+(z) = \exp\left(\frac{1}{4\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log {}_\delta f(e^{i\theta})\, d\theta\right) \tag{28}$$

(see (8)) and prove the following

Lemma 1. *Let* $0 \le f \in L^1(\mathbb{T})$ *and* $\log f \in L^1(\mathbb{T})$*. Suppose* $f_n \in L^1(\mathbb{T})$ *and*

$$\|f_n - f\|_{L^1} \to 0. \tag{29}$$

Then

$$\lim_{\delta\to 0+} \lim_{n\to\infty} \|{}_\delta f_n^+ - f^+\|_{L^2} = 0. \tag{30}$$

Proof. It has been proved in [4] that if $0 \le f_n \in L^1(\mathbb{T})$, (29) holds and $\int_0^{2\pi} \log f_n(e^{i\theta})\, d\theta \to \int_0^{2\pi} \log f(e^{i\theta})\, d\theta$, then $\|f_n^+ - f^+\|_{H^2} \to 0$. Hence, we can show first that

a) $\lim\limits_{n\to\infty} \|{}_\delta f_n - {}_\delta f\|_{L^1} = 0$ for each $\delta > 0$ and

$$\lim_{n\to\infty} \int_{\mathbb{T}} \log {}_\delta f_n(e^{i\theta})\, d\theta = \int_{\mathbb{T}} \log {}_\delta f(e^{i\theta})\, d\theta, \tag{31}$$

which implies that $\lim\limits_{n\to\infty} \|{}_\delta f_n^+ - {}_\delta f^+\|_{L^2} = 0$ and, consequently,

$$\lim_{n\to\infty} \|{}_\delta f_n^+ - f^+\|_{L^2} = \|{}_\delta f^+ - f^+\|_{L^2}, \tag{32}$$

and then

b) $\lim\limits_{\delta\to 0+} \|{}_\delta f - f\|_{L^1} \to 0$ and $\lim\limits_{\delta\to 0+} \int_{\mathbb{T}} \log {}_\delta f(e^{i\theta})\, d\theta = \int_{\mathbb{T}} \log f(e^{i\theta})\, d\theta$, which implies that

$$\lim_{\delta\to 0+} \|{}_\delta f^+ - f^+\|_{L^2} = 0. \tag{33}$$

The relation (30) will then follow from (32) and (33).

Part b) is an easy exercise in Lebesgue integration theory, and we will thus concentrate on a). It is easy to realize that $|{}_\delta f_n - {}_\delta f| \le |f_n - f|$ and therefore the first part of a) follows from (29), which also implies that $\log {}_\delta f_n \rightrightarrows \log {}_\delta f$ as $n \to \infty$. In addition, $[\log {}_\delta f_n]^{(\pm)} \rightrightarrows [\log {}_\delta f]^{(\pm)}$ (see (9)).

The necessary and sufficient condition for (29) is that

$$f_n \rightrightarrows f \ \text{ and } \sup_{n>k,\mu(E)<\varepsilon} \int_E f_n\, dm \to 0 \ \text{ as } \ k\to\infty \ \text{ and } \ \varepsilon \to 0.$$

Therefore, $\|[\log {}_\delta f_n]^{(+)} - [\log {}_\delta f]^{(+)}\|_{L^1} \to 0$ and, in addition, $\|[\log {}_\delta f_n]^{(-)} - [\log {}_\delta f]^{(-)}\|_{L^1} \to 0$ due to the bounded convergence theorem. Thus (31) follows. □

The relation (30) shows that for any sequence f_n, $n = 1, 2, \ldots$, satisfying (29) there exist $\delta_n \to 0+$ such that

$$\lim_{n\to\infty} \|{}_{\delta_n} f_n^+ - f^+\|_{L^2} = 0. \tag{34}$$

However, (34) does not hold for every sequence $\delta_n \to 0+$ and, in general, it is hard to determine conditions on δ_n which would guarantee (34).

An explicit computational procedure is proposed for polynomial case. Namely, for any polynomial $p(z) = \sum\limits_{k=-N}^{N} c_k z^k$ (which might not be positive or even real on $\mathbb{T}$), let

$$\check{p}(t) = \begin{cases} \Re\mathrm{e}\{p(t)\} & \text{when } \Re\mathrm{e}\{p(t)\} > 0, \\ 1 & \text{when } \Re\mathrm{e}\{p(t)\} \le 0, \end{cases} \quad t \in \mathbb{T}, \tag{35}$$

let $\check{p}^+$ be the spectral factor of $\check{p}$:

$$\check{p}^+ = \exp\left(\frac{1}{4\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log \check{p}(e^{i\theta})\, d\theta \right), \tag{36}$$

and let $\widetilde{p}^+$ be its Fourier approximation up to degree N:

$$\widetilde{p}_n^+(z) = \sum_{k=0}^{N} c_k\{\widehat{p}_n^+\} z^k. \tag{37}$$

We prove the following

Lemma 2. *Let*

$$f(t) = \sum_{k=-N}^{N} c_k t^k \geq 0 \ \textit{ for } \ t \in \mathbb{T}, \tag{38}$$

and let

$$f_n(t) = \sum_{k=-N}^{N} c_k^{\{n\}} t^k$$

be such a sequence that

$$f_n \to f. \tag{39}$$

Then

$$\widetilde{f}_n^+ \to f^+. \tag{40}$$

Proof. We will show that

$$\|\check{f}_n^+ - f^+\|_{H^2} \to 0 \tag{41}$$

which implies (40), by virtue of the definition (37).

In order to prove (41), it is sufficient to show that (see [4])

$$\|\check{f}_n - f\|_{L^1} \to 0 \tag{42}$$

and

$$\int_{\mathbb{T}} \log \check{f}_n(t)\, dt \to \int_{\mathbb{T}} \log f(t)\, dt. \tag{43}$$

Let $E_n := \{t \in \mathbb{T} : \Re\mathrm{e}\{f_n(t)\} > 0\}$. Then

$$\check{f}_n = \mathbf{1}_{E_n} \Re\mathrm{e}\{f_n\} + \mathbf{1}_{\mathbb{T}\setminus E_n} = \Re\mathrm{e}\{f_n\}^{(+)} + \mathbf{1}_{\mathbb{T}\setminus E_n}. \tag{44}$$

Since (39) holds, we have $\|\Re\mathrm{e}\{f_n\} - f\|_{L^1} \to 0$, which implies that

$$\left\|\Re\mathrm{e}\{f_n\}^{(+)} - f^{(+)}\right\|_{L^1} = \left\|\Re\mathrm{e}\{f_n\}^{(+)} - f\right\|_{L^1} \to 0. \tag{45}$$

Since $\Re\mathrm{e}\{f_n\} \rightrightarrows f$, $\mu\{f \leq 0\} = 0$, and 0 is the continuity point of the distribution function $t \mapsto \mu\{f \leq t\}$, we have

$$\mu(\mathbb{T} \setminus E_n) \to 0, \tag{46}$$

which implies that $\|\mathbf{1}_{\mathbb{T}\setminus E_n}\|_{L^1} \to 0$ and (42) follows from (44) and (45).

In order to prove (43), we need the following

Lemma 3. *Let $\mathcal{P}'_N \subset \mathcal{P}_N^+$ be the set of monic polynomials with the degrees not exceeding N. Then*

$$\lim_{\delta\to 0+} \sup_{f \in \mathcal{P}'_N} \left| \int_{\{t\in\mathbb{T}:\ |f(t)|<\delta\}} \log|f(t)|\, d\mu \right| = 0. \tag{47}$$

Proof. Let

$$f(z)=\prod_{k=1}^{N}(z-z_k).$$

Then $\{t\in\mathbb{T}:\ |f(t)|<\delta\}\subset\bigcup_{k=1}^{N}\{t\in\mathbb{T}:\ |t-z_k|<\delta^{1/N}\}$ and

$$\Bigg|\int\limits_{\{|f|<\delta\}}\log|f|\,d\mu\Bigg|\le\int\limits_{\bigcup_{k=1}^{N}\{|t-z_k|<\delta^{1/N}\}}\big|\log|f|\big|\,d\mu$$

$$\le\sum_{j=1}^{N}\int\limits_{\bigcup_{k=1}^{N}\{|t-z_k|<\delta^{1/N}\}}\big|\log|t-z_j|\big|\le\sum_{j=1}^{N}\sum_{k=1}^{N}\int\limits_{\{|t-z_k|<\delta^{1/N}\}}\big|\log|t-z_j|\big|$$

$$\le N^2\int\limits_{\{|t-1|<\delta^{1/N}\}}\big|\log|t-1|\big|\,d\mu\to0\ \text{ as }\ \delta\to0+\,.$$

Consequently, (47) holds. □

We continue with the proof of (43) as follows.

Since $\Re\mathrm{e}\{f_n\}\to f$ and $\check{f}_n-\Re\mathrm{e}\{f_n\}\rightrightarrows 0$ by virtue of (35) and (46), the convergence in measure

$$\check{f}_n\rightrightarrows f \tag{48}$$

holds, which implies that ${}_\delta\check{f}_n\rightrightarrows{}_\delta f$ for each $\delta>0$, and since $\check{f}_n$ are uniformly bounded as well, from the above by virtue of (39), we have

$$\int\limits_{\mathbb{T}}\log\,{}_\delta\check{f}_n\,d\mu\to\int\limits_{\mathbb{T}}\log\,{}_\delta f\,d\mu. \tag{49}$$

On the other hand,

$$\lim_{\delta\to0+}\Bigg|\int\limits_{\mathbb{T}}\log\,{}_\delta f\,d\mu-\int\limits_{\mathbb{T}}\log f\,d\mu\Bigg|\le\lim_{\delta\to0+}\int\limits_{\{f\le\delta\}}\log f\,d\mu=0 \tag{50}$$

as $\log f\in L^1(\mathbb{T})$, and

$$\Bigg|\int\limits_{\mathbb{T}}\log\,{}_\delta\check{f}_n\,d\mu-\int\limits_{\mathbb{T}}\log\check{f}_n\,d\mu\Bigg|\le\Bigg|\int\limits_{\{0<\check{f}_n\le\delta\}}\log\check{f}_n\,d\mu\Bigg|$$

$$\le\Bigg|\int\limits_{\{0<\Re\mathrm{e}\{f_n\}\le\delta\}}\log\Re\mathrm{e}\{f_n\}\,d\mu\Bigg|\to0\ \text{ as }\ \delta\to0+, \tag{51}$$

by virtue of Lemma 3, since $\Re\mathrm{e}\{f_n\}(t)=\Re\mathrm{e}\{\sum_{k=-N}^{N}c_k^{\{n\}}t^k\}=\sum_{k=-N}^{N}\check{c}_k^{\{n\}}t^k$ are trigonometric polynomials and $\big(\check{c}_N^{\{n\}}\big)^{-1}t^N\Re\mathrm{e}\{f_n\}(t)\in\mathcal{P}'_{2N}$, while $\check{c}_N^{\{n\}}$

are uniformly bounded. Now (43) follows from (49), (50) and (51) since

$$\lim_{n\to\infty}\left|\int_{\mathbb{T}}\log \check{f}_n\,d\mu-\int_{\mathbb{T}}\log f\,d\mu\right|$$
$$\leq \lim_{n\to\infty}\limsup_{\delta\to 0+}\left|\int_{\mathbb{T}}\log \check{f}_n\,d\mu-\int_{\mathbb{T}}\log {}_\delta\check{f}_n\,d\mu\right|$$
$$+\lim_{n\to\infty}\limsup_{\delta\to 0+}\left|\int_{\mathbb{T}}\log {}_\delta\check{f}_n\,d\mu-\int_{\mathbb{T}}\log {}_\delta f\,d\mu\right|$$
$$+\limsup_{\delta\to 0+}\left|\int_{\mathbb{T}}\log {}_\delta f\,d\mu-\int_{\mathbb{T}}\log f\,d\mu\right|=0. \quad\square$$

6. The Matrix Case

In this section we introduce the computational procedures $\mathfrak{C}_n:\mathcal{P}_N(m\times m)\to\mathcal{P}_N^+(m\times m)$, $n=1,2,\ldots$, and prove Theorem 1. First, we need two auxiliary lemmas.

Lemma 4. *Let*

$$f(z)=\frac{p(z)}{q(z)}\in\mathbb{Q}[z]\cap L^\infty(\mathbb{T}) \tag{52}$$

be a rational function without poles in $\overline{\mathbb{D}}$, *satisfying*

$$|f(z)|<C \ \ \textit{for}\ \ z\in\mathbb{T}, \tag{53}$$

and let

$$p_n\to p \ \ \textit{and}\ \ q_n\to q, \tag{54}$$

where $\deg(p_n)=\deg(p)$ *and* $\deg(q_n)=\deg(q)$, $n=1,2,\ldots.$

Let $\omega_{k,n}=\exp(\frac{2\pi k}{n}i)$, $n=1,2,\ldots$, $k=0,1,\ldots,n-1$, *be the Discrete Fourier Transform nodes. Then*

$$V_n:=\frac{2\pi}{n}\sum_{k=0}^{n-1}|h_n(\omega_{k,n})|^2\to\|f\|_2^2 \ \ \textit{as}\ \ n\to 0, \tag{55}$$

where

$$h_n(\omega)=\begin{cases}p_n(\omega)/q_n(\omega) & \textit{if}\ \ |p_n(\omega)/q_n(\omega)|\le C,\\ 0 & \textit{if}\ \ |p_n(\omega)/q_n(\omega)|>C.\end{cases}$$

Proof. By virtue of (54), for each $R<\infty$, the set of polynomials p_n, $n=1,2,\ldots$, is uniformly bounded on $\overline{\mathbb{D}(0,R)}=\{z\in\mathbb{C}:\ |z|\le R\}$, i.e.,

$$\sup_n\ \sup_{z\in\overline{\mathbb{D}(0,R)}}|p_n(z)|<\infty.$$

Let $q(z)=b\prod_{k=1}^{N}(z-z_k)$, $b\neq 0$. If $q_n(z)=b_n\prod_{k=1}^N(z-z_{k,n})$, $n=1,2,\ldots$, and we label the zeroes of q_n accordingly, then we get

$$b_n\to b \ \text{ and } \ z_{k,n}\to z_k,\ \ k=1,2,\ldots,N, \ \text{ as } \ n\to\infty.$$

Since the zeros of q_n are concentrated around the points z_k, $k = 1, 2, \dots, N$, for each $\varepsilon > 0$, there exist $\delta > 0$ and $n_0 \in \mathbb{N}$ such that

$$\mu\Big\{\mathbb{T} \setminus \bigcup_{k=1}^{N} \overline{D(z_k, \delta)}\Big\} < \varepsilon \tag{56}$$

and the functions

$$f_n := p_n/q_n, \quad n \geq n_0, \tag{57}$$

are uniformly bounded in

$$D_\varepsilon := D(0, 1+\delta) \setminus \bigcup_{k=1}^{N} \overline{D(z_k, \delta)}. \tag{58}$$

Consequently, the set of functions (57) is a normal family and converges uniformly on every compact in (58) which implies, by virtue of (53), that there exists $n_1 \geq n_0$ such that

$$|f_n(z)| < C \quad \text{for} \quad n \geq n_1 \quad \text{and} \quad z \in \mathbb{T} \cap D_\varepsilon.$$

Consequently, $h_n = f_n$ on $\mathbb{T} \cap D_\varepsilon$ for $n \geq n_1$ and h_n converges uniformly to f in $\mathbb{T} \cap D_\varepsilon$.

Since the derivatives of a normal family of functions form a normal family as well, we have that h_n together with h_n' converge uniformly on $\mathbb{T} \cap D_\varepsilon$. Consequently,

$$\frac{2\pi}{n} \sum_{\{k\colon\, \omega_{kn} \in D_\varepsilon\}} |h_n(\omega_{k,n})|^2 \to \int_{\mathbb{T} \cap D_\varepsilon} |f|^2 \, d\mu$$

as $n \to \infty$, while

$$\Bigg| \int_{\mathbb{T} \cap D_\varepsilon} |f|^2 \, d\mu - \int_{\mathbb{T}} |f|^2 \, d\mu \Bigg| \leq \sup_{\mu(E) < \varepsilon} \int_E |f|^2 \, d\mu \to 0 \quad \text{as} \quad \varepsilon \to 0$$

and

$$\limsup_{n\to\infty} \Bigg| \frac{2\pi}{n} \sum_{\{k\colon\, \omega_{kn} \in D_\varepsilon\}} |h_n(\omega_{k,n})|^2 - \frac{2\pi}{n} \sum_{k=0}^{n-1} |h_n(\omega_{k,n})|^2 \Bigg|$$

$$= \limsup_{n\to\infty} \frac{2\pi}{n} \sum_{\{k\colon\, \omega_{kn} \notin D_\varepsilon\}} |h_n(\omega_{k,n})|^2 \leq C\mu\{\mathbb{T} \setminus D_\varepsilon\} \leq C\varepsilon \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Hence (55) holds. □

Lemma 5. *Let f, p_n, q_n and V_n be the same as in Lemma 4. Assume that $q_n(0) \neq 0$, $n = 1, 2, \dots$, and let*

$$\sum_{k=0}^{\infty} \alpha_k z^k \sim \frac{p_n(z)}{q_n(z)}$$

be the Tailor expansion of p_n/q_n *in the neighborhood of zero. Define* $\mathcal{L}_n[p_n,q_n]\in\mathcal{P}_n^+$ *by*

$$\mathcal{L}_n[p_n,q_n](z):=\begin{cases}\sum_{k=0}^{l}\alpha_k z^k & \text{if } \sum_{k=0}^{l}|\alpha_k|^2\le V_n<\sum_{k=0}^{l+1}|\alpha_k|^2 \text{ and } l<n,\\ \sum_{k=0}^{n}\alpha_k z^k & \text{if } \sum_{k=0}^{n}|\alpha_k|^2\le V_n.\end{cases} \tag{59}$$

Then

$$\big\|\mathcal{L}_n[p_n,q_n]-f\big\|_{L^2(\mathbb{T})}\to 0 \text{ as } n\to\infty. \tag{60}$$

Proof. The Tailor coefficients of $f=p/q$ can be expressed recurrently in terms of coefficients of p and q. Thus, because of (54), we have

$$\text{for each } k\ge 0,\quad c_k\big\{\mathcal{L}_n[p_n,q_n]\big\}\to c_k\{f\} \text{ as } n\to\infty. \tag{61}$$

By virtue of Lemma 4 and definition (59), taking into account (61), we also have

$$\big\|\mathcal{L}_n[p_n,q_n]\big\|_{L^2(\mathbb{T})}\to\|f\|_{L^2(\mathbb{T})} \text{ as } n\to\infty.$$

The convergence in (60) now follows from the general fact that in a Hilbert space the weak convergence, when combined with the convergence of norms, implies strong convergence. □

We are ready now to introduce the computational procedure $\mathfrak{C}=\mathfrak{C}_n$ described in the introduction, which can be applied to any $S_n\in\mathcal{P}^{r\times r}$, such that Theorem 1 holds.

Note that if S is a polynomial matrix function (5), then for each m, $1<m\le r$, the first $m-1$ entries $\zeta_1,\zeta_2,\dots,\zeta_{m-1}$ of the m-th row of $M_S\mathbf{U}_2\mathbf{U}_3\dots\mathbf{U}_{m-1}$ in (17) are rational functions, since they can be determined by Cramer's rule from equation (19) as

$$\zeta_i(t)=\overline{p_i(t)/q(t)}=t^N\overline{(t^Np_i(t)/q(t))}, \tag{62}$$

where $q=\det S^+_{[m-1]}\in\mathcal{P}^+_{N(m-1)}$ (it is free of zeros in $\mathbb{D}$) and p_i is the determinant of the matrix $S^+_{[m-1]}$, the i-th column of which is replaced by $[s_{2m},\dots,s_{m-1,m}]^T$, implying $z^Np_i\in\mathcal{P}^+_{Nm}$.

We compute the diagonal entries $\widehat{f}^+_{1,n},\widehat{f}^+_{2,n},\dots,\widehat{f}^+_{r,n}$ of the "triangular factor" of S_n by the formulas: $\widehat{f}^+_{1,n}=\widetilde{(S_n)_{[1]}}^{\,+}$ (see Section 2 for notation $S_{[m]}$ and definitions (35)–(37)) and

$$\widehat{f}^+_{m,n}=\mathcal{L}_n\Big[\det\big(\widetilde{(S_n)_{[m]}}\big)^+,\det\big(\widetilde{(S_n)_{[m-1]}}\big)^+\Big] \tag{63}$$

(see Lemma 5 for definitions). Set

$$(\widehat{S}_n)^+_{[1]}=\widehat{f}^+_{1,n}=\widetilde{(S_n)_{[1]}}^{\,+} \tag{64}$$

and for each $m = 2, 3, \dots, r$ we recurrently construct

$$(\widehat{S}_n)^+_{[m]}(t) = \sum_{k=0}^{N} \widehat{A}_{k,n} t^k, \quad \widehat{A}_{k,n} \in \mathbb{C}^{m\times m},$$

an approximate "spectral factor" of $(S_n)_{[m]}$, making an assumption that $(\widehat{S}_n)^+_{[m-1]}$ has already been constructed and performing the following operations. Let

$$\widehat{\zeta}_{i,n} = t^N \overline{\mathcal{L}_n[t^N \widehat{p}_{i,n}, \widehat{q}_n(t)]}\,, \quad i = 1, 2, \dots, m-1, \tag{65}$$

where $\widehat{p}_{i,n}$ and $\widehat{q}_i$ are defined similar to (62), namely, $\widehat{q}_n = \det\big((\widehat{S}_n)^+_{[m-1]}\big)$ and $\widehat{p}_{i,n}$ is the determinant of the matrix $(\widehat{S}_n)^+_{[m-1]}$ with its i-th column replaced by $[\widehat{s}_{2m}, \dots, \widehat{s}_{m-1,m}]^T$. For the matrix $\widehat{F}_{n,m}$ of the form (18) with the last row

$$(\widehat{\zeta}_{1,n}, \widehat{\zeta}_{2,n}, \dots, \widehat{\zeta}_{m-1,n}, \widehat{f}^+_{m,n}), \tag{66}$$

using Theorem 3, we construct the unitary matrix function $U_{m,n}$ such that $\widehat{F}_{n,m} U_{m,n} \in (\mathcal{P}^+)^{m\times m}$. By virtue of formula (21), the matrix function

$$\widehat{S} \cdot U := \begin{pmatrix} & & & & 0 \\ & & \det\big((\widehat{S}_n)^+_{[m-1]}\big) & & \vdots \\ & & & & 0 \\ \widehat{\zeta}_{1,n} & \widehat{\zeta}_{2,n} & \cdots & \widehat{\zeta}_{m-1,n} & \widehat{f}^+_{m,n} \end{pmatrix} U_{m,n} \tag{67}$$

is a candidate for $(\widehat{S}_n)^+_{[m]}$. Since we know that $S^+_{[m]} \in (\mathcal{P}^+_N)^{m\times m}$, we discard coefficients of the entries in (67) with indices outside the range $[0, N]$ and let

$$(\widehat{S}_n)^+_{[m]}(z) := \sum_{k=0}^{N} C_k\{\widehat{S} \cdot U\} z^k, \quad m = 2, 3, \dots, r. \tag{68}$$

We define

$$\mathfrak{C}_n(S_n) = (\widehat{S}_n)^+_{[r]}. \tag{69}$$

Let us prove now the convergence (7).

Consider the equation (63). Since, because of (6), $\det\big((S_n)_{[m]}\big) \to \det S_{[m]}$ as $n \to \infty$, we have $\det\widetilde{((S_n)_{[m]})}^+ \to \det S^+_{[m]}$, $m = 1, 2, \dots, r$, by virtue of Lemma 2 (in particular,

$$\widetilde{(S_n)_{[1]}}^+ = (\widehat{S}_n)^+_{[1]} \to S^+_{[1]}, \tag{70}$$

see (64)), while the limiting functions $\det S^+_{[m]}$ are free of zeros in $\mathbb{D}$ and $f^+_m = \det S^+_{[m]} / \det S^+_{[m-1]} \in L_2(\mathbb{T})$ (see (25)) do not have poles on $\mathbb{T}$. Consequently, the hypotheses of Lemma 5 hold and therefore

$$\widehat{f}^+_{m,n} \to f^+_m \text{ in } L^2 \ \text{ as } \ n \to \infty, \quad m = 2, 3, \dots, r. \tag{71}$$

Since (70) holds, we assume invoking induction that

$$(\widehat{S}_n)^+_{[m-1]} \to S^+_{[m-1]} \text{ in } L^2 \text{ as } n \to \infty, \tag{72}$$

and prove (72) for $m-1$ replaced by m.

Consider now the equation (65). The sequences of polynomials $p_{i,n}$ and q_n also satisfy the hypothesis of Lemma 5 and therefore

$$\widehat{\zeta}_{i,n} \to \zeta_i \text{ in } L^2 \text{ as } n \to \infty. \tag{73}$$

Thus, taking into account the relation (71) also, we have that the sequence of matrix functions $\widehat{F}_{n,m}$ of the form (18), (66) converges in L^2. Consequently, we can apply Theorem 5 to conclude that the sequence of unitary matrix functions $U_{n,m}$ in the equation (67) is convergent in measure which, by virtue of (10), implies that the product in (67) and, consequently, (68) are convergent. Namely,

$$(\widehat{S}_n)^+_{[m]} \to S^+_{[m]} \text{ in } L^2 \text{ as } n \to \infty. \tag{74}$$

We get (7) if we substitute $m = r$ in (74), and thus the proof of Theorem 1 is completed.

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