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ON THE SOLVABILITY OF THE PERIODIC PROBLEM FOR SYSTEMS OF LINEAR GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

Abstract. A periodic problem for systems of linear generalized differential equations is considered. The Green type theorem on the unique solvability of the problem and the representation of its solution are established. Effective necessary and sufficient conditions (of spectral type) for the unique solvability of the problem are also given.

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1 Statement of the problem and formulation of the results

In the present paper, we investigate the solvability for the system of linear generalized ordinary differential equations

$$dx(t) = dA(t) \cdot x(t) + df(t) \tag{1.1}$$

with the ω-periodic ($\omega > 0$) condition

$$x(t+\omega) = x(t) \ \text{ for } \ t \in \mathbb{R}, \tag{1.2}$$

where $A = (a_{ik})_{i,k=1}^{n} : \mathbb{R} \to \mathbb{R}^{n\times n}$ and $f = (f_i)_{i=1}^{n} : \mathbb{R} \to \mathbb{R}^n$ are, respectively, the matrix- and the vector-functions with bounded variation components on every closed interval from $\mathbb{R}$, and ω is a fixed positive number.

We establish the Green type theorem on the solvability of problem (1.1), (1.2) and the representation of a solution of the problem. In addition, we give effective necessary and sufficient conditions (of spectral type) for the unique solvability of the problem.

The general linear boundary value problem for system (1.1) has been investigated sufficiently well (see, e.g., [6, 7, 15] and the references therein, where the Green type theorems are obtained for the unique solvability). Some questions related to the periodic problem for system (1.1) are investigated in [2–5, 8, 14] (see also the references therein), but in these works no attention is given to the investigation of specific properties analogous to the already established ones for the ordinary differential case (see, e.g., [11]). But some questions concerning the results obtained in [11] for the periodic problem for linear ordinary differential case is not investigated for the periodic problem for the generalized differential case. So, the problem considered in the paper is quite topical.

We establish some special conditions for the unique solvability of the problem.

To a considerable extent, the interest in the theory of generalized ordinary differential equations has also been stimulated by the fact that this theory enables one to investigate ordinary differential, impulsive differential and difference equations from the unified point of view (see [1, 7, 9, 10, 13, 14] and the references therein).

The theory of generalized ordinary differential equations was introduced by J. Kurzweil [13] in connection with the investigation of the well-posed problem for the Cauchy problem for ordinary differential equations.

In the paper, the use will be made of the following notation and definitions.

$\mathbb{R} =]-\infty, +\infty[\,$, $\mathbb{R}_+ = [0, +\infty[\,$.

$\mathbb{R}^{n\times m}$ is the space of all $n \times m$ real matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm

$$\|X\| = \max_{j=1,\dots,m} \sum_{i=1}^{n} |x_{ij}|.$$

$\mathbb{R}_+^{n\times m} = \{(x_{ij})_{i,j=1}^{n,m} : \ x_{ij} \geq 0 \ (i = 1,\dots,n; \ j = 1,\dots,m)\}$.

$O_{n\times m}$ (or O) is the zero $n \times m$ matrix.

If $X = (x_{ij})_{i,j=1}^{n,m} \in \mathbb{R}^{n\times m}$, then $|X| = (|x_{ij}|)_{i,j=1}^{n,m}$.

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^{n}$; $\mathbb{R}_+^n = \mathbb{R}_+^{n\times 1}$.

$x * y$ is the scalar product of the vectors x and $y \in \mathbb{R}^n$.

If $X \in \mathbb{R}^{n\times n}$, then: X^{-1} is the matrix, inverse to X; $\det X$ is the determinant of X; $r(X)$ is the spectral radius of X; X^T is the matrix transposed to X; $\lambda_0(X)$ and $\lambda^0(X)$ are, respectively, the minimal and maximal eigenvalues of the symmetric matrix X.

I_n is the identity $n \times n$-matrix.

The inequalities between the real matrices are understood componentwise.

A matrix-function is said to be continuous, integrable, nondecreasing, etc., if each of its components is such.

If $X : [a,b] \to \mathbb{R}^{n\times m}$ is a matrix-function, then $\bigvee\limits_a^b(X)$ is the sum of variations on $[a,b]$ of its components x_{ij} $(i = 1,\dots,n; \ j = 1,\dots,m)$; $V(X)(t) = (v(x_{ij})(t))_{i,j=1}^{n,m}$, where $v(x_{ij})(a) = 0$, $v(x_{ij})(t) = \bigvee\limits_a^t(x_{ij})$ for $a < t \leq b$.

$X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of X at the point t ($X(a-)=X(a)$, $X(b+)=X(b)$); $d_1X(t)=X(t)-X(t-)$, $d_2X(t)=X(t+)-X(t)$.

$\|X\|_s=\sup\{\|X(t)\|:\ t\in[a,b]\}$, $|X|_s=(\|x_{ij}\|_s)_{i,j=1}^{n,m}$.

$\mathrm{BV}([a,b],\mathbb{R}^{n\times m})$ is the normed space of all bounded variation matrix-functions $X:[a,b]\to\mathbb{R}^{n\times m}$ (i.e., such that $\bigvee\limits_a^b(X)<\infty$) with the norm $\|X\|_s$.

$\mathrm{BV}_{loc}(\mathbb{R},\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X:[a,b]\to\mathbb{R}^{n\times m}$ whose restrictions on every closed interval $[a,b]$ from $\mathbb{R}$ belong to $\mathrm{BV}([a,b],\mathbb{R}^{n\times n})$.

$\mathrm{BV}_\omega(\mathbb{R},\mathbb{R}^{n\times m})$ is the set of all matrix-functions $G:\mathbb{R}\to\mathbb{R}^{n\times m}$ whose restrictions on $[0,\omega]$ belong to $\mathrm{BV}([0,\omega],\mathbb{R}^{n\times m})$ and there exists a constant matrix $C\in\mathbb{R}^{n\times m}$ such that

$$G(t+\omega)=G(t)+C \ \text{ for } \ t\in\mathbb{R}.$$

$\mathrm{BV}([a,b],\mathbb{R}_+^{n\times m})=\{X\in\mathrm{BV}([a,b],\mathbb{R}^{n\times m}):\ X(t)\geq O_{n\times m} \text{ for } t\in[a,b]\}$.

$s_c,s_1,s_2:\mathrm{BV}([a,b],\mathbb{R})\to\mathrm{BV}([a,b],\mathbb{R})$ are the operators defined, respectively, by

$$s_1(x)(a)=s_2(x)(a)=0,$$
$$s_1(x)(t)=\sum_{a<\tau\leq t}d_1x(\tau),\quad s_2(x)(t)=\sum_{a\leq\tau<t}d_2x(\tau) \ \text{ for } \ a<t\leq b,$$

and

$$s_c(x)(t)=x(t)-s_1(x)(t)-s_2(x)(t) \ \text{ for } \ t\in[a,b].$$

If $g:[a,b]\to\mathbb{R}$ is a nondecreasing function, $x:[a,b]\to\mathbb{R}$ and $a\leq s<t\leq b$, then

$$\int_s^t x(\tau)\,dg(\tau)=\int_{]s,t[}x(\tau)\,ds_c(g)(\tau)+\sum_{s<\tau\leq t}x(\tau)d_1g(\tau)+\sum_{s\leq\tau<t}x(\tau)d_2g(\tau),$$

where $\int_{]s,t[}x(\tau)\,ds_c(g)(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s,t[$ with respect to the measure $\mu(s_c(g))$ corresponding to the function $s_c(g)$.

If $a=b$, then we assume

$$\int_a^b x(t)\,dg(t)=0,$$

and if $a>b$, then we assume

$$\int_a^b x(t)\,dg(t)=-\int_b^a x(t)\,dg(t).$$

So, $\int_a^b x(\tau)\,dg(\tau)$ is the Kurzweil–Stieltjes integral (see [13,14]).

If $g(t)\equiv g_1(t)-g_2(t)$, where g_1 and g_2 are nondecreasing functions, then

$$\int_s^t x(\tau)\,dg(\tau)=\int_s^t x(\tau)\,dg_1(\tau)-\int_s^t x(\tau)\,dg_2(\tau) \ \text{ for } \ s\leq t.$$

$L([a,b],\mathbb{R};g)$ is the set of all functions $x:[a,b]\to\mathbb{R}$, measurable and integrable with respect to the measures $\mu(g_i)$ $(i=1,2)$, i.e., such that

$$\int_a^b |x(t)|\,dg_i(t)<+\infty \ \ (i=1,2).$$

If $G=(g_{ik})_{i,k=1}^{l,n}\in \mathrm{BV}([a,b],\mathbb{R}^{l\times n})$ and $X=(x_{kj})_{k,j=1}^{n,m}:[a,b]\to\mathbb{R}^{n\times m}$, then

$$S_c(G)(t)\equiv (s_c(g_{ik})(t))_{i,k=1}^{l,n},\quad S_j(G)(t)\equiv (s_j(g_{ik})(t))_{i,k=1}^{l,n}\ \ (j=1,2)$$

and

$$\int_a^b dG(\tau)\cdot X(\tau)=\bigg(\sum_{k=1}^n\int_a^b x_{kj}(\tau)\,dg_{ik}(\tau)\bigg)_{i,j=1}^{l,m}.$$

We introduce the operator $\mathcal{A}$ as follows. If the matrix-function $X\in \mathrm{BV}_{loc}(\mathbb{R},;\mathbb{R}^{n\times n})$ is such that $\det(I_n+(-1)^j d_jX(t))\neq 0$ for $t\in\mathbb{R}$ $(j=1,2)$, and $Y\in \mathrm{BV}_{loc}(\mathbb{R};\mathbb{R}^{n\times m})$, then

$$\begin{aligned}
&\mathcal{A}(X,Y)(0)=O_{n\times m},\\
&\mathcal{A}(X,Y)(t)=Y(t)-Y(0)+\sum_{0<\tau\le t} d_1X(\tau)\cdot(I_n-d_1X(\tau))^{-1}\,d_1Y(\tau)\\
&\qquad\qquad -\sum_{0\le\tau<t} d_2X(\tau)\cdot(I_n+d_2X(\tau))^{-1}\,d_2Y(\tau)\ \text{ for } t>0,\\
&\mathcal{A}(X,Y)(t)=-\mathcal{A}(X,Y)(t)\ \text{ for } t<0.
\end{aligned}$$

Here, the use will be made of the following formulas:

$$\int_a^b f(t)\,d\bigg(\int_a^t h(s)\,dg(s)\bigg)=\int_a^b f(t)h(t)\,dg(t)\ \ \text{(substitution formula)};$$

$$\begin{aligned}
\int_a^b f(t)\,dg(t)+\int_a^b g(t)\,df(t)&=f(b)g(b)-f(a)g(a)+\sum_{a<t\le b} d_1f(t)\cdot d_1g(t)\\
&\quad-\sum_{a\le t<b} d_2f(t)\cdot d_2g(t)\ \ \text{(integration by parts formula)},
\end{aligned}$$

$$\begin{aligned}
\int_a^b h(t)\,d(f(t)g(t))&=\int_a^b h(t)f(t)\,dg(t)+\int_a^b h(t)g(t)\,df(t)-\sum_{a<t\le b} h(t)d_1f(t)\cdot d_1g(t)\\
&\quad-\sum_{a\le t<b} h(t)d_2f(t)\cdot d_2g(t)\ \ \text{(general integration by parts formula)}
\end{aligned}$$

and

$$d_j\bigg(\int_a^t f(s)\,dg(s)\bigg)=f(t)\,d_jg(t)\ \text{ for } t\in[a,b]\ \ (j=1,2),$$

where f, g and $h\in \mathrm{BV}([a,b],\mathbb{R})$ (see Theorems I.4.25 and I.4.33 in [14]). Further, we use these formulas without special indication.

We say that the matrix-function $X\in \mathrm{BV}([a,b],\mathbb{R}^{n\times n})$ satisfies the Lappo–Danilevskiĭ condition if the matrices $S_c(X)(t)$, $S_1(X)(t)$ and $S_2(X)(t)$ are pairwise permutable for every $t\in[a,b]$ and there exists $t_0\in[a,b]$ such that

$$\int_{t_0}^t S_c(X)(\tau)\,dS_c(X)(\tau)=\int_{t_0}^t dS_c(X)(\tau)\cdot S_c(X)(\tau)\ \text{ for } t\in[a,b].$$

A vector-function $x\in \mathrm{BV}_{loc}(\mathbb{R},\mathbb{R}^{n\times m})$ is said to be a solution of system (1.1) if

$$x(t)-x(s)=\int_s^t dA(\tau)\cdot x(\tau)+f(t)-f(s)\ \text{ for } s<t,\ \ s,t\in\mathbb{R}.$$

We assume that $A \in \mathrm{BV}_\omega(\mathbb{R}, \mathbb{R}^{n\times n})$ and $f \in \mathrm{BV}_\omega(\mathbb{R}, \mathbb{R}^n)$, i.e.,

$$A(t+\omega) = A(t) + C \ \text{ and } \ f(t+\omega) = f(t) + c \ \text{ for } \ t \in \mathbb{R}, \tag{1.3}$$

where $C \in \mathbb{R}^{n\times n}$ and $c \in \mathbb{R}^n$ are, respectively, some constant matrix and vector. Moreover, we assume that

$$\det\big(I_n + (-1)^j d_j A(t)\big) \neq 0 \ \text{ for } \ t \in \mathbb{R} \ \ (j=1,2). \tag{1.4}$$

If a matrix-function $X \in \mathrm{BV}([0,\omega], \mathbb{R}^{n\times n})$ is such that $\det(I_n - d_1 X(t)) \neq 0$ for $t \in [0,\omega]$, then we put

$$[X(t)]_0 = (I_n - d_1 X(t))^{-1}, \quad [X(t)]_i = (I_n - d_1 X(t))^{-1} \int_0^t dX_-(\tau)\cdot [X(\tau)]_{i-1} \quad \text{for } \ t \in [0,\omega] \ \ (i=1,2,\dots), \tag{1.5$_1$}$$

$$(X(t))_0 = O_{n\times n}, \ \ (X(t))_1 = X(t), \ \ (X(t))_{i+1} = \int_0^t dX_-(\tau)\cdot (X(\tau))_i \quad \text{for } \ t \in [0,\omega] \ \ (i=1,2,\dots), \tag{1.6$_1$}$$

and

$$\begin{aligned} V_1(X)(t) &= \big|(I_n - d_1 X(t))^{-1}\big| V(X_-)(t), \\ V_{i+1}(X)(t) &= \big|(I_n - d_1 X(t))^{-1}\big| \int_0^t dV(X_-)(\tau)\cdot V_i(X)(\tau) \ \text{ for } \ t \in [0,\omega] \ \ (i=1,2,\dots), \end{aligned} \tag{1.7$_1$}$$

where $X_-(t) \equiv X(t-)$; and if $X \in \mathrm{BV}([0,\omega], \mathbb{R}^{n\times n})$ is such that $\det(I_n + d_2 X(t)) \neq 0$ for $t \in [0,\omega]$, then we put

$$[X(t)]_0 = (I_n + d_2 X(t))^{-1}, \ \ [X(t)]_i = (I_n + d_2 X(t))^{-1} \int_\omega^t dX_+(\tau)\cdot [X(\tau)]_{i-1} \quad \text{for } \ t \in [0,\omega] \ \ (i=1,2,\dots), \tag{1.5$_2$}$$

$$(X(t))_0 = O_{n\times n}, \ \ (X(t))_1 = X(t), \ \ (X(t))_{i+1} = \int_\omega^t dX_+(\tau)\cdot (X(\tau))_i \quad \text{for } \ t \in [0,\omega] \ \ (i=1,2,\dots) \tag{1.6$_2$}$$

and

$$\begin{aligned} V_1(X)(t) &= \big|(I_n + d_2 X(t))^{-1}\big| \, \big|(V(X_+)(t)(\omega) - V(X_+)(t)\big|, \\ V_{i+1}(X)(t) &= \big|(I_n + d_2 X(t))^{-1}\big| \left| \int_\omega^t dV(X_+)(\tau)\cdot V_i(X)(\tau)\right| \ \text{ for } \ t \in [0,\omega] \ \ (i=1,2,\dots), \end{aligned} \tag{1.7$_2$}$$

where $X_+(t) \equiv X(t+)$.

Alongside with system (1.1), we consider the corresponding homogeneous system

$$dx(t) = dA(t)\cdot x(t). \tag{1.1$_0$}$$

Moreover, along with condition (1.2) we consider the condition

$$x(0) = x(\omega). \tag{1.8}$$

Definition 1.1. Let condition (1.4) hold and let there exist a fundamental matrix Y of problem $(1.1_0),(1.8)$ such that

$$\det(D)\neq 0, \tag{1.9}$$

where $D=Y(\omega)-Y(0)$. A matrix-function $\mathcal{G}:[0,\omega]\times[0,\omega]\to\mathbb{R}^{n\times n}$ is said to be the Green matrix of problem $(1.1_0),(1.8)$ if:

(a) the matrix-function $\mathcal{G}(\,\cdot\,,s)$ satisfies the matrix equation

$$dX(t)=dA(t)\cdot X(t)$$

on both $[0,s[$ and $]s,\omega]$ for every $s\in\,]0,\omega[$;

(b) for $t\in\,]a,b[$,

$$\mathcal{G}(t,t+)-\mathcal{G}(t,t-)=Y(t)D^{-1}\big\{Y(\omega)Y^{-1}(t)(I_n+d_2A(t))^{-1}-Y(0)Y^{-1}(t)(I_n-d_1A(t))^{-1}\big\};$$

(c) $\mathcal{G}(t,\,\cdot\,)\in BV([0,\omega],\mathbb{R}^{n\times n})$ for every $t\in[0,\omega]$;

(d) the equality

$$\int\limits_0^\omega d_s\big(\mathcal{G}(\omega,s)-\mathcal{G}(0,s)\big)\cdot f(s)=0$$

holds for every $f\in BV([0,\omega],\mathbb{R}^n)$.

The Green matrix of problem $(1.1_0),(1.8)$ exists and is unique in the following sense (see [6,15]). If $\mathcal{G}(t,s)$ and $\mathcal{G}_1(t,s)$ are two matrix-functions satisfying conditions (a)–(d) of Definition 1.1, then

$$\mathcal{G}(t,s)-\mathcal{G}_1(t,s)\equiv Y(t)H_*(s),$$

where $H_*\in \mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ is a matrix-function such that

$$H_*(s+)=H_*(s-)=C=const\ \text{ for }\ s\in[0,\omega],$$

and $C\in\mathbb{R}^{n\times n}$ is a constant matrix.

In particular,

$$\mathcal{G}(t,s)=\begin{cases}Y(t)D^{-1}Y(0)Y^{-1}(s) & \text{for }\ 0\le s<t\le\omega,\\ Y(t)D^{-1}Y(\omega)Y^{-1}(s) & \text{for }\ 0\le t<s\le\omega,\\ \text{arbitrary} & \text{for }\ t=s.\end{cases}$$

Theorem 1.1. *System* (1.1) *has a unique ω-periodic solution x if and only if the corresponding homogeneous system* (1.1_0) *has only the trivial solution satisfying condition* (1.8), *i.e., when condition* (1.9) *holds, where Y is a fundamental matrix of system* (1.1_0). *If the last condition holds, then the solution x can be written in the form*

$$x(t)=\int\limits_0^\omega d_s\mathcal{G}(t,s)\cdot f(s)\ \ \textit{for}\ \ t\in[0,\omega], \tag{1.10}$$

where $\mathcal{G}:[a,b]\times[a,b]\to\mathbb{R}^{n\times n}$ is the Green matrix of problem $(1.1_0),(1.8)$.

Corollary 1.1. *Let conditions* (1.3) *and* (1.4) *hold, and the matrix-function A satisfy the Lappo–Danilevskiĭ condition. Then system* (1.1) *has a unique ω-periodic solution if and only if*

$$\det\bigg(\exp(S_0(A)(\omega))\prod_{0\le\tau<\omega}(I_n+d_2A(\tau))\prod_{a<\tau\le\omega}(I_n-d_1A(\tau))^{-1}-I_n\bigg)\neq 0. \tag{1.11}$$

Note that if the matrix-function A satisfies the Lappo–Danilevskiĭ condition, then the matrix-function Y defined by $Y(0) = I_n$ and

$$Y(t) \equiv \exp(S_0(A)(t)) \prod_{0\le\tau<t} (I_n + d_2A(\tau)) \prod_{0<\tau\le t} (I_n - d_1A(\tau))^{-1} \text{ for } t \in [0,\omega] \tag{1.12}$$

is the fundamental matrix of system (1.1_0).

Remark 1.1. Let system (1.1_0) have a nontrivial ω-periodic solution. Then there exists $f \in \mathrm{BV}_\omega(\mathbb{R}, \mathbb{R}^n)$ such that system (1.1) has no ω-periodic solution (see [6]).

In general, it is rather difficult to verify condition (1.9) directly even in the case where one is able to write the fundamental matrix of system (1.1_0) explicitly. Therefore, it is important to find effective conditions which would guarantee the absence of nontrivial ω-periodic solutions of the homogeneous system (1.1_0). Below, we will give the results concerning this question. Analogous results have been obtained by T. Kiguradze for ordinary differential equations (see [11, 12]).

Theorem 1.2. *System* (1.1) *has a unique ω-periodic solution if and only if there exist natural numbers k and m such that the matrix*

$$M_k = -\sum_{i=0}^{k-1} \big([A(\omega)]_i - [A(0)]_i\big) \tag{1.13}$$

is nonsingular and

$$r(M_{k,m}) < 1, \tag{1.14}$$

where

$$M_{k,m} = V_m(A)(c) + \Big(\sum_{i=0}^{m-1} \big|[A(\cdot)]_i\big|_s\Big) \cdot |M_k^{-1}| \cdot \big(V_k(A)(\omega) - V_k(A)(0)\big), \tag{1.15}$$

$[A(t)]_i$ $(i = 0,\dots,m-1)$ *and* $V_i(A)(t)$ $(i = 0,\dots,m-1)$ *are defined, respectively, by* (1.5_l) *and* (1.7_l) *for some* $l \in \{1,2\}$, *and* $c = (2-l)\omega$.

Corollary 1.2. *System* (1.1) *has a unique ω-periodic solution if and only if there exist natural numbers k and m such that the matrix*

$$M_k = -\sum_{i=0}^{k-1} \big((A(\omega))_i - (A(0))_i\big) \tag{1.16}$$

is nonsingular and inequality (1.14) *holds, where*

$$M_{k,m} = (V(A)(c))_m + \Big(I_n + \sum_{i=0}^{m-1} \big|(A(\cdot))_i\big|_s\Big) \cdot |M_k^{-1}| \cdot \big[(V(A)(\omega))_k - (V(A)(0))_k\big], \tag{1.17}$$

$(A(t))_i$ $(i = 0,\dots,m-1)$ *and* $(V(A)(t))_i$ $(i = 0,\dots,m-1)$ *are defined by* (1.6_l) *for some* $l \in \{1,2\}$, *and* $c = (2-l)\omega$.

Corollary 1.3. *Let there exist a natural j such that*

$$(A(0))_i = (A(\omega))_i \ \ (i = 1,\dots,j-1) \tag{1.18}$$

and

$$\det\big((A(\omega))_j - (A(0))_j\big) \ne 0, \tag{1.19}$$

where $(A(t))_i$ $(i = 0,\dots,j)$ *are defined by* (1.6_l) *for some* $l \in \{1,2\}$. *Then there exists* $\varepsilon_0 > 0$ *such that the system*

$$dx(t) = \varepsilon dA(t) \cdot x(t) + df(t) \tag{1.20}$$

has one and only one ω-periodic solution for every $\varepsilon \in]0,\varepsilon_0[$.

Theorem 1.3. *Let a matrix-function $A_0 \in \mathrm{BV}_\omega(\mathbb{R}, \mathbb{R}^{n\times n})$ be such that*

$$\det\big(I_n + (-1)^j d_j A_0(t)\big) \neq 0 \ \textit{ for } \ t \in [0,\omega] \ \ (j = 1,2) \tag{1.21}$$

and the homogeneous system

$$dx(t) = dA_0(t) \cdot x(t) \tag{1.22}$$

has only the trivial ω-periodic solution. Let, moreover, the matrix-function $A \in \mathrm{BV}_\omega(\mathbb{R}, \mathbb{R}^{n\times n})$ admit the estimate

$$\int\limits_0^\omega |\mathcal{G}_0(t,\tau)|\, dV(S_0(A - A_0))(\tau) + \sum_{0<\tau\le\omega} \Big|\mathcal{G}_0(t,\tau-) \cdot d_1\big(A(\tau) - A_0(\tau)\big)\Big| + \sum_{0\le\tau<\omega} \Big|\mathcal{G}_0(t,\tau+) \cdot d_2\big(A(\tau) - A_0(\tau)\big)\Big| \le M, \tag{1.23}$$

where $\mathcal{G}_0(t,\tau)$ is the Green matrix of problem (1.22), (1.8), *and $M \in \mathbb{R}_+^{n\times n}$ is a constant matrix such that*

$$r(M) < 1. \tag{1.24}$$

Then system (1.1) *has one and only one ω-periodic solution.*

Formula (1.10) can be written in a simpler form if we introduce the concept of the Green matrix for problem $(1.1_0), (1.2)$.

Definition 1.2. A matrix-function $\mathcal{G}_\omega : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{n\times n}$ is said to be the Green matrix of problem $(1.1_0), (1.2)$ if:

(a)

$$\mathcal{G}_\omega(t+\omega, \tau+\omega) = \mathcal{G}_\omega(t,\tau), \ \ \mathcal{G}_\omega(t, t+\omega) - \mathcal{G}_\omega(t,t) = I_n \ \text{ for } \ t,\tau \in \mathbb{R}; \tag{1.25}$$

(b) the matrix-function $\mathcal{G}_\omega(\,\cdot\,,\tau) : \mathbb{R} \to \mathbb{R}^{n\times n}$ is a fundamental matrix of system (1.1_0) for every $\tau \in \mathbb{R}$.

Theorem 1.4. *Let condition* (1.3) *hold,*

$$\det\big(I_n \pm d_j A(t)\big) \neq 0 \ \textit{ for } \ t \in \mathbb{R} \ \ (j = 1,2), \tag{1.26}$$

and system (1.1_0) have a unique ω-periodic solution. Then system (1.1) *has likewise a unique ω-periodic solution x which is written in the form*

$$x(t) = \int\limits_t^{t+\omega} \mathcal{G}_\omega(t,\tau)\, d\mathcal{A}\big(A, \mathcal{A}(-A, f)\big)(\tau) \ \textit{ for } \ t \in \mathbb{R}, \tag{1.27}$$

where $\mathcal{G}_\omega$ is the Green matrix of problem $(1.1_0), (1.2)$.

We introduce the following class of matrix-functions.

Let $m, r_1, \dots, r_m$ and $n_1, \dots, n_m$ $(0 = n_0 < n_1 < \dots < n_m = n)$ be natural numbers; $\sigma_j \in \{-1, 1\}$ $(j = 1, \dots, m)$; $g_{lj} : [0,\omega] \to \mathbb{R}$ $(l = 1, \dots, r_j;\ j = 1, \dots, m)$ be nondecreasing functions; $\alpha_{lj} \in L([0,\omega], \mathbb{R}; g_{lj})$ $(l = 1, \dots, r_j;\ j = 1, \dots, m)$, and let matrix-functions $\mathcal{P}_{lj} = (p_{ljik})_{i,k=1}^n$ $(l = 1, \dots, r_j;$ $j = 1, \dots, m)$ be such that $p_{ljik} \in L([0,\omega], \mathbb{R}; g_{lj})$ $(i, k = n_{j-1}+1, \dots, n_j)$ and

$$\sigma_j \sum_{i,k=n_{j-1}+1}^{n_j} p_{ljik}(t) x_i x_k \ge \alpha_{lj}(t) \sum_{i,k=n_{j-1}+1}^{n_j} x_i^2 \ \text{ for } \ \mu(g_{lj})\text{-almost all } \ t \in [0,\omega], \ \ (x_i)_{i=1}^n \in \mathbb{R}^n$$
$$(l = 1, \dots, r_j; \ \ j = 1, \dots, m). \tag{1.28}$$

Then by $Q_\omega((r_j,n_j,\sigma_j)_j^m,(g_{lj},\alpha_{lj},\mathcal{P}_{lj})_{l=1,j=1}^{r_j,m})$ we denote the set of all matrix-functions $A \in \mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ such that

$$a_{ik}(t)\equiv 0 \ \ (i=n_{j-1}+1,\dots,n_j;\ \ k=n_j+1,\dots,n;\ \ j=1,\dots,m), \tag{1.29}$$

$$\sigma_j\bigg(b_{jii}(t)-b_{jii}(s)-\sum_{l=1}^{r_j}\int_s^t p_{ljii}(\tau)dg_{lj}(\tau)\bigg)\geq 0 \ \text{ for } \ 0\leq s\leq t\leq\omega \quad (i=n_{j-1}+1,\dots,n_j;\ \ j=1,\dots,m) \tag{1.30}$$

and

$$b_{jik}(t)=\sum_{l=1}^{r_j}\int_s^t p_{ljik}(\tau)\,dg_{lj}(\tau) \ \text{ for } \ t\in[0,\omega] \ \ (i\neq k;\ \ i,k=n_{j-1}+1,\dots,n_j;\ \ j=1,\dots,m), \tag{1.31}$$

where

$$b_{jik}(t)\equiv a_{ik}(t)-\bigg(\frac{1}{2}\sum_{0<\tau\leq t}\sum_{r=n_{j-1}+1}^{n_j}d_1a_{ri}(\tau)\cdot d_1a_{rk}(\tau)-\sum_{0\leq\tau<t}\sum_{r=n_{j-1}+1}^{n_j}d_2a_{ri}(\tau)\cdot d_2a_{rk}(\tau)\bigg) \quad (i,k=n_{j-1}+1,\dots,n_j;\ \ j=1,\dots,m). \tag{1.32}$$

If $s\in\mathbb{R}$ and $\beta\in\mathrm{BV}([0,\omega],\mathbb{R})$ are such that

$$1+(-1)^jd_j\beta(t)\neq 0 \ \text{ for } \ (-1)^j(t-s)<0 \ \ (j=1,2),$$

then by $\gamma_s(\beta)$ we write a unique solution of the Cauchy problem

$$d\gamma(t)=\gamma(t)\,d\beta(t),\ \ \gamma(s)=1.$$

Notice that condition (1.4) guarantees the unique solvability of the Cauchy problem for system (1.1) (see, e.g., [13,14]).

It is known (see [9,10]) that

$$\gamma_s(\beta)(t)=\begin{cases}\exp\big(s_0(\beta)(t)-s_0(\beta)(s)\big)\displaystyle\prod_{s<\tau\leq t}(1-d_1\beta(\tau))^{-1}\prod_{s\leq\tau<t}(1+d_2\beta(\tau)) & \text{for } \ s<t\leq\omega,\\ \exp\big(s_0(\beta)(t)-s_0(\beta)(s)\big)\displaystyle\prod_{t<\tau\leq s}(1-d_1\beta(\tau))\prod_{t\leq\tau<s}(1+d_2\beta(\tau))^{-1} & \text{for } \ 0\leq t<s.\end{cases} \tag{1.33}$$

Theorem 1.5. *Let there exist natural numbers m, $r_1,\dots,r_m$ and $n_1,\dots,n_m$ $(0=n_0<n_1<\cdots<n_m=n)$, $\sigma_j\in\{-1,1\}$ $(j=1,\dots,m)$, nondecreasing functions $g_{lj}:[0,\omega]\to\mathbb{R}$ $(l=1,\dots,r_j;$ $j=1,\dots,m)$, functions $\alpha_{lj}\in L([0,\omega],\mathbb{R};g_{lj})$ $(l=1,\dots,r_j;\ j=1,\dots,m)$ and matrix-functions $\mathcal{P}_{lj}=(p_{ljik})_{i,k=1}^n$ $(l=1,\dots,r_j;\ j=1,\dots,m)$, $p_{ljik}\in L([0,\omega],\mathbb{R};g_{lj})$ $(i,k=n_{j-1}+1,\dots,n_j)$ such that*

$$A\in Q_\omega\big((r_j,n_j,\sigma_j)_j^m,(g_{lj},\alpha_{lj},\mathcal{P}_{lj})_{l=1,j=1}^{r_j,m}\big). \tag{1.34}$$

Let, moreover,

$$(1-\sigma_j)d_1g_j(t)+(1+\sigma_j)d_2g_j(t)\neq -2 \ \textit{ for } \ t\in[0,\omega] \ \ (j=1,\dots,m) \tag{1.35}$$

and

$$\gamma_{t_j}(\sigma_jg_j)(\omega-t_j)<1 \ \ (j=1,\dots,m), \tag{1.36}$$

where $t_j=\frac{1}{2}\,(1+\sigma_j)\omega$, the functions $\gamma_{t_j}(\sigma_jg_j)$ $(j=1,\dots,m)$ are defined by (1.33), and

$$g_j(t)\equiv 2\sum_{l=1}^{r_j}\int_0^t\alpha_{lj}(\tau)\,dg_{lj}(\tau).$$

Then system (1.1) has a unique ω-periodic solution.

Remark 1.2. In the above theorem, if in addition to condition (1.35), the condition

$$(1+\sigma_j)\,d_1g_j(t)+(1-\sigma_j)\,d_2g_j(t)<2 \tag{1.37}$$

holds, then, by (1.33), inequality (1.36) is equivalent to

$$\exp\big(s_0(g_j)(\omega)\big)>-\frac{1}{2}\bigg((1+\sigma_j)\prod_{0<\tau\le\omega}(1-d_1g_j(\tau))\prod_{0\le\tau<\omega}(1+d_1g_j(\tau))^{-1}$$
$$+(1-\sigma_j)\prod_{0<\tau\le\omega}(1+d_1g_j(\tau))^{-1}\prod_{0\le\tau<\omega}(1-d_2g_j(\tau))\bigg)\ \text{ for } t\in[0,\omega]\ \ (j=1,\dots,m).$$

Let $g:[0,\omega]\to\mathbb{R}$ be a nondecreasing function and $P=(p_{ik})_{i,k=1}^n$, where $p_{ik}\in L([0,\omega],\mathbb{R};g)$ $(i,k=1,\dots,n)$. Then we denote by $Q_\omega(P;g)$ the set of all matrix-functions $A=(a_{ik})_{i,k=1}^n\in\mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ such that

$$b_{ik}(t)=\int\limits_0^t p_{ik}(\tau)\,dg(\tau)\ \text{ for } t\in[0,\omega]\ \ (i,k=1,\dots,n), \tag{1.38}$$

where

$$b_{ik}(t)\equiv a_{ik}(t)-\frac{1}{2}\bigg(\sum_{l=1}^n\sum_{0<\tau\le t}d_1a_{li}(\tau)\cdot d_1a_{lk}(\tau)-\sum_{0\le\tau<t}d_2a_{li}(\tau)\cdot d_2a_{lk}(\tau)\bigg)\ \ (i,k=1,\dots,n). \tag{1.39}$$

Theorem 1.6. *Let $A\in Q_\omega(P;g)$. Let, moreover, either*

(a)

$$\sum_{i,k=1}^n p_{ik}(t)x_ix_k\ge\alpha(t)\sum_{i=1}^n x_i^2\ \text{ for }\ \mu(g)-a.a.\ t\in[0,\omega],\ \ (x_i)_{i=1}^n\in\mathbb{R}^n, \tag{1.40}$$

$$1-2\alpha(t)d_1g(t)>0,\quad 1+2\alpha(t)d_2g(t)\ne0\ \text{ for } 0\le t<\omega, \tag{1.41}$$

$$\gamma_\omega(\alpha)(0)<1 \tag{1.42}$$

or

(b)

$$\sum_{i,k=1}^n p_{ik}(t)x_ix_k\le\beta(t)\sum_{i=1}^n x_i^2\ \text{ for }\ \mu(g)-a.a.\ t\in[0,\omega],\ \ (x_i)_{i=1}^n\in\mathbb{R}^n, \tag{1.43}$$

$$1+2\beta(t)d_2g(t)>0,\quad 1-2\beta(t)d_1g(t)\ne0\ \text{ for } 0<t\le\omega, \tag{1.44}$$

$$\gamma_0(\beta)(\omega)<1, \tag{1.45}$$

where $\alpha,\beta\in L([0,\omega],\mathbb{R};g)$, the function $\gamma_0(\beta)$ is defined by (1.33), *and*

$$g_\alpha(t)\equiv2\int\limits_0^t\alpha(\tau)\,dg(\tau)\ \text{ and }\ g_\beta(t)\equiv2\int\limits_0^t\beta(\tau)\,dg(\tau). \tag{1.46}$$

Then system (1.1) *has a unique ω-periodic solution.*

Corollary 1.4. *Let $A\in Q_\omega(P;g)$. Let, moreover, either* (a) *conditions* (1.41) *and* (1.42) *hold, or* (b) *conditions* (1.44) *and* (1.45) *hold, where the functions g_α and g_β are defined by* (1.46), *$\alpha(t)\equiv\lambda_0(P^*(t))$, $\beta(t)\equiv\lambda^0(P^*(t))$, and $P^*(t)\equiv P(t)+P^T(t)$. Then system* (1.1) *has a unique ω-periodic solution.*

2 Auxiliary propositions

Lemma 2.1. *The following statements are valid:*

(a) *if x is a solution of system* (1.1)*, then the vector-function $y(t) = x(t+\omega)$ $(t \in \mathbb{R})$ will be a solution of system* (1.1)*, as well;*

(b) *problem* (1.1), (1.2) *is solvable if and only if system* (1.1) *has on the closed interval $[0, \omega]$ a solution satisfying the boundary condition* (1.8)*. Moreover, the set of restrictions of solutions of problem* (1.1), (1.2) *on $[0, \omega]$ coincides with the set of solutions of problem* (1.1), (1.8).

Proof. Let x be an arbitrary solution of system (1.1). Assume $y(t) = x(t+\omega)$ for $t \in \mathbb{R}$. Then, by (1.3), we have

$$\begin{aligned} y(t) &= x(0) + \int_0^{t+\omega} dA(\tau) \cdot x(\tau) + f(t+\omega) - f(0) \\ &= x(0) + \int_0^{\omega} dA(\tau) \cdot x(\tau) + f(\omega) - f(0) + \int_{\omega}^{t+\omega} dA(\tau) \cdot x(\tau) + f(t+\omega) - f(\omega) \\ &= x(\omega) + \int_0^{t} dA(\tau+\omega) \cdot x(\tau+\omega) + f(t+\omega) - f(\omega) \\ &= y(0) + \int_0^{t} dA(\tau) \cdot y(\tau) + f(t) - f(0) \ \text{ for } \ t \in \mathbb{R}. \end{aligned}$$

Therefore, y is likewise a solution of system (1.1). Thus statement (a) is proved.

Let us show statement (b). It is evident that the restrictions of every solution of problem (1.1), (1.2) on the interval $[0, \omega]$ will be a solution of problem (1.1), (1.8). Consider now an arbitrary solution x of problem (1.1), (1.8). Any continuation of this solution we again denote by x. According to statement (a), the vector-function $y(t) = x(t+\omega)$ will be a solution of system (1.1), as well. On the other hand, in view of (1.8), we have

$$y(0) = x(\omega) = x(0).$$

This implies that the functions x and y are the solutions of system (1.1) under the common initial value condition. So, $x(t) \equiv y(t)$. Therefore, x is a solution of problem (1.1), (1.2). □

Lemma 2.2. *An arbitrary fundamental matrix Y of system* (1.1_0) *satisfies the identity*

$$Y(t+\omega) = Y(t)Y^{-1}(0)Y(\omega) \ \text{ for } \ t \in \mathbb{R}. \tag{2.1}$$

Proof. By Lemma 2.1, the columns of the matrix-function $Z(t) = Y(t+\omega)$ are the solutions of system (1.1_0). Therefore, there exists a constant matrix $C \in \mathbb{R}$ such that

$$Z(t) = Y(t)C \ \text{ for } \ t \in \mathbb{R}.$$

Thus it is clear that

$$C = Y^{-1}(0)Z(0) = Y^{-1}(0)Y(\omega).$$

Hence equality (2.1) holds. □

Lemma 2.3. *Let problem* (1.1_0), (1.2) *have only the trivial solution. Then there exists a unique Green matrix of the problem having the following form:*

$$\mathcal{G}_\omega(t,\tau) = Y(t)\big(Y^{-1}(\omega)Y(0) - I_n\big)^{-1}Y^{-1}(\tau) \ \text{ for } \ t, \tau \in \mathbb{R}, \tag{2.2}$$

where Y is a fundamental matrix of system (1.1_0).

Proof. Let Y be an arbitrary fundamental matrix of system (1.1_0). Then, by Lemma 2.1, condition (1.9) holds because the lemma guarantees the validity of Theorem 1.1 (see the proof of Theorem 1.1 below). According to Definition 1.2, the matrix-function $\mathcal{G}_\omega : \mathbb{R}\times\mathbb{R}\to\mathbb{R}^{n\times n}$ is the Green matrix if and only if

$$\mathcal{G}_\omega(t,\tau)=Y(t)C(\tau)\ \text{ for }\ t,\tau\in\mathbb{R}, \tag{2.3}$$

where the matrix-function $C:\mathbb{R}\to\mathbb{R}^{n\times n}$ is such that equalities (1.25) hold, i.e.,

$$Y(t+\omega)C(\tau+\omega)=Y(t)C(\tau),\quad Y(t)\big(C(t+\omega)-C(t)\big)=I_n\ \text{ for }\ t,\tau\in\mathbb{R}. \tag{2.4}$$

By equality (2.1), equalities (2.4) hold if and only if

$$Y^{-1}(0)Y(\omega)C(\tau+\omega)=C(\tau),\quad C(\tau+\omega)-C(\tau)=Y^{-1}(\tau)\ \text{ for }\ \tau\in\mathbb{R}.$$

Clearly,

$$\big(I_n-Y^{-1}(0)Y(\omega)\big)C(\tau)=Y^{-1}(0)Y(\omega)Y^{-1}(\tau)\ \text{ for }\ \tau\in\mathbb{R}.$$

Therefore, taking into account condition (1.9), we conclude that

$$C(\tau)=\big(Y^{-1}(\omega)Y(0)-I_n\big)^{-1}Y^{-1}(\tau)\ \text{ for }\ \tau\in\mathbb{R}.$$

Putting the obtained value of $C(t)$ in (2.4), we obtain equality (2.2). □

Lemma 2.4. *If $X\in \mathrm{BV}_\omega(\mathbb{R},\mathbb{R}^{n\times n})$ and $Y\in \mathrm{BV}_\omega(\mathbb{R},\mathbb{R}^{n\times m})$, then*

(a)

$$d_jX(t+\omega)=d_jX(t)\ \textit{ for }\ t\in\mathbb{R}\ \ (j=1,2); \tag{2.5}$$

(b)

$$\mathcal{A}(X,Y)\in \mathrm{BV}_\omega(\mathbb{R},\mathbb{R}^{n\times m}),\ \ \textit{i.e.},\ \ \mathcal{A}(X,Y)(t+\omega)=\mathcal{A}(X,Y)(t)+C\ \textit{ for }\ t\in\mathbb{R}, \tag{2.6}$$

where C is some constant $n\times n$-matrix.

Proof. Consider equality (2.5). Let $j=1$. Then by the definition of the set $\mathrm{BV}_\omega(\mathbb{R},\mathbb{R}^{n\times m})$, we have

$$\begin{aligned}d_1X(t+\omega)&=\lim_{\varepsilon\to0,\ \varepsilon>0}\big(X(t+\omega)-X(t+\omega-\varepsilon)\big)\\&=\lim_{\varepsilon\to0,\ \varepsilon>0}\big(X(t)-X(t-\varepsilon)\big)=d_1X(t)\ \text{ for }\ t\in\mathbb{R}.\end{aligned}$$

Analogously, we show equality (2.5) for $j=2$.

Let us show (2.6). From the definition of the operator $\mathcal{A}$ and equalities (2.5), we conclude that

$$\begin{aligned}\mathcal{A}(X,Y)(t+\omega)&=Y(t+\omega)-Y(0)+\sum_{0<\tau\le t+\omega}d_1X(\tau)\cdot(I_n-d_1X(\tau))^{-1}\,d_1Y(\tau)\\&\quad-\sum_{0\le\tau<t+\omega}d_2X(\tau)\cdot(I_n+d_2X(\tau))^{-1}\,d_2Y(\tau)\\&=Y(t+\omega)-Y(0)+C_0+\sum_{\omega<\tau\le t+\omega}d_1X(\tau)\cdot(I_n-d_1X(\tau))^{-1}\,d_1Y(\tau)\\&\quad-\sum_{0\le\tau<t}d_2X(\tau+\omega)\cdot(I_n+d_2X(\tau+\omega))^{-1}\,d_2Y(\tau+\omega)\\&=Y(t+\omega)-Y(0)+C_0+\sum_{0<\tau\le t}d_1X(\tau+\omega)\cdot(I_n-d_1X(\tau+\omega))^{-1}\,d_1Y(\tau+\omega)\\&\quad-\sum_{0\le\tau<t}d_2X(\tau)\cdot(I_n+d_2X(\tau+\omega))^{-1}\,d_2Y(\tau+\omega)=\mathcal{A}(X,Y)(t)+C,\end{aligned}$$

where

$$C_0=\sum_{0<\tau\le\omega}d_1X(\tau)\cdot(I_n-d_1X(\tau))^{-1}\,d_1Y(\tau)-\sum_{0\le\tau<\omega}d_2X(\tau)\cdot(I_n+d_2X(\tau))^{-1}\,d_2Y(\tau),$$

and C is some constant matrix. □

Lemma 2.5. *Let $g : [a,b] \to \mathbb{R}$ be a nondecreasing function, $t_0 \in [a,b]$ and $c_0 \in \mathbb{R}$. Let, in addition, $z \in \mathrm{BV}([a,b],\mathbb{R})$ be such that*

$$\begin{gathered}\big(dz(t) - z(t)\,dg(t)\big)\operatorname{sgn}(t-t_0) \le 0 \ \ \text{for} \ \ t \in [a,b],\\ 1 - d_1 g(t) > 0, \quad 1 + d_2 g(t) \neq 0 \ \ \text{for} \ \ a \le t < t_0,\\ 1 + d_2 g(t) > 0, \quad 1 - d_1 g(t) \neq 0 \ \ \text{for} \ \ t_0 < t \le b,\\ (-1)^j\big(d_j z(t_0) - c_0 d_j g(t_0)\big) \le 0 \ \ (j=1,2)\end{gathered}$$

and $z(t_0) \le c_0$. Then

$$z(t) \le x(t) \ \ \text{for} \ \ t \in [a,b],$$

where x is a unique solution of the problem

$$\begin{gathered}dx(t) = x(t)\,dg(t) \ \ \text{for} \ \ t \in [a,b],\\ x(t_0) = c_0.\end{gathered}$$

The above lemma is a particular case of Lemma 2.4 from the paper [1].

Lemma 2.6. *If C is a symmetric matrix, then the inequalities*

$$\lambda_0(C)(x*x) \le Cx*x \le \lambda^0(C)(x*x)$$

hold for every $x \in \mathbb{R}$.

The lemma is proved in [11, Lemma 1.9].

3 Proof of the results

By Lemma 2.1, Theorem 1.1 follows immediately from the corresponding results of the papers [6, 15], and Theorems 1.2, 1.3 and Corollaries 1.1–1.3 follow immediately from Theorems 2.1, 2.2 and Corollaries 2.2–2.4 of [7], respectively, if we assume that the linear operator l appearing there has the form $l(x) \equiv x(0) - x(\omega)$. Note that condition (1.9) has form (1.11) when the fundamental matrix of system (1.1_0) is given by (1.12) in Corollary 1.1.

Proof of Theorem 1.4. By Theorem 1.1 and Lemma 2.3, problem (1.1), (1.2) is uniquely solvable, and problem (1.1_0), (1.2) has the unique Green matrix $\mathcal{G}_\omega$. Therefore, for the proof it is sufficient to verify that the vector-function given by (1.27) is the ω-periodic solution of system (1.1).

Assume

$$\varphi(t) = \mathcal{A}(-A,f)(t) \ \ \text{for} \ \ t \in \mathbb{R}.$$

Let us show that the vector-function x defined by (1.27) satisfies condition (1.2). By Lemma 2.4, it is evident that $\mathcal{A}(A,\varphi) \in \mathrm{BV}_\omega(\mathbb{R},\mathbb{R}^n)$ and, therefore,

$$\mathcal{A}(A,\varphi)(t+\omega) = \mathcal{A}(A,\varphi)(t) + c \ \ \text{for} \ \ t \in \mathbb{R}, \tag{3.1}$$

where c is some constant n-vector. Taking into account (3.1) and (1.27), due to (1.25) we have

$$x(t+\omega) = \int\limits_{t+\omega}^{t+2\omega} \mathcal{G}_\omega(t+\omega,\tau)\,d\mathcal{A}(A,\varphi)(\tau) = \int\limits_{t}^{t+\omega} \mathcal{G}_\omega(t+\omega,\tau+\omega)\,d\mathcal{A}(A,\varphi)(\tau+\omega) = x(t).$$

Let us verify that the vector-function x satisfies system (1.1). By equality (2.2),

$$\mathcal{G}_\omega(t,\tau) = Y(t)C_\omega Y^{-1}(\tau) \ \ \text{for} \ \ t,\tau \in \mathbb{R},$$

where Y is a fundamental matrix of system (1.1_0), and

$$C_\omega = \big(Y^{-1}(\omega)Y(0) - I_n\big)^{-1}.$$

Thus, using the general integration by parts formula, we find that

$$\begin{aligned} x(t)-x(s)=\int_s^t dx(\tau)=\int_s^t d\Big(\int_\tau^{\tau+\omega}\mathcal{G}_\omega(\tau,\eta)\,d\mathcal{A}(A,\varphi)(\eta)\Big)=\int_s^t d\Big(Y(\tau)C_\omega\int_\tau^{\tau+\omega}Y^{-1}(\eta)\,d\mathcal{A}(A,\varphi)(\eta)\Big)\\ =\int_s^t dY(\tau)\cdot C_\omega\int_\tau^{\tau+\omega}Y^{-1}(\eta)\,d\mathcal{A}(A,\varphi)(\eta)+\int_s^t Y(\tau)C_\omega\,d\Big(\int_\tau^{\tau+\omega}Y^{-1}(\eta)d\mathcal{A}(A,\varphi)(\eta)\Big)\\ -\sum_{s<\eta\le t}d_1Y(\tau)\cdot C_\omega\,d_1\Big(\int_\tau^{\tau+\omega}Y^{-1}(\eta)\,d\mathcal{A}(A,\varphi)(\eta)\Big)\\ +\sum_{s\le\eta<t}d_2Y(\tau)\cdot C_\omega\,d_2\Big(\int_\tau^{\tau+\omega}Y^{-1}(\eta)\,d\mathcal{A}(A,\varphi)(\eta)\Big)\ \text{ for }\ s<t. \end{aligned}\tag{3.2}$$

On the other hand, due to (2.1),

$$Y^{-1}(t+\omega)-Y^{-1}(t)\equiv C_\omega^{-1}Y^{-1}(t).\tag{3.3}$$

By (3.1), for $\tau\in\mathbb{R}$, we conclude that

$$\begin{aligned} \int_\tau^{\tau+\omega}Y^{-1}(\eta)\,d\mathcal{A}(A,\varphi)(\eta)=\int_\tau^{\omega}Y^{-1}(\eta)\,d\mathcal{A}(A,\varphi)(\eta)+\int_\omega^{\tau+\omega}Y^{-1}(\eta)\,d\mathcal{A}(A,\varphi)(\eta)\\ =\int_\tau^{\omega}Y^{-1}(\eta)\,d\mathcal{A}(A,\varphi)(\eta)+\int_0^{\tau}Y^{-1}(\eta+\omega)\,d\mathcal{A}(A,\varphi)(\eta+\omega)\\ =\int_0^{\omega}Y^{-1}(\eta)\,d\mathcal{A}(A,\varphi)(\eta)+\int_0^{\tau}\big(Y^{-1}(\eta+\omega)-Y^{-1}(\eta)\big)\,d\mathcal{A}(A,\varphi)(\eta). \end{aligned}$$

Hence, taking into account (3.3), we get

$$\int_\tau^{\tau+\omega}Y^{-1}(\eta)\,d\mathcal{A}(A,\varphi)(\eta)\equiv\int_0^{\omega}Y^{-1}(\eta)\,d\mathcal{A}(A,\varphi)(\eta)+C_\omega^{-1}\int_0^{\tau}Y^{-1}(\eta)\,d\mathcal{A}(A,\varphi)(\eta).$$

Due to the last equality and the general integration-by-parts formula, taking into account the equalities

$$dY(t)=dA(t)\cdot Y(t)\ \text{ and }\ d_jY(t)=d_jA(t)\cdot Y(t)\ \text{ for }\ t\in\mathbb{R}\ \ (j=1,2),$$

it follows from (3.2) that

$$\begin{aligned} x(t)-x(s)=\int_s^t dA(\tau)\cdot Y(\tau)C_\omega\int_\tau^{\tau+\omega}Y^{-1}(\eta)\,d\mathcal{A}(A,\varphi)(\eta)+F(s,t)\\ =\int_s^t dA(\tau)\cdot x(\tau)+F(s,t)\ \text{ for }\ s,t\in\mathbb{R},\ \ s<t, \end{aligned}\tag{3.4}$$

where

$$\begin{aligned} F(s,t)=\mathcal{A}(A,\varphi)(t)-\mathcal{A}(A,\varphi)(s)\\ -\sum_{s<\tau\le t}d_1A(\tau)\cdot d_1\mathcal{A}(A,\varphi)(\tau)+\sum_{s\le\tau<t}d_2A(\tau)\cdot d_2\mathcal{A}(A,\varphi)(\tau)\ \text{ for }\ s,t\in\mathbb{R},\ \ s<t. \end{aligned}$$

Moreover, taking into account condition (1.26), according to the definition of the operator $\mathcal{A}$ and the function φ, we conclude that

$$d_1\varphi(\tau) = d_1 f(\tau) - \sum_{s<\tau\leq t} d_1A(\tau)\cdot\big(I_n + d_1A(\tau)\big)^{-1} d_1 f(\tau) \ \text{ for } \ \tau\in\mathbb{R},$$

and

$$d_2\varphi(\tau) = d_2 f(\tau) + \sum_{s\leq\tau<t} d_2A(\tau)\cdot\big(I_n - d_2A(\tau)\big)^{-1} d_2 f(\tau) \ \text{ for } \ \tau\in\mathbb{R}.$$

Using the last equalities, we can easily show that

$$\begin{aligned}F(s,t) = \varphi(t) - \varphi(s) &+ \sum_{s<\tau\leq t} d_1A(\tau)\cdot(I_n - d_1A(\tau))^{-1} d_1\varphi(\tau)\\ &- \sum_{s\leq\tau<t} d_2A(\tau)\cdot(I_n + d_2A(\tau))^{-1} d_2\varphi(\tau)\\ &- \sum_{s<\tau\leq t} (d_1A(\tau))^2\cdot(I_n - d_1A(\tau))^{-1} d_1\varphi(\tau)\\ &- \sum_{s\leq\tau<t} (d_2A(\tau))^2\cdot(I_n + d_2A(\tau))^{-1} d_2\varphi(\tau)\\ = \varphi(t) - \varphi(s) &+ \sum_{s<\tau\leq t} d_1A(\tau)\cdot d_1\varphi(\tau) - \sum_{s\leq\tau<t} d_2A(\tau)\cdot d_2\varphi(\tau)\\ = f(t) - f(s) &\quad \text{for } \ s,t\in\mathbb{R}, \ \ s<t.\end{aligned}$$

Consequently, due to (3.4), the vector-function x satisfies equation (1.1). □

Proof of Theorem 1.5. According to Theorem 1.1, to prove the theorem, it suffices to show that the homogeneous system (1.1_0) has only the trivial ω-periodic solution. Let $x = (x_i)_{i=1}^n$ be an arbitrary solution of the latter problem. Assume

$$u_j(t) = \sum_{i=n_{j-1}+1}^{n_j} x_i^2(t) \ \text{ for } \ t\in[0,\omega] \ \ (j=1,\dots,m).$$

By condition (1.34), conditions (1.28)–(1.32) are fulfilled. In view of (1.29) and the formula of integration by parts, we find that

$$\begin{aligned}\sigma_1(u_1(t) - u_1(s)) &= \sigma_1\sum_{i=1}^{n_1}\bigg(2\int_s^t x_i(\tau)\,dx_i(\tau) - \sum_{s<\tau\leq t}(d_1x_i(t))^2 + \sum_{s\leq\tau<t}(d_2x_i(t))^2\bigg)\\ &= \sigma_1\sum_{i=1}^{n_1}\bigg(2\int_s^t x_i(\tau)x_k(\tau)\,da_{ik}(\tau) + \sum_{s<\tau\leq t}\big(x_i^2(\tau) - x_i^2(\tau-) - 2x_i(\tau)\,d_1x_i(\tau)\big)\\ &\qquad + \sum_{s\leq\tau<t}\big(x_i^2(\tau+) - x_i^2(\tau) - 2x_i(\tau)\,d_2x_i(\tau)\big)\bigg)\\ &= 2\sigma_1\sum_{i,k=1}^{n_1}\bigg(\int_s^t x_i(\tau)x_k(\tau)\,da_{ik}(\tau) - \sum_{s<\tau\leq t} x_i(\tau)x_k(\tau)\,d_1a_{ik}(\tau) - \sum_{s\leq\tau<t} x_i(\tau)x_k(\tau)\,d_2a_{ik}(\tau)\bigg)\\ &\qquad + \sigma_1\sum_{j=1}^{2}\big(s_j(u_1)(t) - s_j(u_1)(s)\big) \ \text{ for } \ 0\leq s\leq t\leq\omega.\end{aligned}$$

Hence,

$$\sigma_1\big(u_1(t)-u_1(s)\big)=2\sigma_1\sum_{i,k=1}^{n_1}\int_s^t x_i(\tau)x_k(\tau)\,ds_0(a_{ik})(\tau)$$
$$+\sigma_1\sum_{j=1}^{2}\big(s_j(u_1)(t)-s_j(u_1)(s)\big)\ \text{ for }\ 0\le s\le t\le\omega.$$

On the other hand,

$$\sigma_1\sum_{j=1}^{2}\big(s_j(u_1)(t)-s_j(u_1)(s)\big)$$
$$=\sum_{i,k=1}^{n_1}\bigg\{\sum_{s<\tau\le t}d_1x_i(\tau)\cdot\big(2x_i(\tau)-d_1x_i(\tau)\big)+\sum_{s\le\tau<t}d_2x_i(\tau)\cdot\big(2x_i(\tau)+d_2x_i(\tau)\big)\bigg\}$$
$$=2\sum_{i,k=1}^{n_1}\bigg\{\sum_{s<\tau\le t}x_i(\tau)x_k(\tau)\Big(d_1a_{ik}(\tau)-\frac{1}{2}\sum_{r=1}^{n_1}d_1a_{ri}(\tau)\cdot d_1a_{rk}(\tau)\Big)$$
$$+\sum_{s\le\tau<t}x_i(\tau)x_k(\tau)\Big(d_2a_{ik}(\tau)-\frac{1}{2}\sum_{r=1}^{n_1}d_2a_{ri}(\tau)\cdot d_2a_{rk}(\tau)\Big)\bigg\}\ \text{ for }\ 0\le s\le t\le\omega.$$

From this and (1.32), we obtain

$$\sigma_1(u_1(t)-u_1(s))=2\sigma_1\sum_{i,k=1}^{n_1}\int_s^t x_i(\tau)x_k(\tau)\,db_{1ik})(\tau)\ \text{ for }\ 0\le s\le t\le\omega. \tag{3.5}$$

With regard for (1.28)–(1.31), it follows from (3.5) that

$$\sigma_1(u_1(t)-u_1(s))=2\sigma_1\sum_{i=1}^{n_1}\int_s^t x_i^2(\tau)\,db_{1ii}(\tau)+2\sigma_1\sum_{i\neq k;i,k=1}^{n_1}\int_s^t x_i(\tau)x_k(\tau)\,db_{1ik}(\tau)$$
$$\ge 2\sigma_1\sum_{l=1}^{r_1}\sum_{i,k=1}^{n_1}\int_s^t p_{l1ik}(\tau)x_i(\tau)x_k(\tau)\,dg_{l1}(\tau)\ge 2\sum_{l=1}^{r_1}\int_s^t \alpha_{l1}(\tau)\sum_{i=1}^{n_1}x_i^2(\tau)\,dg_{l1}(\tau),$$

i.e.,

$$\sigma_1(u_1(t)-u_1(s))\ge\int_s^t u_1(\tau)\,dg_1(\tau)\ \text{ for }\ 0\le s\le t\le\omega.$$

Moreover, by (1.35), the conditions of Lemma 2.5 are fulfilled for $t_0=t_1$, $c_0=u_1(t_0)$ and $g(t)\equiv g_1(t)$. In addition, by (1.35),

$$1+(-1)^jd_jg_1(t)\neq 0\ \text{ for }\ t\in[0,\omega]\ \ (j=1,2)$$

and, therefore, the problem

$$dx(t)=\sigma_1x(t)\,dg_1(t),\quad x(t_0)=c_0$$

has a unique solution x given by

$$x(t)=c_0\gamma_{t_0}(\sigma_1g_1)(t)\ \text{ for }\ t\in[0,\omega],$$

where the function $\gamma_{t_0}(\sigma_1g_1)(t)$ is defined by (1.33). According to Lemma 2.5, we have

$$u_1(t)\le c_0\gamma_{t_0}(\sigma_1g_1)(t)\ \text{ for }\ t\in[0,\omega]. \tag{3.6}$$

Due to (1.8), we have $u_1(0) = u_1(\omega)$. Hence, it follows from (3.6) that

$$u_1(\omega - t_1) \le u_1(t_1)\gamma_{t_1}(\sigma_1 g_1)(\omega - t_1) = u_1(\omega - t_1)\,\gamma_{t_1}(\sigma_1 g_1)(\omega - t_1).$$

Therefore, due to (1.36),

$$c_0 = u_1(0) = u_1(\omega) = 0,$$

and thus, by (3.6), we have

$$u_1(t) \equiv 0.$$

Using this identity and (1.28)–(1.32), by induction, we prove $u_j(t) \equiv 0$ $(j = 1, \dots, m)$. Consequently, $x_i(t) = 0$ for $t \in [0, \omega]$ $(i = 1, \dots, n)$. □

Proof of Theorem 1.6. According to Theorem 1.1, to prove the theorem, it suffices to show that the homogeneous system (1.1_0) has only the trivial ω-periodic solution. Let $x = (x_i)_{i=1}^n$ be an arbitrary solution of the latter problem. Assume

$$u(t) = \sum_{i=1}^n x_i^2(t) \ \text{ for } \ t \in [0, \omega].$$

Consider the case (a). Analogously to the proof of equality (3.5) in Theorem 1.5, using (1.39), we can show that the equality

$$u(t) - u(s) = 2\sum_{i=1}^n \int_s^t x_i(\tau)x_k(\tau)\,db_{ik}(\tau) \ \text{ for } \ 0 \le s \le t \le \omega$$

is valid. Thus, by (1.38), we have

$$u(t) - u(s) = 2\sum_{i=1}^n \int_s^t p_{ik}(\tau)x_i(\tau)x_k(\tau)\,dg(\tau) \ \text{ for } \ 0 \le s \le t \le \omega.$$

Therefore, due to (1.40), we find

$$u(t) - u(s) \ge \int_s^t u(\tau)\,dg_\alpha(\tau) \ \text{ for } \ 0 \le s \le t \le \omega,$$

i.e.,

$$\big(du(t) - u(t)\,dg_\alpha(t)\big)\,\mathrm{sgn}(t - \omega) \le 0 \ \text{ for } \ 0 \le s \le t \le \omega$$

and

$$d_1 u(\omega) - u(\omega)\,d_1 g_\alpha(\omega) \ge 0.$$

Now, taking into account condition (1.41), due to Lemma 2.5, we find

$$u(t) \le u(\omega)\gamma_\omega(\alpha)(t) \ \text{ for } \ t \in [0, \omega], \tag{3.7}$$

whence, by equality $u(0) = u(\omega)$ and (1.42), we have

$$u(\omega) = u(0) \le u(\omega)\gamma_\omega(\alpha)(0)$$

and $u(\omega) = 0$. Hence by (3.7) we find $u(t) \equiv 0$ and $x_i(t) \equiv 0$ $(i = 1, \dots, n)$.

In a similar way we can prove the theorem in the case (b) as well. It should only be noted that due to (1.43), (1.44) and Lemma 2.5, we have the estimate

$$u(t) \le u(0)\gamma_0(\beta)(t) \ \text{ for } \ t \in [0, \omega]$$

instead of (3.7). Thus

$$u(\omega) = u(0) \le u(0)\gamma_0(\beta)(\omega)$$

and, therefore, by (1.45), we get $u(0) = 0$, $u(t) \equiv 0$ and $x_i(t) \equiv 0$ $(i = 1, \dots, n)$. □

Proof of Corollary 1.4. It is evident that

$$\sum_{i,k=1}^{n} p_{ik}(t)x_i x_k \equiv \frac{1}{2}\sum_{i,k=1}^{n} \big(p_{ik}(t)+p_{ki}(t)\big) \;\text{ for }\; \mu(g)\text{-almost all }\; t\in[0,\omega],\;\; (x_i)_{i=1}^n\in\mathbb{R}^n,$$

from which by Lemma 2.6, we have

$$\lambda_0(P^*(t))\sum_{i=1}^{n} x_i^2 \le \sum_{i,k=1}^{n} p_{ik}(t)x_i x_k \le \lambda^0(P^*(t))\sum_{i=1}^{n} x_i^2 \;\text{ for }\; \mu(g)\text{-almost all }\; t\in[0,\omega],\;\; (x_i)_{i=1}^n\in\mathbb{R}^n.$$

Therefore, the corollary follows immediately from Theorem 1.6. □

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