Memoirs on Differential Equations and Mathematical Physics

Volume 73, 2018, 1–20

Khalida Aissani, Mouffak Benchohra, Mustapha Meghnafi

CONTROLLABILITY FOR IMPULSIVE FRACTIONAL EVOLUTION EQUATIONS WITH STATE-DEPENDENT DELAY

Abstract. In this paper, we prove the controllability for a class of impulsive fractional evolution equations with state-dependent delay in a Banach space. Our study is based on the Sadovskii's fixed point theorem. For the illustration of the main result, an example is given.

2010 Mathematics Subject Classification. 26A33, 34A08, 34A37, 34G20, 34K30.

Key words and phrases. Impulsive fractional differential, α-resolvent family, solution operator, Caputo fractional derivative, mild solution, state-dependent delay, fixed point, Banach space.

რეზიუმე. ნაშრომში მდგომარეობებზე დამოკიდებული დაგვიანებით იმპულსურ ფრაქციონალურ ევოლუციური განტოლებების ერთი კლასისთვის ბანახის სივრცეში დამტკიცებულია მართვადობა. საკითხის შესწავლა ეფუძნება სადოვსკის უძრავი წერტილის თეორემას. ძირითადი შედეგის საილუსტრაციოდ მოყვანილია მაგალითი.

1 Introduction

Fractional order differential equations are generalizations of classical integer order differential equations. These are increasingly used to model problems in fluid flow, mechanics, viscoelasticity, biology, physics, engineering and other applications. In recent years, there has been a significant development in ordinary and partial fractional differential equations; see the monographs by Abbas *et al.* [1, 2], Baleanu *et al.* [9], Diethelm [14], Hilfer [22], Kilbas *et al.* [25], Miller and Ross [28], Podlubny [30], Samko *et al.* [33], Tarasov [38], and Zhou [41, 42] and the references therein.

Functional differential equations with state-dependent delay appear frequently in applications as a model of equations and for this reason the study of this type of equations has received great attention in the last years (see [3, 4, 6, 11, 17–21, 24, 27, 35, 39, 40]).

The problem of controllability of linear and nonlinear systems represented by ordinary differential equations in finite dimensional space has been extensively studied. Several authors have extended the controllability concept to infinite dimensional systems in Banach space. Mophou *et al.* [29] studied the controllability of semilinear neutral fractional functional evolution equations with infinite delay, whereas Tai and Wang [37] discussed the controllability of fractional-order impulsive neutral functional infinite delay integrodifferential systems. Controllability of impulsive fractional differential equations with infinite delay is studied by Aissani and Benchohra [5].

Motivated by the previous literature, the purpose of this work is to establish the controllability for a class of impulsive fractional equations with state-dependent delay described by

$$\begin{gathered} D_t^\alpha x(t) = Ax(t) + Bu(t) + f(t, x_{\rho(t,x_t)}, x(t)), \;\; t \in J_k = (t_k, t_{k+1}], \;\; k = 0, 1, \dots, m, \\ \Delta x(t_k) = I_k(x(t_k^-)), \;\; k = 1, 2, \dots, m, \\ x(t) = \phi(t), \;\; t \in (-\infty, 0], \end{gathered} \tag{1.1}$$

where D_t^α is the Caputo fractional derivative of order α, $0 < \alpha < 1$, $A : D(A) \subset E \to E$ is the infinitesimal generator of an α-resolvent family $(S_\alpha(t))_{t\geq 0}$, $f : J \times \mathcal{B} \times E \to E$ is a given function, $J = [0, T]$, $T > 0$, and $\rho : J \times \mathcal{B} \to (-\infty, T]$ is an appropriate function, B is a bounded linear operator from E into E, the control $u \in L^2(J; E)$, the Banach space of admissible controls. Here, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $I_k : E \to E$, $k = 1, 2, \dots, m$, are the given functions, $(E, \|\cdot\|)$ is a complex Banach space, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $x(t_k^+) = \lim\limits_{h\to 0} x(t_k + h)$ and $x(t_k^-) = \lim\limits_{h\to 0} x(t_k - h)$ denotes the right and the left limit of $x(t)$ at $t = t_k$, respectively. We denote by x_t the element of $\mathcal{B}$ defined by $x_t(\theta) = x(t + \theta)$, $\theta \in (-\infty, 0]$. Here x_t represents the history of the state up to the present time t. We assume that the histories x_t belong to some abstract phase space $\mathcal{B}$, to be specified later, and $\phi \in \mathcal{B}$.

2 Preliminaries

In what follows, we recall some notations, definitions, and results that we will need in the sequel.

Let $C = C(J, E)$ be the Banach space of continuous functions from J into E with the norm

$$\|y\|_C = \sup\big\{\|y(t)\| : \; t \in J\big\}.$$

$L(E)$ is the Banach space of all linear and bounded operators on E.

A measurable function $y : J \to E$ is Bochner integrable if and only if $\|y\|$ is Lebesgue integrable.

$L^1(J, E)$ is the Banach space of measurable functions $y : J \to E$ that are Bochner integrable, with the norm

$$\|y\|_{L^1} = \int\limits_0^T \|y(t)\|\, dt \;\; \text{for all} \;\; y \in L^1(J, E).$$

$B_r(x, E)$ represents the closed ball in E with the center at x and of radius r.

We need some basic definitions and properties of the fractional calculus theory which will be used further in this paper.

Definition 2.1. Let $\alpha > 0$ and $f : \mathbb{R}_+ \to E$ be in $L^1(\mathbb{R}_+, E)$. Then the Riemann–Liouville integral is given by

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int\limits_0^t \frac{f(s)}{(t-s)^{1-\alpha}}\, ds.$$

For more details on the Riemann–Liouville fractional derivative, we refer the reader to [13].

Definition 2.2 ([30])**.** The Caputo derivative of order α for a function $f : [0, +\infty) \to E$ can be written as

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int\limits_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}}\, ds = I^{n-\alpha} f^{(n)}(t), \quad t > 0, \quad n-1 \le \alpha < n.$$

If $0 \le \alpha < 1$, then

$$D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int\limits_0^t \frac{f'(s)}{(t-s)^{\alpha}}\, ds.$$

Obviously, the Caputo derivative of a constant is equal to zero.

In order to define a mild solution of problem (1.1), we recall the following

Definition 2.3. A closed linear operator A is said to be sectorial if there are constants $\omega \in \mathbb{R}$, $\theta \in [\frac{\pi}{2}, \pi]$, $M > 0$, such that the following two conditions are satisfied:

1. $\sum\limits_{(\theta,\omega)} := \{\lambda \in C : \ \lambda \neq \omega, \ |arg(\lambda - \omega)| < \theta\} \subset \rho(A)$ ($\rho(A)$ being the resolvent set of A).
2. $\|R(\lambda, A)\|_{L(E)} \le \frac{M}{|\lambda - \omega|}$, $\lambda \in \sum\limits_{(\theta,\omega)}$.

Sectorial operators are well studied in the literature. For details see [15].

Definition 2.4 ([8])**.** Let A be a closed linear operator with domain $D(A)$ defined on a Banach space E and $\alpha > 0$. We say that A is the generator of an α-resolvent family if there exist $\omega \ge 0$ and a strongly continuous function $S_\alpha : \mathbb{R}_+ \to L(E)$ such that $\{\lambda^\alpha : \ \mathrm{Re}(\lambda) > \omega\} \subset \rho(A)$ and

$$(\lambda^\alpha I - A)^{-1} x = \int\limits_0^\infty e^{-\lambda t} S_\alpha(t) x \, dt, \quad \mathrm{Re}\, \lambda > \omega, \quad x \in E.$$

In this case, $S_\alpha(t)$ is called the α-resolvent family generated by A.

Definition 2.5 (see Definition 2.1 in [7])**.** Let A be a closed linear operator with domain $D(A)$ defined on a Banach space E and $\alpha > 0$. We say that A is the generator of a solution operator if there exist $\omega \ge 0$ and a strongly continuous function $S_\alpha : \mathbb{R}_+ \to L(E)$ such that $\{\lambda^\alpha : \ \mathrm{Re}(\lambda) > \omega\} \subset \rho(A)$ and

$$\lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1} x = \int\limits_0^\infty e^{-\lambda t} S_\alpha(t) x \, dt, \quad \mathrm{Re}\, \lambda > \omega, \quad x \in E.$$

In this case, $S_\alpha(t)$ is called the solution operator generated by A. For more details see [26, 31].

In this paper, we will employ an axiomatic definition for the phase space $\mathcal{B}$ which is similar to those introduced by Hale and Kato [16]. Specifically, $\mathcal{B}$ will be a linear space of functions mapping $(-\infty, 0]$ into E endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$, and satisfying the following axioms:

(A1) If $x : (-\infty, T] \to E$ is such that $x_0 \in \mathcal{B}$, then for every $t \in J$, $x_t \in \mathcal{B}$ and

$$\|x(t)\| \le C \|x_t\|_{\mathcal{B}},$$

where $C > 0$ is a constant.

(A2) There exist a continuous function $C_1(t) > 0$ and a locally bounded function $C_2(t) \geq 0$ in $t \geq 0$ such that

$$\|x_t\|_{\mathcal{B}} \leq C_1(t) \sup_{s\in[0,t]} \|x(s)\| + C_2(t)\|x_0\|_{\mathcal{B}},$$

for $t \in J$ and x as in $(A1)$.

(A3) The space $\mathcal{B}$ is complete.

Example 2.6. The phase space $C_r \times L^p(g, X)$.
Let $r \geq 0$, $1 \leq p < \infty$, and let $g : (-\infty, -r) \to \mathbb{R}$ be a nonnegative measurable function which satisfies the conditions $(g-5), (g-6)$ in the terminology of [23]. Briefly, this means that g is locally integrable and there exists a nonnegative locally bounded function Λ on $(-\infty, 0]$ such that $g(\xi+\theta) \leq \Lambda(\xi)g(\theta)$ for all $\xi \leq 0$ and $\theta \in (-\infty, -r) \setminus N_\xi$, where $N_\xi \subset (-\infty, -r)$ is a set with Lebesgue measure zero.

The space $C_r \times L^p(g, X)$ consists of all classes of functions $\varphi : (-\infty, 0] \to X$ such that φ is continuous on $[-r, 0]$, Lebesgue-measurable, and $g\|\varphi\|^p$ on $(-\infty, -r)$. The seminorm in $\|\cdot\|_{\mathcal{B}}$ is defined by

$$\|\varphi\|_{\mathcal{B}} = \sup_{\theta\in[-r,0]} \|\varphi(\theta)\| + \left(\int_{-\infty}^{-r} g(\theta)\|\varphi(\theta)\|^p \, d\theta \right)^{\frac{1}{p}}.$$

The space $\mathcal{B} = C_r \times L^p(g, X)$ satisfies axioms (A1), (A2), (A3). Moreover, for $r = 0$ and $p = 2$, this space coincides with

$$C_0 \times L^2(g, X), \quad H = 1, \quad M(t) = \Lambda(-t)^{\frac{1}{2}}, \quad K(t) = 1 + \left(\int_{-r}^{0} g(\tau) \, d\tau \right)^{\frac{1}{2}}.$$

For more details see [23, Theorem 1.3.8].

Definition 2.7. A function $f : J \times \mathcal{B} \times E \to E$ is said to be a Carathéodory function if it satisfies:

(i) for each $t \in J$, the function $f(t, \cdot, \cdot) : \mathcal{B} \times E \to E$ is continuous;

(ii) for each $(v, w) \in \mathcal{B} \times E$, the function $f(\cdot, v, w) : J \to E$ is measurable.

Definition 2.8. Problem (1.1) is said to be controllable on the interval J if for every initial function $\phi \in \mathcal{B}$ and $x_1 \in E$ there exists a control $u \in L^2(J, E)$ such that the mild solution $x(\cdot)$ of (1.1) satisfies $x(T) = x_1$.

Next, we give the concept of a measure of noncompactness [10].

Definition 2.9. Let B be a bounded subset of a seminormed linear space Y. The Kuratowski's measure of noncompactness of B is defined as

$$\alpha(B) = \inf\big\{d > 0 : \ B \text{ has a finite cover by sets of diameter } \leq d\big\}.$$

We need to use the following basic properties of α measure and Sadovskii's fixed point theorem (see [34]).

Lemma 2.10. *Let A and B be two bounded sets of the Banach space E. Then:*

1. *If $A \subseteq B$, then $\alpha(A) \leq \alpha(B)$;*
2. $\alpha(A) = 0 \Longleftrightarrow \overline{A}$ *is compact (A is relatively compact);*
3. $\alpha(A + B) \leq \alpha(A) + \alpha(B)$.

Theorem 2.11 (Sadovskii's fixed point Theorem)**.** *Let $\mathcal{N}$ be a condensing operator on the Banach space X, i.e., $\mathcal{N}$ is continuous and takes bounded sets into bounded sets, and $\alpha(\mathcal{N}(D)) < \alpha(D)$ for every bounded set D of E with $\alpha(D) > 0$. If $\mathcal{N}(S) \subset S$ for a convex, closed and bounded set S of X, then $\mathcal{N}$ has a fixed point in S.*

3 Controllability results

Before going further, we need the following lemma [36].

Lemma 3.1. *Consider the Cauchy problem*

$$\begin{aligned} D_t^\alpha x(t) &= Ax(t) + Bu(t) + f(t), \quad 0 < \alpha < 1, \\ x(0) &= x_0, \end{aligned} \tag{3.1}$$

if f satisfies the uniform Hölder condition with exponent $\beta \in (0,1]$ and A is a sectorial operator, then the unique solution of the Cauchy problem (3.1) is given by

$$x(t) = T_\alpha(t)x_0 + \int_0^t S_\alpha(t-s)Bu(s)\,ds + \int_0^t S_\alpha(t-s)f(s)\,ds,$$

where

$$T_\alpha(t) = \frac{1}{2\pi i}\int_{\widehat{B_r}} e^{\lambda t}\,\frac{\lambda^{\alpha-1}}{\lambda^\alpha - A}\,d\lambda, \quad S_\alpha(t) = \frac{1}{2\pi i}\int_{\widehat{B_r}} e^{\lambda t}\,\frac{1}{\lambda^\alpha - A}\,d\lambda,$$

$\widehat{B_r}$ denotes the Bromwich path, $S_\alpha(t)$ is called the α-resolvent family and $T_\alpha(t)$ is the solution operator generated by A.

Theorem 3.2 ([12, 36]). *If $\alpha \in (0,1)$ and $A \in \mathbb{A}^\alpha(\theta_0, \omega_0)$, then for any $x \in E$ and $t > 0$, we have*

$$\|T_\alpha(t)\|_{L(E)} \leq Me^{\omega t} \;\; and \;\; \|S_\alpha(t)\|_{L(E)} \leq Ce^{\omega t}(1 + t^{\alpha-1}), \;\; t > 0, \;\; \omega > \omega_0.$$

Let

$$\widetilde{M}_T = \sup_{0\leq t\leq T}\|T_\alpha(t)\|_{L(E)}, \quad \widetilde{M}_s = \sup_{0\leq t\leq T} Ce^{\omega t}(1 + t^{\alpha-1}),$$

hence we have

$$\|T_\alpha(t)\|_{L(E)} \leq \widetilde{M}_T, \quad \|S_\alpha(t)\|_{L(E)} \leq t^{\alpha-1}\widetilde{M}_s.$$

Let us consider the set of functions

$$\mathcal{B}_1 = \Big\{x : \; (-\infty, T] \to E \text{ such that } x\big|_{J_k} \in C(J_k, E) \text{ and there exist} \\ x(t_k^+) \text{ and } x(t_k^-) \text{ with } x(t_k) = x(t_k^-), \; x_0 = \phi, \; k = 1, 2, \ldots, m\Big\}.$$

Endowed with the seminorm,

$$\|x\|_{\mathcal{B}_1} = \sup\big\{\|x(s)\| : \; s \in [0,T]\big\} + \|\phi\|_{\mathcal{B}}, \;\; x \in \mathcal{B}_1,$$

where $x|_{J_k}$ is the restriction of x to $J_k = (t_k, t_{k+1}]$, $k = 1, 2, \ldots, m$.

From Lemma 3.1 we can define a mild solution of system (1.1) as follows.

Definition 3.3. A function $x \in \mathcal{B}_1$ is called a mild solution of (1.1) if it satisfies the following integral equation:

$$x(t)=\begin{cases}\phi(t), & t\in(-\infty,0],\\ \int_0^t S_\alpha(t-s)Bu(s)\,ds+\int_0^t S_\alpha(t-s)f\big(s,x_{\rho(s,x_s)},x(s)\big)\,ds, & t\in[0,t_1],\\ T_\alpha(t-t_1)\big(x(t_1^-)+I_1(x(t_1^-))\big)+\int_{t_1}^t S_\alpha(t-s)Bu(s)\,ds & \\ \qquad +\int_{t_1}^t S_\alpha(t-s)f\big(s,x_{\rho(s,x_s)},x(s)\big)\,ds, & t\in(t_1,t_2],\\ \qquad\vdots & \\ T_\alpha(t-t_m)\big(x(t_m^-)+I_m(x(t_m^-))\big)+\int_{t_m}^t S_\alpha(t-s)Bu(s)\,ds & \\ \qquad +\int_{t_m}^t S_\alpha(t-s)f\big(s,x_{\rho(s,x_s)},x(s)\big)\,ds, & t\in(t_m,T].\end{cases}\tag{3.2}$$

Set

$$\mathcal{R}(\rho^-)=\big\{\rho(s,\varphi):\ (s,\varphi)\in J\times\mathcal{B},\ \rho(s,\varphi)\le 0\big\}.$$

We always assume that $\rho: J\times\mathcal{B}\to(-\infty,T]$ is continuous. Additionally, we introduce the following hypothesis:

(H_φ) The function $t\to\varphi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$ and there exists a continuous and bounded function $L^\phi:\mathcal{R}(\rho^-)\to(0,\infty)$ such that

$$\|\phi_t\|_{\mathcal{B}}\le L^\phi(t)\|\phi\|_{\mathcal{B}}\ \text{ for every }\ t\in\mathcal{R}(\rho^-).$$

Remark 3.4. Condition (H_φ) is frequently verified by the continuous and bounded functions. For more details see, e.g., [23].

Remark 3.5. In the rest of this section, C_1^* and C_2^* are the constants

$$C_1^*=\sup_{s\in J}C_1(s)\ \text{ and }\ C_2^*=\sup_{s\in J}C_2(s).$$

Lemma 3.6 ([21]). *If $x:(-\infty,T]\to X$ is a function such that $x_0=\phi$, then*

$$\|x_s\|_{\mathcal{B}}\le(C_2^*+L^\phi)\|\phi\|_{\mathcal{B}}+C_1^*\sup\big\{\|y(\theta)\|:\ \theta\in[0,max\{0,s\}]\big\},\quad s\in\mathcal{R}(\rho^-)\cup J,$$

where $L^\phi=\sup_{t\in\mathcal{R}(\rho^-)}L^\phi(t)$.

Let us introduce the following hypotheses:

(H1) The semigroup $S(t)$ is compact for $t>0$.

(H2) $f: J\times\mathcal{B}\times E\to E$ satisfies the Carathéodory conditions.

(H3) There exist a continuous function $\mu\in L^1(J,\mathbb{R}^+)$ and a continuous nondecreasing function $\psi:\mathbb{R}^+\to(0,+\infty)$ such that

$$\|f(t,x,y)\|\le\mu(t)\psi\big(\|x\|_{\mathcal{B}}+\|y\|\big),\quad(t,x,y)\in J\times\mathcal{B}\times E.$$

(H4) The function $I_k : E \to E$ is continuous, and there exists $\Omega > 0$ such that

$$\Omega = \max_{1 \le k \le m} \big\{ \|I_k(x)\| : \ x \in B_r \big\}.$$

(H5) The linear operator $W : L^2(J, E) \to E$ defined by

$$Wu = \int_0^T S_\alpha(T-s)Bu(s)\, ds,$$

has an inverse operator $\widetilde{W}^{-1}$, which takes values in $L^2(J, E)/\ker W$ and there exist two positive constants M_1 and M_2 such that

$$\|B\|_{L(E)} \le M_1, \ \ \|\widetilde{W}^{-1}\|_{L(E)} \le M_2.$$

Remark 3.7. The construction of the operator $\widetilde{W}^{-1}$ and its properties are discussed in [32].

Theorem 3.8. *Assume that Hypotheses* (H_φ), $(H1)$–$(H5)$ *are satisfied with* $\widetilde{M_T} < 1$, *then the IVP* (1.1) *is controllable on* $(-\infty, T]$.

Proof. We transform problem (1.1) into a fixed-point problem. Consider the operator $N : \mathcal{B}_1 \to \mathcal{B}_1$ defined by:

$$Nx(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ \displaystyle\int_0^t S_\alpha(t-s)Bu(s)\, ds + \int_0^t S_\alpha(t-s) f\big(s, x_{\rho(s,x_s)}, x(s)\big)\, ds, & t \in [0, t_1], \\ \displaystyle T_\alpha(t-t_1)(x(t_1^-) + I_1(x(t_1^-))) + \int_{t_1}^t S_\alpha(t-s)Bu(s) & \\ \qquad \displaystyle + \int_{t_1}^t S_\alpha(t-s) f\big(s, x_{\rho(s,x_s)}, x(s)\big)\, ds, & t \in (t_1, t_2], \\ \qquad \vdots & \\ \displaystyle T_\alpha(t-t_m)\big(x(t_m^-) + I_m(x(t_m^-))\big) + \int_{t_m}^t S_\alpha(t-s)Bu(s)\, ds & \\ \qquad \displaystyle + \int_{t_m}^t S_\alpha(t-s) f\big(s, x_{\rho(s,x_s)}, x(s)\big)\, ds, & t \in (t_m, T]. \end{cases}$$

Using hypothesis $(H5)$, for an arbitrary function $x(\cdot)$, we define the control

$$u(t)=\begin{cases}\widetilde{W}^{-1}\Big[x_1-\int_0^T S_\alpha(t-s)f\big(s,x_{\rho(s,x_s)},x(s)\big)\,ds\Big](t), & t\in[0,t_1],\\ \widetilde{W}^{-1}\Big[x_1-T_\alpha(T-t_1)\big(x(t_1^-)+I_1(x(t_1^-))\big)\\ \qquad -\int_{t_1}^T S_\alpha(t-s)f\big(s,x_{\rho(s,x_s)},x(s)\big)\,ds\Big](t), & t\in(t_1,t_2],\\ \quad\vdots\\ \widetilde{W}^{-1}\Big[x_1-T_\alpha(T-t_m)\big(x(t_m^-)+I_m(x(t_m^-))\big)\\ \qquad -\int_{t_m}^T S_\alpha(t-s)f\big(s,x_{\rho(s,x_s)},x(s)\big)\,ds\Big](t), & t\in(t_m,T].\end{cases}\tag{3.3}$$

Clearly, fixed points of the operator N are mild solutions of problem (1.1).

Let us define $y(\cdot):(-\infty,T]\to E$ as

$$y(t)=\begin{cases}\phi(t), & t\in(-\infty,0],\\ 0, & t\in J.\end{cases}$$

Then $y_0=\phi$. For each $z\in C(J,E)$ with $z(0)=0$, we denote by $\overline{z}$ the function defined by

$$\overline{z}(t)=\begin{cases}0, & t\in(-\infty,0],\\ z(t), & t\in J.\end{cases}$$

If $x(\cdot)$ satisfies (3.2), we can decompose it as $x(t)=y(t)+\overline{z}(t)$ for $t\in J$, which implies $x_t=y_t+\overline{z}_t$ for every $t\in J$, the expression of the control given by (3.3) becomes

$$u(t)=\begin{cases}\widetilde{W}^{-1}\Big[x_1-\int_0^T S_\alpha(t-s)f\big(s,y_{\rho(s,y_s+\overline{z}(s))}+\overline{z}_{\rho(s,y_s+\overline{z}(s))},y(s)+\overline{z}(s)\big)\,ds\Big](t), & t\in[0,t_1],\\ \widetilde{W}^{-1}\Big[x_1-T_\alpha(T-t_1)\big[y(t_1^-)+\overline{z}(t_1^-)+I_1(y(t_1^-)+\overline{z}(t_1^-))\big]\\ \qquad -\int_{t_1}^T S_\alpha(t-s)f\big(s,y_{\rho(s,y_s+\overline{z}(s))}+\overline{z}_{\rho(s,y_s+\overline{z}(s))},y(s)+\overline{z}(s)\big)\,ds\Big](t), & t\in(t_1,t_2],\\ \quad\vdots\\ \widetilde{W}^{-1}\Big[x_1-T_\alpha(T-t_m)\big[y(t_m^-)+\overline{z}(t_m^-)+I_m(y(t_m^-)+\overline{z}(t_m^-))\big]\\ \qquad -\int_{t_m}^T S_\alpha(t-s)f\big(s,y_{\rho(s,y_s+\overline{z}(s))}+\overline{z}_{\rho(s,y_s+\overline{z}(s))},y(s)+\overline{z}(s)\big)\,ds\Big](t), & t\in(t_m,T],\end{cases}$$

and

$$z(t)=\begin{cases}\int_0^t S_\alpha(t-s)Bu(s)\,ds \\ \qquad +\int_0^t S_\alpha(t-s)f\big(s,y_{\rho(s,y_s+\overline{z}(s))}+\overline{z}_{\rho(s,y_s+\overline{z}(s))},y(s)+\overline{z}(s)\big)\,ds, & t\in[0,t_1],\\ T_\alpha(t-t_1)\big[y(t_1^-)+\overline{z}(t_1^-)+I_1(y(t_1^-)+\overline{z}(t_1^-))\big]+\int_{t_1}^t S_\alpha(t-s)Bu(s)\,ds \\ \qquad +\int_{t_1}^t S_\alpha(t-s)f\big(s,y_{\rho(s,y_s+\overline{z}(s))}+\overline{z}_{\rho(s,y_s+\overline{z}(s))},y(s)+\overline{z}(s)\big)\,ds, & t\in(t_1,t_2],\\ \qquad \vdots \\ T_\alpha(t-t_m)\big[y(t_m^-)+\overline{z}(t_m^-)+I_m(y(t_m^-)+\overline{z}(t_m^-))\big]+\int_{t_m}^t S_\alpha(t-s)Bu(s)\,ds \\ \qquad +\int_{t_m}^t S_\alpha(t-s)f\big(s,y_{\rho(s,y_s+\overline{z}(s))}+\overline{z}_{\rho(s,y_s+\overline{z}(s))},y(s)+\overline{z}(s)\big)\,ds, & t\in(t_m,T].\end{cases}$$

Moreover, $z_0=0$.

Let

$$\mathcal{B}_2=\big\{z\in\mathcal{B}_1:\ z_0=0\big\}.$$

For any $z\in\mathcal{B}_2$, we have

$$\|z\|_{\mathcal{B}_2}=\sup_{t\in J}\|z(t)\|+\|z_0\|_{\mathcal{B}}=\sup_{t\in J}\|z(t)\|.$$

Thus $(\mathcal{B}_2,\|\cdot\|_{\mathcal{B}_2})$ is a Banach space. We define the operator $P:\mathcal{B}_2\to\mathcal{B}_2$ by

$$P(z)(t)=\begin{cases}\int_0^t S_\alpha(t-s)Bu(s)\,ds \\ \qquad +\int_0^t S_\alpha(t-s)f\big(s,y_{\rho(s,y_s+\overline{z}(s))}+\overline{z}_{\rho(s,y_s+\overline{z}(s))},y(s)+\overline{z}(s)\big)\,ds, & t\in[0,t_1],\\ T_\alpha(t-t_1)\big[y(t_1^-)+\overline{z}(t_1^-)+I_1(y(t_1^-)+\overline{z}(t_1^-))\big]+\int_{t_1}^t S_\alpha(t-s)Bu(s)\,ds \\ \qquad +\int_{t_1}^t S_\alpha(t-s)f\big(s,y_{\rho(s,y_s+\overline{z}(s))}+\overline{z}_{\rho(s,y_s+\overline{z}(s))},y(s)+\overline{z}(s)\big)\,ds, & t\in(t_1,t_2],\\ \qquad \vdots \\ T_\alpha(t-t_m)\big[y(t_m^-)+\overline{z}(t_m^-)+I_m(y(t_m^-)+\overline{z}(t_m^-))\big]+\int_{t_m}^t S_\alpha(t-s)Bu(s)\,ds \\ \qquad +\int_{t_m}^t S_\alpha(t-s)f\big(s,y_{\rho(s,y_s+\overline{z}(s))}+\overline{z}_{\rho(s,y_s+\overline{z}(s))},y(s)+\overline{z}(s)\big)\,ds, & t\in(t_m,T].\end{cases}$$

Obviously, the operator N has a fixed point is equivalent to P to have a fixed point, so it remains to prove that P has a fixed point. Let

$$B_r=\big\{z\in\mathcal{B}_2:\ \|z\|_{\mathcal{B}_2}\le r\big\},$$

where r is any fixed finite real number which satisfies the inequality

$$r \geq \frac{\widetilde{M}_T \Omega}{1-\widetilde{M}_T} + \frac{\widetilde{M}_S}{1-\widetilde{M}_T}\, \psi\big((C_2^* + L^{\phi})\|\phi\|_{\mathcal{B}} + (C_1^*+1)r\big)\|\mu\|_{L^1}.$$

Clearly, the subset B_r is a closed, bounded and convex set of $\mathcal{B}_2$. We need the following

Lemma 3.9. *If $x \in B_r$, then we have*

$$\|y_{\rho(s,y_s+\overline{z}(s))} + \overline{z}_{\rho(s,y_s+\overline{z}(s))}\|_{\mathcal{B}} \leq (C_2^* + L^{\phi})\|\phi\|_{\mathcal{B}} + C_1^* r,$$

and

$$\|u(s)\| \leq \begin{cases} M_2\Big[\|x_1\| + \widetilde{M}_S \int\limits_0^T (t-\tau)^{\alpha-1}\mu(\tau) \\ \qquad \times\psi\Big(\big\|y_{\rho(\tau,y_\tau+\overline{z}(\tau))} + \overline{z}_{\rho(\tau,y_\tau+\overline{z}(\tau))}\big\|_{\mathcal{B}} + \|y(\tau)+\overline{z}(\tau)\|\Big)\,d\tau\Big], & t \in [0,t_1], \\ M_2\Big[\|x_1\| + \widetilde{M}_T(r+\Omega) + \widetilde{M}_S \int\limits_0^T (t-\tau)^{\alpha-1}\mu(\tau) \\ \qquad \times\psi\Big(\big\|y_{\rho(\tau,y_\tau+\overline{z}(\tau))} + \overline{z}_{\rho(\tau,y_\tau+\overline{z}(\tau))}\big\|_{\mathcal{B}} + \|y(\tau)+\overline{z}(\tau)\|\Big)\,d\tau\Big], & t \in (t_1,t_2], \\ \qquad \vdots \\ M_2\Big[\|x_1\| + \widetilde{M}_T(r+\Omega) + \widetilde{M}_S \int\limits_0^T (t-\tau)^{\alpha-1}\mu(\tau) \\ \qquad \times\psi\Big(\big\|y_{\rho(\tau,y_\tau+\overline{z}(\tau))} + \overline{z}_{\rho(\tau,y_\tau+\overline{z}(\tau))}\big\|_{\mathcal{B}} + \|y(\tau)+\overline{z}(\tau)\|_E\Big)\,d\tau\Big], & t \in (t_m,T]. \end{cases} \tag{3.4}$$

Proof. Using Lemma 3.6, (H3) and (H5), we obtain

$$\begin{aligned} &\big\|y_{\rho(s,y_s+\overline{z}(s))} + \overline{z}_{\rho(s,y_s+\overline{z}(s))}\big\|_{\mathcal{B}} \\ &\qquad \leq (C_2^* + L^{\phi})\|\phi\|_{\mathcal{B}} + C_1^* \sup\big\{|y(\theta)| : \ \theta \in [0, max\{0,t\}]\big\} \leq (C_2^* + L^{\phi})\|\phi\|_{\mathcal{B}} + C_1^* r. \end{aligned}$$

Also, we get

$$\|u(s)\| \leq \begin{cases} \|\widetilde{W}^{-1}\| \Big[\|x_1\| + \widetilde{M}_S \int\limits_0^T (t-\tau)^{\alpha-1} \\ \qquad \times \Big\| f\big(\tau, y_{\rho(\tau, y_\tau + \overline{z}(\tau))} + \overline{z}_{\rho(\tau, y_\tau + \overline{z}(\tau))}, y(\tau) + \overline{z}(\tau)\big) \Big\| \, d\tau \Big], \quad t \in [0, t_1], \\ \|\widetilde{W}^{-1}\| \Big[\|x_1\| + \widetilde{M}_T (r + \Omega) + \widetilde{M}_S \int\limits_0^T (t-\tau)^{\alpha-1} \\ \qquad \times \Big\| f\big(\tau, y_{\rho(\tau, y_\tau + \overline{z}(\tau))} + \overline{z}_{\rho(\tau, y_\tau + \overline{z}(\tau))}, y(\tau) + \overline{z}(\tau)\big) \Big\| \, d\tau \Big], \quad t \in (t_1, t_2], \\ \qquad \vdots \\ \|\widetilde{W}^{-1}\| \Big[\|x_1\| + \widetilde{M}_T (r + \Omega) + \widetilde{M}_S \int\limits_0^T (t-\tau)^{\alpha-1} \\ \qquad \times \Big\| f\big(\tau, y_{\rho(\tau, y_\tau + \overline{z}(\tau))} + \overline{z}_{\rho(\tau, y_\tau + \overline{z}(\tau))}, y(\tau) + \overline{z}(\tau)\big) \Big\| \, d\tau \Big], \quad t \in (t_m, T] \end{cases}$$

$$\leq \begin{cases} M_2 \Big[\|x_1\| + \widetilde{M}_S \int\limits_0^T (t-\tau)^{\alpha-1} \mu(\tau) \\ \qquad \times \psi\Big(\big\| y_{\rho(\tau, y_\tau + \overline{z}(\tau))} + \overline{z}_{\rho(\tau, y_\tau + \overline{z}(\tau))} \big\|_{\mathcal{B}} + \big\| y(\tau) + \overline{z}(\tau) \big\|_E \Big) \, d\tau \Big], \quad t \in [0, t_1], \\ M_2 \Big[\|x_1\| + \widetilde{M}_T (r + \Omega) + \widetilde{M}_S \int\limits_0^T (t-\tau)^{\alpha-1} \mu(\tau) \\ \qquad \times \psi\Big(\big\| y_{\rho(\tau, y_\tau + \overline{z}(\tau))} + \overline{z}_{\rho(\tau, y_\tau + \overline{z}(\tau))} \big\|_{\mathcal{B}} + \big\| y(\tau) + \overline{z}(\tau) \big\| \Big) \, d\tau \Big], \quad t \in (t_1, t_2], \\ \qquad \vdots \\ M_2 \Big[\|x_1\| + \widetilde{M}_T (r + \Omega) + \widetilde{M}_S \int\limits_0^T (t-\tau)^{\alpha-1} \mu(\tau) \\ \qquad \times \psi\Big(\big\| y_{\rho(\tau, y_\tau + \overline{z}(\tau))} + \overline{z}_{\rho(\tau, y_\tau + \overline{z}(\tau))} \big\|_{\mathcal{B}} + \big\| y(\tau) + \overline{z}(\tau) \big\|_E \Big) \, d\tau \Big], \quad t \in (t_m, T]. \end{cases}$$

Thus the lemma is proved. □

Now, we define two operators P_1 and P_2 on B_r as

$$P_1(z)(t) = \begin{cases} \int_0^t S_\alpha(t-s) f\big(s, y_{\rho(s,y_s+\overline{z}(s))} + \overline{z}_{\rho(s,y_s+\overline{z}(s))}, y(s)+\overline{z}(s)\big)\, ds, & t \in [0,t_1], \\ T_\alpha(t-t_1)\Big[y(t_1^-)+\overline{z}(t_1^-)+I_1\big(y(t_1^-)+\overline{z}(t_1^-)\big)\Big] & \\ \qquad + \int_{t_1}^t S_\alpha(t-s) f\big(s, y_{\rho(s,y_s+\overline{z}(s))} + \overline{z}_{\rho(s,y_s+\overline{z}(s))}, y(s)+\overline{z}(s)\big)\, ds, & t \in (t_1,t_2], \\ \qquad \vdots & \\ T_\alpha(t-t_m)\big[y(t_m^-)+\overline{z}(t_m^-)+I_m(y(t_m^-)+\overline{z}(t_m^-))\big] & \\ \qquad + \int_{t_m}^t S_\alpha(t-s) f\big(s, y_{\rho(s,y_s+\overline{z}(s))} + \overline{z}_{\rho(s,y_s+\overline{z}(s))}, y(s)+\overline{z}(s)\big)\, ds, & t \in (t_m,T], \end{cases}$$

$$P_2(z)(t) = \begin{cases} \int_0^t S_\alpha(t-s)Bu(s)\, ds, \quad t \in [0,t_1], \\ \int_{t_1}^t S_\alpha(t-s)Bu(s)\, ds, \quad t \in (t_1,t_2], \\ \qquad \vdots \\ \int_{t_m}^t S_\alpha(t-s)Bu(s)\, ds, \quad t \in (t_m,T]. \end{cases}$$

Firstly, we show that the operator P_1 maps B_r into itself, next, we prove that P_2 is completely continuous.

Step 1: Let $z \in B_r$, then show that $P_1 z \in B_r$. For $t \in [0,t_1]$, we have

$$\begin{aligned} \|P_1(z)(t)\| &\leq \int_0^t \|S_\alpha(t-s)\|_{L(E)} \Big\| f\big(s, y_{\rho(s,y_s+\overline{z}(s))} + \overline{z}_{\rho(s,y_s+\overline{z}(s))}, y(s)+\overline{z}(s)\big)\Big\|\, ds \\ &\leq \widetilde{M}_S \int_0^t (t-s)^{\alpha-1}\mu(s)\psi\Big(\big\|y_{\rho(s,y_s+\overline{z}(s))} + \overline{z}_{\rho(s,y_s+\overline{z}(s))}\|_{\mathcal{B}} + \|y(s)+\overline{z}(s)\big\|\Big)\, ds \\ &\leq \widetilde{M}_S \int_0^t (t-s)^{\alpha-1}\mu(s)\psi\big((C_2^*+L^\phi)\|\phi\|_{\mathcal{B}} + C_1^* r + r\big)\, ds \\ &\leq \widetilde{M}_S \psi\big((C_2^*+L^\phi)\|\phi\|_{\mathcal{B}} + (C_1^*+1)r\big) \int_0^t \mu(s)\, ds \\ &\leq \widetilde{M}_S \psi\big((C_2^*+L^\phi)\|\phi\|_{\mathcal{B}} + (C_1^*+1)r\big)\|\mu\|_{L^1} \\ &\leq r. \end{aligned}$$

Moreover, when $t \in (t_i, t_{i+1}]$, $i = 1, \ldots, m$, we have the estimate

$$\begin{aligned}\|P_1(z)(t)\| &\le T_\alpha(t-t_i)\big[z(t_i^-) + I_i(z(t_i^-))\big] \\ &\quad + \int_0^t \|S_\alpha(t-s)\|_{L(E)} \Big\| f\big(s, y_{\rho(s,y_s+\overline{z}(s))} + \overline{z}_{\rho(s,y_s+\overline{z}(s))}, y(s)+\overline{z}(s)\big)\Big\| \, ds \\ &\le \widetilde{M}_T(r+\Omega) + \widetilde{M}_S \psi\big((C_2^* + L^\phi)\|\phi\|_{\mathcal{B}} + (C_1^*+1)r\big)\|\mu\|_{L^1} \\ &\le r.\end{aligned}$$

Step 2: P_2 is completely continuous. This will be given in several claims.

Claim 1: P_2 is continuous.

Let $\{z^n\}_{n\in\mathbb{N}}$ be a sequence such that $z^n \to z$ in $\mathcal{B}_2$ as $n \to \infty$. Since f satisfies $(H2)$, we get

$$f\big(\tau, y_\tau + \overline{z}^n_\tau, y(\tau) + \overline{z}^n(\tau)\big) \longrightarrow f\big(\tau, y_\tau + \overline{z}_\tau, y(\tau) + \overline{z}(\tau)\big) \text{ as } n \to \infty.$$

Now for all $t \in [0, t_1]$, we have

$$\begin{aligned}\big\|P_2(z^n)(t) - P_2(z)(t)\big\| &\le \int_0^t \big\|S_\alpha(t-s)B(u_n(s)-u(s))\big\|_{L(E)}\, ds \\ &\le \int_0^t \|S_\alpha(t-s)\|_{L(E)}\|B\|_{L(E)}\|u_n(s)-u(s)\|\, ds \le M_1\widetilde{M}_S \int_0^t (t-s)^{\alpha-1}\|(u_n(s)-u(s))\|\, ds \\ &\le M_1M_2\widetilde{M}_S^2 \int_0^t (t-s)^{\alpha-1}\int_0^T (T-\tau)^{\alpha-1}\Big\|f\big(\tau, y_{\rho(\tau,y_\tau+\overline{z}^n(\tau))} + \overline{z}^n_{\rho(\tau,y_\tau+\overline{z}^n(\tau))}, y(\tau)+\overline{z}^n(\tau)\big) \\ &\qquad - f\big(\tau, y_{\rho(\tau,y_\tau+\overline{z}(\tau))} + \overline{z}_{\rho(\tau,y_\tau+\overline{z}(\tau))}, y(\tau)+\overline{z}(\tau)\big)\Big\|\, d\tau\, ds \le M_1M_2\widetilde{M}_S^2\,\frac{T^{2\alpha}}{\alpha^2}\,\varepsilon,\end{aligned}$$

where $\varepsilon > 0$, $\varepsilon \to 0$ as $n \to \infty$. Moreover,

$$\begin{aligned}\big\|P_2(z^n)(t) - P_2(z)(t)\big\| &\le M_1M_2\widetilde{M}_S \int_{t_i}^t (t-s)^{\alpha-1}\Bigg[\widetilde{M}_T\big\|z^n(t_i^-) - z(t_i^-)\big\| + \big\|I_i(z^n(t_i^-)) - I_i(z(t_i^-))\big\| \\ &\quad + \widetilde{M}_S\int_{t_i}^T (T-\tau)^{\alpha-1}\Big\|f\big(\tau, y_{\rho(\tau,y_\tau+\overline{z}^n(\tau))} + \overline{z}^n_{\rho(\tau,y_\tau+\overline{z}^n(\tau))}, y(\tau)+\overline{z}^n(\tau)\big) \\ &\quad - f\big(\tau, y_{\rho(\tau,y_\tau+\overline{z}(\tau))} + \overline{z}_{\rho(\tau,y_\tau+\overline{z}(s))}, y(s)+\overline{z}(s)\big)\Big\|\, d\tau\Bigg]\, ds \\ &\le M_1M_2\widetilde{M}_S\widetilde{M}_T\,\frac{T^\alpha}{\alpha}\Big[\big\|z^n(t_i^-) - z(t_i^-)\big\| + \big\|I_i(z^n(t_i^-)) - I_i(z(t_i^-))\big\|\Big] + M_1M_2\widetilde{M}_S^2\,\frac{T^{2\alpha}}{\alpha^2}\,\varepsilon,\end{aligned}$$

where $\varepsilon > 0$, $\varepsilon \to 0$ as $n \to \infty$, for all $t \in (t_i, t_{i+1}]$, $i = 1, \ldots, m,$. The impulsive functions I_k, $k = 1, \ldots, m$, are continuous, and we get

$$\lim_{n\to\infty}\|P_2z^n - P_2z\|_{\mathcal{B}_2} = 0.$$

This means that P_2 is continuous.

Claim 2: P_2 maps bounded sets of $\mathcal{B}_2$ into bounded sets in $\mathcal{B}_2$. So, let us prove that for any $r > 0$, there exists $\xi > 0$ such that for each $z \in B_r = \{z \in \mathcal{B}_2 : \ \|z\|_{\mathcal{B}_2} \le r\}$, $\|P_2z\|_{\mathcal{B}_2} \le \xi$. Indeed, for any

$z \in B_r$, $t \in [0, t_1]$, we have

$$
\begin{aligned}
\|P_2(z)(t)\| &\le \int_0^t \|S_\alpha(t-s)\|_{L(E)} \|B\|_{L(E)} \|u(s)\| \, ds \\
&\le M_1 M_2 \widetilde{M}_S \int_0^t (t-s)^{\alpha-1} \bigg[\|x_1\| + \widetilde{M}_S \int_0^T (T-\tau)^{\alpha-1} \mu(\tau) \\
&\qquad \times \psi\Big(\big\| y_{\rho(\tau, y_\tau + \overline{z}(\tau))} + \overline{z}_{\rho(s, y_\tau + \overline{z}(\tau))} \|_{\mathcal{B}} + \|y(\tau) + \overline{z}(\tau)\big\| \Big) \, d\tau \bigg] \, ds \\
&\le M_1 M_2 \widetilde{M}_S \int_0^t (t-s)^{\alpha-1} \bigg[\|x_1\| + \widetilde{M}_S \int_0^T (T-\tau)^{\alpha-1} \mu(\tau) \\
&\qquad \times \psi\big((C_2^* + L^\phi)\|\phi\|_{\mathcal{B}} + C_1^* r + r \big) \, d\tau \bigg] \, ds \\
&\le M_1 M_2 \widetilde{M}_S \, \frac{T^\alpha}{\alpha} \, \|x_1\| + M_1 M_2 \widetilde{M}_S^2 \, \frac{T^{2\alpha}}{\alpha^2} \, \psi\big((C_2^* + L^\phi)\|\phi\|_{\mathcal{B}} + (C_1^* + 1) r \big) \int_0^t \mu(s) \, ds \\
&\le M_1 M_2 \widetilde{M}_S \, \frac{T^\alpha}{\alpha} \, \|x_1\| + M_1 M_2 \widetilde{M}_S^2 \, \frac{T^{2\alpha}}{\alpha^2} \, \psi\big((C_2^* + L^\phi)\|\phi\|_{\mathcal{B}} + (C_1^* + 1) r \big) \|\mu\|_{L^1}.
\end{aligned}
$$

Moreover, when $t \in (t_i, t_{i+1}]$, $i = 1, \dots, m$, we have the estimate

$$
\begin{aligned}
\|P_2(z)(t)\| \le M_1 M_2 \widetilde{M}_S \, \frac{T^\alpha}{\alpha} \, \|x_1\| &+ M_1 M_2 \widetilde{M}_S \widetilde{M}_T (r + \Omega) \, \frac{T^\alpha}{\alpha} \\
&+ M_1 M_2 \widetilde{M}_S^2 \, \frac{T^{2\alpha}}{\alpha^2} \, \psi\big((C_2^* + L^\phi)\|\phi\|_{\mathcal{B}} + (C_1^* + 1) r \big) \|\mu\|_{L^1}.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\|P_2 z\|_{\mathcal{B}_2} \le M_1 M_2 \widetilde{M}_S \, \frac{T^\alpha}{\alpha} \, \|x_1\| &+ M_1 M_2 \widetilde{M}_S \widetilde{M}_T (r + \Omega) \, \frac{T^\alpha}{\alpha} \\
&+ M_1 M_2 \widetilde{M}_S^2 \, \frac{T^{2\alpha}}{\alpha^2} \, \psi\big((C_2^* + L^\phi)\|\phi\|_{\mathcal{B}} + (C_1^* + 1) r \big) \|\mu\|_{L^1}.
\end{aligned}
$$

Claim 3: $P_2(B_r)$ is bounded and equicontinuous. Letting $u, v \in [0, T]$, with $u < v$, we have

$$
\begin{aligned}
\|P_2(z)(v) - P_2(z)(u)\| &\le Q_1 + Q_2, \\
Q_1 &= \int_u^v \|S_\alpha(v-s)\|_{L(E)} \|B\|_{L(E)} \|u(s)\| \, ds, \\
Q_2 &= \int_0^u \|S_\alpha(v-s) - S_\alpha(u-s)\|_{L(E)} \|B\|_{L(E)} \|u(s)\| \, ds.
\end{aligned}
$$

In view of (3.4), for $t \in [0, t_1]$, we have

$$\begin{aligned} Q_1 &= \int_u^v \|S_\alpha(v-s)\|_{L(E)} \|B\|_{L(E)} \|u(s)\|\, ds \\ &\le M_1 M_2 \widetilde{M}_S \int_u^v (v-s)^{\alpha-1} \\ &\quad \times \left[\|x_1\| + \widetilde{M}_S \int_0^T (T-\tau)^{\alpha-1} f\big(\tau, y_{\rho(\tau, y_\tau + \overline{z}(\tau))} + \overline{z}_{\rho(\tau, y_\tau+\overline{z}(\tau))}, y(\tau) + \overline{z}(\tau)\big)\, d\tau \right] ds \\ &\le M_1 M_2 \widetilde{M}_S \frac{(v-u)^\alpha}{\alpha} \left[\|x_1\| + \widetilde{M}_S \frac{T^\alpha}{\alpha} \psi\big((C_2^* + L^\phi)\|\phi\|_{\mathcal{B}} + (C_1^*+1)r\big) \|\mu\|_{L^1} \right]. \end{aligned}$$

Hence, $\lim_{u \to v} Q_1 = 0$. Similarly, for $u, v \in (t_i, t_{i+1}]$, with $u < v$, $i = 1, \ldots, m$, we get

$$\begin{aligned} Q_1 &= \int_u^v \|S_\alpha(v-s)\|_{L(E)} \|B\|_{L(E)} \|u(s)\|\, ds \\ &\le M_1 M_2 \widetilde{M}_S \int_u^v (v-s)^{\alpha-1} \Big[\|x_1\| + \widetilde{M_T}(r+\Omega) \\ &\quad + \widetilde{M}_S \int_0^T (T-\tau)^{\alpha-1} f\big(\tau, y_{\rho(\tau, y_\tau + \overline{z}(\tau))} + \overline{z}_{\rho(\tau, y_\tau+\overline{z}(\tau))}, y(\tau) + \overline{z}(\tau)\big)\, d\tau \Big] ds \\ &\le M_1 M_2 \widetilde{M}_S \frac{(v-u)^\alpha}{\alpha} \left[\|x_1\| + \widetilde{M_T}(r+\Omega) + \widetilde{M}_S \frac{T^\alpha}{\alpha} \psi\big((C_2^* + L^\phi)\|\phi\|_{\mathcal{B}} + (C_1^*+1)r\big) \|\mu\|_{L^1} \right]. \end{aligned}$$

Hence, we deduce that $\lim_{u \to v} Q_1 = 0$.

Using (3.4), for all $t \in [0, t_1]$ we get

$$\begin{aligned} Q_2 &= \int_0^u \big\|S_\alpha(v-s) - S_\alpha(u-s)\big\|_{L(E)} \|B\|_{L(E)} \|u(s)\|\, ds \\ &\le M_1 M_2 \Big[\|x_1\| + \widetilde{M}_S \frac{T^\alpha}{\alpha} \psi\big((C_2^* + L^\phi)\|\phi\|_{\mathcal{B}} + (C_1^*+1)r\big) \|\mu\|_{L^1} \Big] \\ &\quad \times \int_0^u \big\|S_\alpha(v-s) - S_\alpha(u-s)\big\|_{L(E)}\, ds. \end{aligned}$$

Similarly, when $u, v \in (t_i, t_{i+1}]$, $i = 1, \ldots, m$, we have the estimate

$$\begin{aligned} Q_2 &= \int_0^u \big\|S_\alpha(v-s) - S_\alpha(u-s)\big\|_{L(E)} \|B\|_{L(E)} \|u(s)\|\, ds \\ &\le M_1 M_2 \Big[\|x_1\| + \widetilde{M_T}(r+\Omega) + \widetilde{M}_S \frac{T^\alpha}{\alpha} \psi\big((C_2^* + L^\phi)\|\phi\|_{\mathcal{B}} + (C_1^*+1)r\big) \|\mu\|_{L^1} \Big] \\ &\quad \times \int_0^u \big\|S_\alpha(v-s) - S_\alpha(u-s)\big\|_{L(E)}\, ds. \end{aligned}$$

Since

$$\big\|S_\alpha(v-s) - S_\alpha(u-s)\big\|_{L(E)} \le 2\widetilde{M}_s (t_i - s)^{\alpha-1},$$

which belongs to $L^1(J,\mathbb{R}_+)$ and $S_\alpha(v-s)-S_\alpha(u-s)\to 0$ as $u\to v$, S_α is strongly continuous. This implies that $\lim\limits_{u\to v} Q_2=0$. Thus, from the above inequalities, we have

$$\lim_{u\to v}\|P(z)(v)-P(z)(u)\|=0.$$

So, $P_2(B_r)$ is equicontinuous.

Finally, combining Claims 1 and 3 together with the Arzelà–Ascoli's theorem, we conclude that the operator P_2 is compact. In fact, by Step 1–Step 2 and Lemma 2.10, one can conclude that $P=P_1+P_2$ is continuous and takes bounded sets into bounded sets. Meanwhile, it is easy to see that $\alpha(P_2(B_r))=0$, since $P_2(B_r)$ is relatively compact. It comes from $P_1(B_r)\subseteq B_r$ and $\alpha(P_2(B_r))=0$ that

$$\alpha(P(B_r))\le\alpha(P_1(B_r))+\alpha(P_2(B_r))\le\alpha(B_r)$$

for every bounded set B_r of $\mathcal{B}_2$ with $\alpha(B_r)>0$.

Since $P(B_r)\subset B_r$ for a convex, closed and bounded set B_r of $\mathcal{B}_2$, using Theorem 2.11, P has a fixed point z in $B_r\subset\mathcal{B}_2$. It is easy to see that x is a fixed point of the operator N which is a mild solution of (1.1) satisfying $x(T)=x_1$. Thus, system (1.1) is controllable on $(-\infty,T]$. □

4 An example

To apply our abstract results, we consider the impulsive fractional integro-differential system:

$$\begin{aligned}
&\frac{\partial_t^q}{\partial t^q}v(t,\zeta)=\frac{\partial^2}{\partial\zeta^2}v(t,\zeta)+\omega\mu(t,\zeta)\\
&+\int_{-\infty}^{t}a_1(s-t)v\big(s-\rho_1(t)\rho_2(|v(t)|),\xi\big)\,ds+t^2\cos|v(t,\zeta)|,\quad t\in[0,T],\quad \zeta\in[0,\pi],\\
&v(t,0)=v(t,\pi)=0,\quad t\in[0,T],\\
&v(t,\zeta)=v_0(\theta,\zeta),\quad \theta\in(-\infty,0],\quad \zeta\in[0,\pi],\\
&\Delta v(t_k)(\zeta)=\int_{-\infty}^{t_k}p_k(t_k-y)\,dy\cos(v(t_k)(\zeta)),\quad k=1,2,\dots,m,
\end{aligned}\tag{4.1}$$

where $0<q<1$, $\omega>0$, $\mu:[0,T]\times[0,\pi]\to[0,\pi]$, $p_k:\mathbb{R}\to\mathbb{R}$, $k=1,2,\dots,m$, and $a_1:(-\infty,0]\to\mathbb{R}$, $\rho_i:[0,+\infty)\to[0,+\infty)$, $i=1,2$, $v_0:(-\infty,0]\times[0,\pi]\to\mathbb{R}$ are continuous functions.

Set $E=L^2([0,\pi])$ and let $D(A)\subset E\to E$ be the operator $A\omega=\omega''$ with the domain

$$D(A)=\big\{\omega\in E:\ \omega,\ \omega'\text{ are absolutely continuous, }\omega''\in E,\ \omega(0)=\omega(\pi)=0\big\},$$

then

$$A\omega=\sum_{n=1}^{\infty}n^2(\omega,\omega_n)\omega_n,\quad \omega\in D(A),$$

where $\omega_n(x)=\sqrt{\frac{2}{\pi}}\sin(nx)$, $n\in\mathbb{N}$, is the orthogonal set of eigenvectors of A. It is well known that A is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\ge0}$ in E and is given by

$$T(t)\omega=\sum_{n=1}^{\infty}e^{-n^2t}(\omega,\omega_n)\omega_n\ \text{ for all }\ \omega\in E\ \text{ and all }\ t>0.$$

From these expressions it follows that $\{T(t)\}_{t\ge0}$ is a uniformly bounded compact semigroup such that $R(\lambda,A)=(\lambda-A)^{-1}$ is a compact operator for all $\lambda\in\rho(A)$, that is, $A\in\mathbb{A}^\alpha(\theta_0,\omega_0)$.

For the phase space, we choose $\mathcal{B}=C_0\times L^2(g,X)$ (for details, see Example 2.6).

Set

$$x(t)(\zeta) = v(t,\zeta), \quad t \in [0,T], \quad \zeta \in [0,\pi];$$
$$\phi(\theta)(\zeta) = v_0(\theta,\zeta), \quad \theta \in (-\infty,0], \quad \zeta \in [0,\pi];$$
$$f(t,\varphi,x(t))(\zeta) = \int_{-\infty}^{0} a_1(s)\varphi(s,\xi)\,ds + t^2\cos|x(t)(\zeta)|, \quad t \in [0,T], \quad \zeta \in [0,\pi];$$
$$\rho(s,\varphi) = s - \rho_1(s)\rho_2(|\varphi(0)|);$$
$$I_k(x(t_k^-))(\zeta) = \int_{-\infty}^{0} p_k(t_k-y)\,dy\,\cos(x(t_k)(\zeta)), \quad k=1,2,\dots,m;$$
$$Bu(t)(\zeta) = \omega\mu(t,\zeta).$$

Under the above conditions, we can represent system (4.1) in the abstract form (1.1). Assume that the operator $W : L^2(J,E) \to X$ defined by

$$Wu(\,\cdot\,) = \int_0^T S_\alpha(T-s)\omega\mu(s,\,\cdot\,)\,ds$$

has a bounded invertible operator $\widetilde{W}^{-1}$ in $L^2(J,E)/\ker W$.

The following result is a direct consequence of Theorem 3.8.

Proposition 4.1. *Let $\varphi \in \mathcal{B}$ be such that (H_φ) holds, and assume that the above conditions are fulfilled, then system* (4.1) *is controllable on* $(-\infty,T]$.

References

[1] S. Abbas, M. Benchohra and G. M. N'Guérékata, *Topics in Fractional Differential Equations*. Developments in Mathematics, 27. Springer, New York, 2012.

[2] S. Abbas, M. Benchohra and G. M. N'Guérékata, *Advanced Fractional Differential and Integral Equations*. Mathematics Research Developments Series. Nova Science Publishers, Inc., New York, 2015.

[3] W. G. Aiello, H. I. Freedman and J. Wu, Analysis of a model representing stage-structured population growth with state-dependent time delay. *SIAM J. Appl. Math.* **52** (1992), no. 3, 855–869.

[4] K. Aissani and M. Benchohra, Global existence results for fractional integro-differential equations with state-dependent delay. *An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)* **62** (2016), no. 2, vol. 1, 411–422.

[5] K. Aissani and M. Benchohra, Controllability of impulsive fractional differential equations with infinite delay. *Lib. Math. (N.S.)* **33** (2013), no. 2, 47–64.

[6] K. Aissani and M. Benchohra, Fractional integro-differential equations with state-dependent delay. *Adv. Dyn. Syst. Appl.* **9** (2014), no. 1, 17–30.

[7] R. P. Agarwal, B. de Andrade and G. Siracusa, On fractional integro-differential equations with state-dependent delay. *Comput. Math. Appl.* **62** (2011), no. 3, 1143–1149.

[8] D. Araya and C. Lizama, Almost automorphic mild solutions to fractional differential equations. *Nonlinear Anal.* **69** (2008), no. 11, 3692–3705.

[9] D. Baleanu, K. Diethelm, E. Scalas and J. J. Trujillo, *Fractional Calculus. Models and Numerical Methods*. Series on Complexity, Nonlinearity and Chaos 3. World Scientific, Hackensack, NJ, 2012.

[10] J. Banaś and K. Goebel, *Measures of Noncompactness in Banach Spaces*. Lecture Notes in Pure and Applied Mathematics, 60. Marcel Dekker, Inc., New York, 1980.

[11] M. Bartha, Periodic solutions for differential equations with state-dependent delay and positive feedback. *Nonlinear Anal.* **53** (2003), no. 6, 839–857.

[12] E. Bazhiekova. *Fractional Evolution Equations in Banach Spaces.* Ph.D. Thesis, Eindhoven University of Technology, Netherlands, 2001.

[13] L. Debnath and D. Bhatta, *Integral Transforms and their Applications.* Second edition. Chapman & Hall/CRC, Boca Raton, FL, 2007.

[14] K. Diethelm, *The Analysis of Fractional Differential Equations. An Application-Oriented Exposition using Differential Operators of Caputo Type.* Lecture Notes in Mathematics, 2004. Springer-Verlag, Berlin, 2010.

[15] M. Haase, *The Functional Calculus for Sectorial Operators.* Operator Theory: Advances and Applications, 169. Birkhäuser Verlag, Basel, 2006.

[16] J. K. Hale and J. Kato, Phase space for retarded equations with infinite delay. *Funkcial. Ekvac.* **21** (1978), no. 1, 11–41.

[17] F. Hartung, Parameter estimation by quasilinearization in functional differential equations with state-dependent delays: a numerical study. *Nonlinear Anal.* **47** (2001), no. 7, 4557–4566.

[18] F. Hartung and J. Turi, Identification of parameters in delay equations with state-dependent delays. *Nonlinear Anal.* **29** (1997), no. 11, 1303–1318.

[19] E. Hernández Morales and M. A. McKibben, On state-dependent delay partial neutral functional-differential equations. *Appl. Math. Comput.* **186** (2007), no. 1, 294–301.

[20] E. Hernández Morales, M. A. McKibben and H. R. Henríquez, Existence results for partial neutral functional differential equations with state-dependent delay. *Math. Comput. Modelling* **49** (2009), no. 5-6, 1260–1267.

[21] E. Hernández, A. Prokopczyk and L. Ladeira, A note on partial functional differential equations with state-dependent delay. *Nonlinear Anal. Real World Appl.* **7** (2006), no. 4, 510–519.

[22] R. Hilfer (Ed.), *Applications of Fractional Calculus in Physics.* World Scientific, Singapore, 2000.

[23] Y. Hino, S. Murakami and T. Naito, *Functional-Differential Equations with Infinite Delay.* Lecture Notes in Mathematics, 1473. Springer-Verlag, Berlin, 1991.

[24] V. Kavitha, P.-Z. Wang and R. Murugesu, Existence results for neutral functional fractional differential equations with state dependent-delay. *Malaya J. Mat.* **1** (2012), no. 1, 50–61.

[25] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations.* North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[26] C. Lizama, Regularized solutions for abstract Volterra equations. *J. Math. Anal. Appl.* **243** (2000), no. 2, 278–292.

[27] M. Meghnafi, M. Benchohra and K. Aissani, Impulsive fractional evolution equations with state-dependent delay. *Nonlinear Stud.* **22** (2015), no. 4, 659–671.

[28] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations.* A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1993.

[29] G. M. Mophou and G. M. N'Guérékata, Controllability of semilinear neutral fractional functional evolution equations with infinite delay. *Nonlinear Stud.* **18** (2011), no. 2, 195–209.

[30] I. Podlubny, *Fractional Differential Equations. An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications.* Mathematics in Science and Engineering, 198. Academic Press, Inc., San Diego, CA, 1999.

[31] J. Prüss, *Evolutionary Integral Equations and Applications.* Monographs in Mathematics, 87. Birkhäuser Verlag, Basel, 1993.

[32] M. D. Quinn and N. Carmichael, An approach to nonlinear control problems using fixed-point methods, degree theory and pseudo-inverses. *Numer. Funct. Anal. Optim.* **7** (1984/85), no. 2-3, 197–219.

[33] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications.* Edited and with a foreword by S. M. Nikol'skiĭ. Translated from the 1987 Russian original. Revised by the authors. Gordon and Breach Science Publishers, Yverdon, 1993.

[34] A. P. Sadovskii and T. V. Shcheglova, Center conditions for a polynomial differential system. (Russian) *Differ. Uravn.* **49** (2013), no. 2, 151–164; translation in *Differ. Equ.* **49** (2013), no. 2, 151–165.

[35] A. V. Rezounenko, Partial differential equations with discrete and distributed state-dependent delays. *J. Math. Anal. Appl.* **326** (2007), no. 2, 1031–1045.

[36] X.-B. Shu, Y. Lai and Y. Chen, The existence of mild solutions for impulsive fractional partial differential equations. *Nonlinear Anal.* **74** (2011), no. 5, 2003–2011.

[37] Z. Tai and X. Wang, Controllability of fractional-order impulsive neutral functional infinite delay integrodifferential systems in Banach spaces. *Appl. Math. Lett.* **22** (2009), no. 11, 1760–1765.

[38] V. E. Tarasov, *Fractional Dynamics. Applications of Fractional Calculus to Dynamics of Particles, Fields and Media.* Nonlinear Physical Science. Springer, Heidelberg; Higher Education Press, Beijing, 2010.

[39] Z. Yang and J. Cao, Existence of periodic solutions in neutral state-dependent delays equations and models. *J. Comput. Appl. Math.* **174** (2005), no. 1, 179–199.

[40] Z. Yan and H. Zhang, Existence of solutions to impulsive fractional partial neutral stochastic integro-differential inclusions with state-dependent delay. *Electron. J. Differ. Equ.* **2013**, Paper No. 81, 21 pp.

[41] Y. Zhou, *Basic Theory of Fractional Differential Equations.* World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2014.

[42] Y. Zhou, *Fractional Evolution Equations and Inclusions: Analysis and Control.* Elsevier/Academic Press, London, 2016.

(Received 12.07.2017)

Authors' addresses:

Khalida Aissani, **Mustapha Meghnafi**
University of Bechar, PO Box 417, 08000, Bechar, Algeria.
E-mail: megnafi3000@yahoo.fr; aissani_k@yahoo.fr

Mouffak Benchohra
Laboratory of Mathematics, Djillali Liabes University of Sidi Bel-Abbès, PO Box 89, 22000, Sidi Bel-Abbès, Algeria.
E-mail: benchohra@yahoo.com