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ON THE STABILITY OF INVARIANT TORI OF A CLASS OF DYNAMICAL SYSTEMS WITH THE LAPPO–DANILEVSKII CONDITION

Abstract. The sufficient conditions for the existence of an asymptotically stable invariant toroidal manifolds of linear extensions of dynamical system on torus are obtained in the case where the matrix of the system commutes with its integral. New theorem requires the conditions to hold only in a nonwandering set of the corresponding dynamical system in order to guarantee the existence and stability of the invariant manifold. Additionally, the proposed approach is applied to the investigation of invariant sets of a certain class of discontinuous dynamical systems.

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რეზიუმე. დადგენილია დინამიკური სისტემის წრფივი გაფართოების ასიმპტოტურად მდგრადი ინვარიანტული ტოროიდული მრავალსახეობების არსებობის საკმარისი პირობები ტორზე იმ შემთხვევაში, როცა სისტემის მატრიცა მისი ინტეგრალის კომუტატიურია. ინვარიანტული მრავალსახეობის არსებობისა და მდგრადობისთვის ახალ თეორემაში მოითხოვება გარკვეული პირობების შესრულება შესაბამისი დინამიკური სისტემის მხოლოდ არამოხეტიალე სიმრავლის შემთხვევაში. შემოთავაზებული მიდგომა გამოყენებულია წყვეტილი დინამიკური სისტემების გარკვეული კლასის ინვარიანტული სიმრავლეების გამოკვლევისთვის.

1 Introduction and preliminaries

One of the important questions within mathematical theory of multi-frequency oscillations is the problem of the existence and stability of invariant toroidal manifolds of the systems of differential equations that are defined in the direct product of a torus and Euclidean space. Such manifolds serve as carriers of multi-frequency oscillations in the system. The basics of this theory are systematically developed in [8, 13].

In this paper, we establish new sufficient conditions for the existence of an asymptotically stable invariant torus of a particular class of dynamical systems subjected to Lappo–Danilevskii condition [1, Chap. II, § 13]. We propose an approach that relaxes conventional constraints and requires the conditions to hold only in nonwandering set of the corresponding dynamical system in order to guarantee the existence and stability of invariant manifold. This extends the result in [3], where the analogous conditions are being imposed on the whole surface of the torus. In the last section, we extend this approach to a certain class of discontinuous dynamical system [4] defined in the direct product of a torus and Euclidean space.

We consider the following system of ordinary differential equations defined in the direct product of a torus $\mathcal{T}_m$ and Euclidean space $\mathbb{R}^n$

$$\frac{d\varphi}{dt} = a(\varphi)\,, \quad \frac{dx}{dt} = P(\varphi)x + f(\varphi), \tag{1.1}$$

where $\varphi = (\varphi_1, \dots, \varphi_m)^T \in \mathcal{T}_m$, $x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$, $P(\varphi), f(\varphi) \in C(\mathcal{T}_m)$; $C(\mathcal{T}_m)$ stands for a space of continuous 2π-periodic with respect to each of the component φ_v, $v = 1, \dots, m$, functions defined on the m-dimensional torus $\mathcal{T}_m$. The function $a(\varphi) \in C(\mathcal{T}_m)$ and satisfies the Lipschitz condition

$$\|a(\varphi'') - a(\varphi')\| \le L\|\varphi'' - \varphi'\| \tag{1.2}$$

for any $\varphi', \varphi'' \in \mathcal{T}_m$ and some positive constant $L > 0$.

Condition (1.2) guarantees that the system

$$\frac{d\varphi}{dt} = a(\varphi) \tag{1.3}$$

generates a dynamical system on the torus $\mathcal{T}_m$, which we will denote as $\varphi_t(\varphi)$.

Along with system (1.1), we consider a linear system of equations

$$\frac{dx}{dt} = P(\varphi_t(\varphi))x + f(\varphi_t(\varphi)), \tag{1.4}$$

that depends on $\varphi \in \mathcal{T}_m$ as a parameter.

Definition 1.1 ([13])**.** A manifold $\mathcal{M}$ is called an invariant manifold of system (1.1) if $\mathcal{M}$ is defined by $x = u(\varphi)$, $\varphi \in \mathcal{T}_m$, with the function $u(\varphi) \in C(\mathcal{T}_m)$ such that $x(t, \varphi) = u(\varphi_t(\varphi))$ is a solution to (1.4) for any $\varphi \in \mathcal{T}_m$.

The main approach to investigate the properties of invariant toroidal manifolds of system (1.1) is based on the notion of a Green–Samoilenko function [13]. Consider the homogeneous system of differential equations

$$\frac{dx}{dt} = P(\varphi_t(\varphi))x, \tag{1.5}$$

that depends on $\varphi \in \mathcal{T}_m$ as a parameter and denote by $\Omega_\tau^t(\varphi)$ the fundamental matrix of (1.5) satisfying $\Omega_\tau^\tau(\varphi) \equiv E$.

Let $C(\varphi)$ be a matrix from the space $C(\mathcal{T}_m)$. Denote

$$G_0(\tau, \varphi) = \begin{cases} \Omega_\tau^0(\varphi) C(\varphi_\tau(\varphi)), & \tau \le 0, \\ -\Omega_\tau^0(\varphi)(E - C(\varphi_\tau(\varphi))), & \tau > 0. \end{cases}$$

Definition 1.2 ([13])**.** The function $G_0(\tau,\varphi)$ is called a Green–Samoilenko function of the system

$$\frac{d\varphi}{dt} = a(\varphi), \quad \frac{dx}{dt} = P(\varphi)x,$$

if $\int_{-\infty}^{+\infty} \|G_0(\tau,\varphi)\|\, d\tau$ is bounded uniformly with respect to φ,

$$\sup_{\varphi\in\mathcal{T}_m} \int_{-\infty}^{+\infty} \|G_0(\tau,\varphi)\|\, d\tau < \infty.$$

The existence of the Green–Samoilenko function guarantees the existence of an invariant toroidal manifold of system (1.1) for any inhomogeneity $f(\varphi) \in C(\mathcal{T}_m)$ and can be presented as [13]

$$x = u(\varphi) = \int_{-\infty}^{+\infty} G_0(\tau,\varphi) f(\varphi_\tau(\varphi))\, d\tau, \quad \varphi \in \mathcal{T}_m.$$

2 Main results

Consider system (1.1) for the case when the matrix $P(\varphi_t(\varphi))$ commutes with its integral (the so-called Lappo–Danilevskii case [1, Chap. II, § 13]): for any $t \geq \tau$,

$$P(\varphi_t(\varphi)) \int_{\tau}^{t} P(\varphi_{t_1}(\varphi))\, dt_1 = \int_{\tau}^{t} P(\varphi_{t_1}(\varphi))\, dt_1 \cdot P(\varphi_t(\varphi)). \tag{2.1}$$

Then the equality

$$\Omega_\tau^t(\varphi) = e^{\int_\tau^t P(\varphi_{t_1}(\varphi))\, dt_1}$$

is actually the fundamental matrix of the homogeneous system (1.5) that depends on $\varphi \in \mathcal{T}_m$ as a parameter. Really, taking into account that

$$\frac{d}{dt} \int_{\tau}^{t} P(\varphi_{t_1}(\varphi))\, dt_1 = P(\varphi_t(\varphi)),$$

we have

$$\frac{d}{dt}\, \Omega_\tau^t(\varphi) = e^{\int_\tau^t P(\varphi_{t_1}(\varphi))\, dt_1} P(\varphi_t(\varphi)) = P(\varphi_t(\varphi)) e^{\int_\tau^t P(\varphi_{t_1}(\varphi))\, dt_1} = P(\varphi_t(\varphi))\Omega_\tau^t(\varphi).$$

Additionally, $\Omega_\tau^\tau(\varphi) = E$.

Note also [2] that a (2×2)-matrix of the form

$$P(\varphi_t(\varphi)) = \begin{bmatrix} p(\varphi_t(\varphi)) & q(\varphi_t(\varphi)) \\ q(\varphi_t(\varphi)) & p(\varphi_t(\varphi)) \end{bmatrix}$$

satisfies the Lappo–Danilevskii condition (2.1).

Definition 2.1 ([9])**.** A point $\varphi \in \mathcal{T}_m$ is called a wandering point of the dynamical system (1.3) if there exist a neighbourhood $U(\varphi)$ and a time $T = T(\varphi) > 0$ such that

$$U(\varphi) \cap \varphi_t(U(\varphi)) = \varnothing \quad \forall\, t \geq T.$$

Let W be a set of all wandering points of the dynamical system (1.3) and let $M = \mathcal{T}_m \setminus W$ be a set of all nonwandering points. Since $\mathcal{T}_m$ is a compact set, it is known [9] that $M \neq \varnothing$ is invariant and compact subset of $\mathcal{T}_m$. Moreover, the following theorem holds.

Theorem 2.1 ([9])**.** *For any $\varepsilon > 0$, there exists $T(\varepsilon) > 0$ such that for any $\varphi \notin M$, the corresponding trajectory $\varphi_t(\varphi)$ remains outside the ε-neighbourhood of nonwandering set M for a time, not exceeding $T(\varepsilon)$.*

Now we are in position to state the main result of the paper.

Theorem 2.2. *Let the Lappo–Danilevskii condition* (2.1) *hold and uniformly with respect to $\varphi \in \mathcal{T}_m$ there exist*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t P(\varphi_s(\varphi))\, ds := A(\varphi). \tag{2.2}$$

If for every $\varphi \in M$

$$\operatorname{Re} \lambda(A(\varphi)) < 0 \tag{2.3}$$

for all eigenvalues $\lambda(A(\varphi))$ of the matrix $A(\varphi)$, then system (1.1) *has an asymptotically stable invariant toroidal manifold for any $f(\varphi) \in C(\mathcal{T}_m)$.*

Proof. From condition (2.1) it follows that for $t \geq s \geq 0$,

$$P(\varphi_t(\varphi)) \int_s^t P(\varphi_{t_1}(\varphi))\, dt_1 = \int_s^t P(\varphi_{t_1}(\varphi))\, dt_1 \cdot P(\varphi_t(\varphi)). \tag{2.4}$$

After differentiating equality (2.4) by s, we get $P(\varphi_t(\varphi))P(\varphi_s(\varphi)) = P(\varphi_s(\varphi))P(\varphi_t(\varphi))$. Hence,

$$\int_\tau^t P(\varphi_{t_1}(\varphi))\, dt_1 \cdot \frac{1}{s} \int_\tau^s P(\varphi_{t_2}(\varphi))\, dt_2 = \frac{1}{s} \int_\tau^t \int_\tau^s P(\varphi_{t_1}(\varphi))P(\varphi_{t_2}(\varphi))\, dt_2\, dt_1$$
$$= \frac{1}{s} \int_\tau^t \int_\tau^s P(\varphi_{t_2}(\varphi))P(\varphi_{t_1}(\varphi))\, dt_2\, dt_1 = \frac{1}{s} \int_\tau^s P(\varphi_{t_2}(\varphi))\, dt_2 \cdot \int_\tau^t P(\varphi_{t_1}(\varphi))\, dt_1.$$

Taking the limit $s \to \infty$ in the last equality, we get

$$\int_\tau^t P(\varphi_{t_1}(\varphi))\, dt_1 \cdot A(\varphi) = A(\varphi) \cdot \int_\tau^t P(\varphi_{t_1}(\varphi))\, dt_1. \tag{2.5}$$

It means that the limit matrix $A(\varphi)$ commutes with its integral $\int_\tau^t P(\varphi_{t_1}(\varphi))\, dt_1$.

Due to (2.2), we may introduce the matrix B such that

$$\frac{1}{t} \int_0^t P(\varphi_{t_1}(\varphi))\, dt_1 = A(\varphi) + B(t, \varphi),$$

where

$$\sup_{\varphi \in \mathcal{T}_m} \|B(t, \varphi)\| \longrightarrow 0, \quad t \to \infty.$$

The next step of the proof is to prove that the matrices $A(\varphi)$ and $B(t, \varphi)$ commute. Indeed, from (2.5) we get

$$A(\varphi) \cdot B(t, \varphi) = A(\varphi) \cdot \left[\frac{1}{t} \int_\tau^t P(\varphi_{t_1}(\varphi))\, dt_1 - A(\varphi) \right]$$
$$= \frac{1}{t} \int_\tau^t P(\varphi_{t_1}(\varphi))\, dt_1 \cdot A(\varphi) - A^2(\varphi) = B(t, \varphi)A(\varphi).$$

Then from the equality

$$\int_0^t P(\varphi_s(\varphi))\, ds = A(\varphi)\cdot t + B(t,\varphi)\cdot t$$

we derive that the fundamental matrix of the homogeneous system (1.5) has a representation

$$\Omega_0^t(\varphi) = e^{\int_0^t P(\varphi_{t_1}(\varphi))\, dt_1} = e^{tA(\varphi)+tB(t,\varphi)} = e^{tA(\varphi)}\cdot e^{tB(t,\varphi)}. \tag{2.6}$$

The aim of the further steps of the proof is to prove that condition (2.3) guarantees the following estimate

$$\|\Omega_0^t(\varphi)\| \le Ke^{-\eta t} \quad \forall\, t\ge 0, \tag{2.7}$$

for any $\varphi\in\mathcal{T}_m$ and for some positive constants $K,\eta>0$ which do not depend on $\varphi\in\mathcal{T}_m$.

Due to the uniformity of the limit in (2.2), we find that

$$\text{map}\,\varphi\longmapsto A(\varphi)\ \text{ is continuous on } \mathcal{T}_m.$$

Then, from [5], the eigenvalues of $A(\varphi)$ depend continuously on φ. Hence, from (2.3), it follows that there exist $\gamma>0$ and $\varepsilon\in(0,\gamma)$ such that

$$\forall\,\varphi\in\overline{O_\varepsilon(M)},\ \ \operatorname{Re}\lambda(A(\varphi)) < -2\gamma,$$

where $O_\varepsilon(M)$ is an ε-neighbourhood of M.

By a picked $\varepsilon>0$, we choose $T_1=T_1(\varepsilon)$ such that

$$\sup_{\varphi\in\mathcal{T}_m}\|B(t,\varphi\| < \varepsilon \quad \forall\, t\ge T_1. \tag{2.8}$$

Next we prove that there exists $K_1>0$ such that for any $\varphi\in\overline{O_\varepsilon(M)}$ and for any $t\ge 0$ the inequality

$$\|e^{A(\varphi)t}\| \le K_1\cdot e^{-\gamma t} \tag{2.9}$$

holds.

Choose some $\varphi_0\in\overline{O_\varepsilon(M)}$. Due to the properties of the exponent, there exists $C(\varphi_0)>0$ such that for any $t\ge 0$,

$$\|e^{A(\varphi_0)t}\| \le C(\varphi_0)e^{-\frac{3\gamma}{2}\,t}. \tag{2.10}$$

Due to the continuity of $A(\varphi)$, there exists $\delta=\delta(\varphi_0)>0$ such that for any $\varphi\in O_\delta(\varphi_0)$,

$$\|A(\varphi)-A(\varphi_0)\| < \frac{\gamma}{2C(\varphi_0)}\,. \tag{2.11}$$

The matrix $X(t)=e^{A(\varphi)t}$ is a solution to the Cauchy problem

$$\begin{cases} \dot X = A(\varphi_0)X + (A(\varphi)-A(\varphi_0))X,\\ X(0)=E. \end{cases}$$

Using the variation of the constant method, we obtain

$$X(t) = e^{A(\varphi_0)t} + \int_0^t e^{(t-s)A(\varphi_0)}\cdot(A(\varphi)-A(\varphi_0))X(s)\, ds.$$

Then from (2.10), (2.11) we get

$$\|X(t)\| \le C(\varphi_0)e^{-\frac{3\gamma}{2}\,t} + \int_0^t e^{-\frac{3\gamma}{2}\,(t-s)}\cdot\frac{\gamma}{2}\cdot\|X(s)\|\, ds,$$

$$\|X(t)\|\cdot e^{\frac{3\gamma}{2}\,t} \le C(\varphi_0) + \int_0^t \frac{\gamma}{2}\cdot e^{\frac{3\gamma}{2}\,s}\|X(s)\|\, ds.$$

Applying the Gronwall inequality to the last inequality, we finally get

$$\forall\, t \geq 0,\ \ \forall\, \varphi \in O_\delta(\varphi_0) \quad \|e^{A(\varphi)t}\| \leq C(\varphi_0)e^{-\gamma t}.$$

From a cover $\{O_{\delta(\varphi_0)}(\varphi_0)\}_{\varphi_0 \in \overline{O_\varepsilon(M)}}$ of the compact set $\overline{O_\varepsilon(M)}$ let us pick a finite subcover $\{O_{\delta(\varphi_i)}(\varphi_i)\}_{i=1}^N$. Letting $K_1 = \max\limits_{1\leq i\leq N} C(\varphi_i)$, we get (2.9).

Finally, for $\varphi \in \overline{O_\varepsilon(M)}$, due to equality (2.6) and inequalities (2.8), (2.9), we get: for all $t \geq T_1$,

$$\left\|e^{\int_0^t P(\varphi_s(\varphi))\,ds}\right\| = \|e^{A(\varphi)t+B(t,\varphi)t}\| \leq K_1 e^{(-\gamma+\varepsilon)t}. \tag{2.12}$$

In the case for $\varphi \notin O_\varepsilon(M)$, we use Theorem 2.1, which says that there exists $\tau = \tau(\varphi,\varepsilon) < T(\varepsilon)$ such that $\varphi_\tau(\varphi) \in O_\varepsilon(M)$. Hence, for $t > T(\varepsilon) + T_1$,

$$\begin{aligned}\left\|e^{\int_0^t P(\varphi_s(\varphi))\,ds}\right\| &= \left\|e^{\int_0^\tau P(\varphi_s(\varphi))\,ds} \cdot e^{\int_\tau^t P(\varphi_s(\varphi))\,ds}\right\| \leq e^{d\cdot T(\varepsilon)} \cdot \left\|e^{\int_0^{t-\tau} P(\varphi_s(\varphi_\tau(\varphi)))\,ds}\right\| \\ &\leq e^{d\cdot T(\varepsilon)} K_1 e^{(-\gamma+\varepsilon)(t-\tau)} \leq e^{(d+\gamma)\cdot T(\varepsilon)} K_1 e^{(-\gamma+\varepsilon)t},\end{aligned} \tag{2.13}$$

where $d = \max\limits_{\varphi \in T_m} \|P(\varphi)\|$.

From estimates (2.12), (2.13) we derive the desired inequality (2.7).

From (2.7) it directly follows that the function $G_0(\tau,\varphi) = \begin{cases} \Omega_\tau^0(\varphi), & \tau \leq 0, \\ 0, & \tau > 0 \end{cases}$ satisfies the estimate $\|G_0(\tau,\varphi)\| \leq Ke^{-\eta|\tau|}$, $\tau \in \mathbb{R}$, and it is a Green–Samoilenko function of the invariant tori problem. Moreover, estimate (2.7) is sufficient for the existence of an asymptotically stable invariant toroidal manifold of system (1.1) of the form

$$x = u(\varphi) = \int_{-\infty}^{0} \Omega_\tau^0(\varphi) f(\varphi_\tau(\varphi))\,d\tau, \quad \varphi \in \mathcal{T}_m.$$

This completes the proof. □

Remark 2.1. From the proof of Theorem 2.2 it follows that the constant $\eta > 0$ in (2.7) can be chosen as

$$\eta = -\max_{\varphi \in M} \max_{j=1,\dots,n} \operatorname{Re} \lambda_j(A(\varphi)) - \varepsilon,$$

where $\varepsilon > 0$ is arbitrarily small.

Remark 2.2. Since $\forall\, t \geq \tau$, $\forall\, \theta \in \mathbb{R}$ $\Omega_\tau^t(\varphi_\theta(\varphi)) = \Omega_{\tau+\theta}^{t+\theta}(\varphi)$, from (2.7) it follows that

$$\forall\, t \geq \tau,\ \ \forall\, \varphi \in \mathcal{T}_m \quad \|\, \Omega_\tau^t(\varphi)\, \| \leq Ke^{-\eta(t-\tau)}.$$

Example 2.1. Consider the following system:

$$\begin{aligned} \frac{d\varphi}{dt} &= -\sin^2\left(\frac{\varphi}{2}\right), \\ \begin{pmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \end{pmatrix} &= \begin{pmatrix} -\cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix} x + \begin{pmatrix} f_1(\varphi) \\ f_2(\varphi) \end{pmatrix}, \end{aligned} \tag{2.14}$$

where $\varphi \in \mathcal{T}_1$, $x = (x_1,x_2) \in \mathbb{R}^2$, $f(\varphi) = (f_1(\varphi), f_2(\varphi)) \in C(\mathcal{T}_1)$.

Note that the symmetric matrix $P(\varphi) = \begin{pmatrix} -\cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix}$ satisfies the Lappo–Danilevskii condition (2.1).

For the dynamical system $\frac{d\varphi}{dt} = -\sin^2(\frac{\varphi}{2})$ on the torus $\mathcal{T}_1$, the set M of nonwandering points consists of a single point $\varphi = 0$. The point $\varphi = 0$ is a fixed point, and all other trajectories tend to 0 as $t \to +\infty$. Hence, uniformly with respect to $\varphi \in \mathcal{T}_1$,

$$\lim_{t\to\infty} P(\varphi_t(\varphi)) = \lim_{t\to\infty} \begin{pmatrix} -\cos(\varphi_t(\varphi)) & \sin(\varphi_t(\varphi)) \\ \sin(\varphi_t(\varphi)) & -\cos(\varphi_t(\varphi)) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$A = \lim_{t\to\infty} \frac{1}{t} \int_{\tau}^{t} P(\varphi_{t_1}(\varphi))\, dt_1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Since

$$\operatorname{Re} \lambda_j A = \operatorname{Re} \lambda_j \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -1 < 0, \quad j = 1, 2,$$

system (2.14) satisfies the conditions of Theorem 2.2 and has an asymptotically stable invariant toroidal manifold.

3 Application to discontinuous dynamical systems

Let us apply the proposed approach to a certain class of discontinuous dynamical systems [6, 7, 12, 14]

$$\begin{aligned} \frac{d\varphi}{dt} &= a(\varphi), \quad \varphi \in \mathcal{T}_m, \\ \frac{dx}{dt} &= P(\varphi)x + f(\varphi), \quad \varphi \notin \Gamma, \\ \Delta x\big|_{\varphi\in\Gamma} &= I(\varphi)x + g(\varphi), \end{aligned} \tag{3.1}$$

where $\Gamma \subset \mathcal{T}_m$, $a(\varphi) \in C(\mathcal{T}_m)$ satisfies (1.2), $P(\varphi), f(\varphi) \in C(\mathcal{T}_m)$, $I(\varphi), g(\varphi) \in C(\Gamma)$.

We assume that the set Γ is a smooth submanifold of a torus $\mathcal{T}_m$ of dimension $m-1$ and is defined by the equation $\Phi(\varphi) = 0$, where $\Phi(\varphi)$ is a continuous scalar 2π-periodic with respect to each of the components φ_v, $v = 1, \dots, m$, function.

Denote by $t_i(\varphi)$, $i \in \mathbb{Z}$, the solutions of the equation $\Phi(\varphi_t(\varphi)) = 0$, which are the moments of impulsive perturbations in system (3.1). We assume that for every $\varphi \in \mathcal{T}_m$ the corresponding solutions $t = t_i(\varphi)$ exist, $\lim\limits_{i\to\pm\infty} t_i(\varphi) = \pm\infty$, and uniformly with respect to $t \in \mathbb{R}$ and $\varphi \in \mathcal{T}_m$,

$$\lim_{T\to\pm\infty} \frac{i(t, t+T)}{T} = p < \infty, \tag{3.2}$$

where $i(a, b)$ is the number of points $t_i(\varphi)$ in the interval (a, b).

Along with system (3.1), we consider a linear system

$$\begin{aligned} \frac{dx}{dt} &= P(\varphi_t(\varphi))x + f(\varphi_t(\varphi)), \quad t \neq t_i(\varphi), \\ \Delta x\big|_{t=t_i(\varphi)} &= I(\varphi_{t_i(\varphi)}(\varphi))x + g(\varphi_{t_i(\varphi)}(\varphi)), \end{aligned} \tag{3.3}$$

that depends on $\varphi \in \mathcal{T}_m$ as a parameter.

Let $C_\Gamma(\mathcal{T}_m)$ be a space of piecewise continuous 2π-periodic with respect to each of the components φ_v, $v = 1, \dots, m$, functions that are defined on the m-dimensional torus $\mathcal{T}_m$.

Definition 3.1. A set $\mathcal{M}$ is called an invariant set of system (3.1) if $\mathcal{M}$ is defined by $x = u(\varphi)$, $\varphi \in \mathcal{T}_m$, where a piecewise continuous function $u(\varphi) \in C_\Gamma(\mathcal{T}_m)$ is such that $x(t, \varphi) = u(\varphi_t(\varphi))$ is a solution to (3.3) for any $\varphi \in \mathcal{T}_m$.

The problems of the existence and stability of invariant toroidal sets of (3.1) have been studied in [10, 11]. Consider the homogeneous system of equations

$$\begin{aligned} \frac{dx}{dt} &= P(\varphi_t(\varphi))x, \quad t \neq t_i(\varphi), \\ \Delta x\big|_{t=t_i(\varphi)} &= I(\varphi_{t_i(\varphi)}(\varphi))x. \end{aligned} \tag{3.4}$$

Let $X_\tau^t(\varphi)$ be a fundamental matrix of (3.4) with $X_\tau^\tau(\varphi) \equiv E$.

Let $C(\varphi)$ be some matrix from the space $C_\Gamma(\mathcal{T}_m)$. Denote

$$G_0(\tau,\varphi) = \begin{cases} X_\tau^0(\varphi)C(\varphi_\tau(\varphi)), & \tau \le 0, \\ -X_\tau^0(\varphi)(E - C(\varphi_\tau(\varphi))), & \tau > 0. \end{cases}$$

Definition 3.2. A function $G_0(\tau,\varphi)$ is called a Green–Samoilenko function of the impulsive system

$$\begin{aligned} \frac{d\varphi}{dt} &= a(\varphi), \quad \varphi \in \mathcal{T}_m, \\ \frac{dx}{dt} &= P(\varphi)x, \quad \varphi \notin \Gamma, \\ \Delta x\big|_{\varphi\in\Gamma} &= I(\varphi)x, \end{aligned}$$

if

$$\|X_\tau^t(\varphi)\| \le Ke^{-\eta|t-\tau|}, \quad t,\tau \in \mathbb{R}, \tag{3.5}$$

for some $K \ge 1$, $\eta > 0$, not depending on $\varphi \in \mathcal{T}_m$.

Then the invariant toroidal set of system (3.1) can be presented as

$$x = u(\varphi) = \int_{-\infty}^{+\infty} G_0(\tau,\varphi)f(\varphi_\tau(\varphi))\,d\tau + \sum_{-\infty<t_i(\varphi)<\infty} G_0(t_i(\varphi)+0,\varphi)g(\varphi_{t_i(\varphi)}(\varphi)), \quad \varphi \in \mathcal{T}_m.$$

Conditions (3.2), (3.5) guarantee the convergence of the integral and sum. Hence, the existence of the Green–Samoilenko function $G_0(\tau,\varphi)$ is a sufficient condition for the existence of invariant toroidal set of system (3.1).

Theorem 3.1. *Let for system* (3.1) *conditions* (3.2) *hold, the matrix* $P(\varphi)$ *satisfy conditions* (2.1) *and* (2.2)*, the matrices* $A(\varphi)$ *and* $I(\varphi)$ *commute* $\forall\, \varphi \in \mathcal{T}_m$ *and, additionally,*

$$\gamma + p\ln\alpha < 0,$$

where

$$\gamma = \max_{\varphi\in M}\max_{j=1,\dots,n} \operatorname{Re}\lambda_j(A(\varphi)), \qquad \alpha = \sup_{\varphi\in\Gamma}\|E + I(\varphi)\|.$$

Then system (3.1) *has an asymptotically stable invariant toroidal set.*

Proof. Choose $\varepsilon > 0$ such that $\gamma + p\ln\alpha + 3\varepsilon < 0$. The fundamental matrix of the impulsive system (3.4) can be presented in the form [14]

$$X_0^t(\varphi) = \Omega_{t_i(\varphi)}^t(\varphi) \prod_{0<t_j(\varphi)<t_i(\varphi)} \big(E + I(\varphi_{t_j(\varphi)}(\varphi))\big)\Omega_{t_{j-1}(\varphi)}^{t_j(\varphi)}(\varphi), \tag{3.6}$$

where $t_0(\varphi) = 0$, $t_i(\varphi) < t \le t_{i+1}(\varphi)$, $\Omega_\tau^t(\varphi)$ is the fundamental matrix of unperturbed system for which the estimate

$$\sup_{\varphi\in\mathcal{T}_m}\|\Omega_\tau^t(\varphi)\| \le K_1e^{(\gamma+\varepsilon)(t-\tau)} \ \text{ for } \ t \ge \tau \tag{3.7}$$

holds with an arbitrarily small ε and some constant $K_1 = K_1(\varepsilon) > 0$ (see Remark 2.1). Due to (2.6), we have

$$\Omega_\tau^t(\varphi) = e^{\int_\tau^t P(\varphi_{t_1}(\varphi))\,dt_1} = e^{(t-\tau)A(\varphi_\tau(\varphi))} \cdot e^{(t-\tau)B(t-\tau,\varphi_\tau(\varphi))}.$$

From a commutativity of the matrices $A(\varphi)$ and $I(\varphi)$ it follows that matrices $E + I(\varphi_{t_{j-1}}(\varphi))$ and $\Omega_{t_{j-1}(\varphi)}^{t_j(\varphi)}(\varphi)$ commute. Then from representation (3.6) and estimates (3.7), (2.8) we get the estimate

$$\|X_0^t(\varphi)\| \le K_2 e^{(\gamma+2\varepsilon)t} \alpha^{i(0,t)} \quad \text{for} \quad t \ge 0,$$

where $K_2 = K_2(\varepsilon) > 0$ does not depend on φ.

From the existence of the uniform limit (3.2) it follows that there exists some $K_3 = K_3(\varepsilon) \ge 1$, not depending on φ, such that $\alpha^{i(0,t)} \le K_3 e^{(\varepsilon+p\ln\alpha)t}$. Then for the fundamental matrix we get the estimate

$$\|X_0^t(\varphi)\| \le K \cdot e^{(3\varepsilon+\gamma+p\ln\alpha)t} \quad \text{for} \quad t \ge 0,$$

where $K = K(\varepsilon) > 0$ does not depend on φ. This means that the function $G_0(\tau,\varphi) = \begin{cases} X_\tau^0(\varphi), & \tau \le 0, \\ 0, & \tau > 0 \end{cases}$ is a Green–Samoilenko function of the invariant tori problem. Hence, system (3.1) has an asymptotically stable invariant toroidal set defined by

$$x = u(\varphi) = \int_{-\infty}^{0} X_\tau^0(\varphi) f(\varphi_\tau(\varphi))\,d\tau + \sum_{-\infty < t_i(\varphi) < 0} X_{t_i(\varphi)+0}^0(\varphi) g(\varphi_{t_i(\varphi)}(\varphi)), \quad \varphi \in \mathcal{T}_m.$$

This completes the proof. □

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