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INVESTIGATION AND NUMERICAL SOLUTION OF NONLINEAR PARTIAL DIFFERENTIAL AND INTEGRO-DIFFERENTIAL MODELS BASED ON SYSTEM OF MAXWELL EQUATIONS

Abstract. The present monograph is concerned with the investigation and numerical solution of the initial-boundary value problems for some nonlinear partial differential and parabolic type integro-differential models. The models are based on the well-known system of Maxwell equations which describes the process of propagation of an electromagnetic field into a medium. The existence, uniqueness and asymptotic behavior of solutions, as time tends to infinity, for some types of initial-boundary value problems are studied. The examples of one-dimensional nonlinear systems and their analytical solutions are given which show that those systems do not, in general, have global solutions. Consequently, the case of a blow-up solution is observed. Linear stability of the stationary solution of the initial-boundary value problem for one nonlinear system is proved. The possibility of occurrence of the Hopf-type bifurcation is established. Semi-discrete and finite difference approximations are discussed. The splitting-up scheme with respect to physical processes for one-dimensional case as well as additive Rothe-type semi-discrete schemes for multi-dimensional cases are investigated. The stability and convergence properties for those schemes are studied. Algorithms for finding approximate solutions are constructed. Results of numerical experiments with tables and graphical illustrations are given. Their analysis is carried out.

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Key words and phrases. Nonlinear parabolic integro-differential equations, nonlinear partial differential systems, Maxwell equations, unique solvability, asymptotic behavior, blow-up, Hopf-type bifurcation, semi-discrete and discrete models, splitting-up schemes, stability, convergence.

რეზიუმე. მონოგრაფია ეძღვნება ზოგიერთი არაწრფივი კერძოწარმოებულებიანი და პარაბოლური ტიპის ინტეგრო-დიფერენციალური მოდელებისათვის საწყის-სასაზღვრო ამოცანების გამოკვლევას და რიცხვით ამოხსნას. ეს მოდელები დაფუძნებულია მაქსველის ცნობილ განტოლებათა სისტემაზე, რომელიც აღწერს გარემოში ელექტრომაგნიტური ველის გავრცელების პროცესს. რამდენიმე ტიპის საწყის-სასაზღვრო ამოცანისათვის შესწავლილია ამონახსნების არსებობა, ერთადერთობა და ასიმპტოტიკური ყოფაქცევა დროითი ცვლადის უსასრულოდ ზრდისას. ერთგანზომილებიანი სისტემების მაგალითები და მათი ანალიზური ამონახსნები აჩვენებენ, რომ ამ სისტემებს საზოგადოდ არ გააჩნიათ გლობალური ამონახსნები. შესაბამისად, დაფიქსირებულია ფეთქებადი ამონახსნის არსებობის შემთხვევაც. ერთი სისტემის საწყის-სასაზღვრო ამოცანისათვის დამტკიცებულია სტაციონალური ამონახსნის წრფივად მდგრადობა. აღმოჩენილია ჰოფის ტიპის ბიფურკაციას შესაძლებლობა. ერთგანზომილებიანი შემთხვევისათვის შესწავლილია ფიზიკური პროცესების მიმართ გახლეჩილი სქემები, მრავალგანზომილებიანი შემთხვევებისათვის კი ადიტიური როტეს-ტიპის ნახევრად-დისკრეტული სქემები. გამოკვლეულია ამ სქემების მდგრადობისა და კრებადობის თვისებები. შექმნილია მიახლოებითი ამონახსნების მოძებნის ალგორითმები. გრაფიკებისა და ცხრილების სახით მოცემულია რიცხვითი ექსპერიმენტების შედეგები. ჩატარებულია მათი ანალიზი.

Introduction

Differential, integral and integro-differential equations (IDEs) occur in many applications. Numerous scientific works, monographs and textbooks are devoted to the investigation of differential equations. There are lots of publications in the field of integral and integro-differential models, as well. Differential equations connect unknown functions, their derivative and an independent variable or variables. On the other hand, integral equations contain unknown functions under the integral sign. In the literature, the term IDE is used in the case if the equation contains an unknown function together with its derivative and if either an unknown function, or its derivative, or both appear under the integral sign.

Differential equations are, naturally, divided into two classes: ordinary differential equations (ODEs) and partial differential equations (PDEs). If the derivative is always taken with respect to one variable, the differential equation is called ODE. Other differential equations, on the contrary, which often occur in the mathematical physics, contain derivatives with respect to different variables are called PDEs.

Integro-differential models are also divided into two classes: ordinary integro-differential equations (OIDEs) and partial integro-differential equations (PIDEs). Let us give a general classification of IDEs. If the equation contains derivatives of an unknown function of one variable, then the IDE is called OIDE. The order of an equation is the same as that of the higher-order derivative of the unknown function in the equation.

The IDEs, encountered often in physics and mathematics, contain derivative of various variables; therefore, those equations are called as the integro-differential equations with partial derivatives or PIDEs.

The ODEs and OIDEs are the special cases of the PDEs and PIDEs, but the behavior of their solutions is quite different, in general. It is much more complicated in the case of PDEs and PIDEs and caused by the fact that the functions which we are looking for are the functions of more than one independent variable.

The advantage of the IDEs representation for a variety of problems is witnessed by their increasing frequency in the literature and in many texts dealing with the method of advanced applied mathematics. Also, the suitability of the solution method for machine computation, combined with the inherent simplicity of the subject structure make the IDE approach very valuable for many applications (see, e.g., [12, 19, 39, 48] and the references therein).

The IDEs are classified as two types of the OIDEs and PIDEs; namely, the Fredholm and Voltera types, as it is usually given in the well-known theory of integral equations. The Fredholm and Volterra type IDEs can be classified into the first, second and third kinds as it is in the theory of integral equations.

The theory of PDEs and PIDEs considers three main classes: elliptic, parabolic and hyperbolic type equations. Although, there are models not belonging to those classes.

All the above-mentioned differential, integral and integro-differential models are broadly classified as linear and nonlinear. For example, a linear PDE is one in which all of the partial derivatives appear in a linear form and none of the coefficients depends on the dependent variables. The coefficient may be a function of the independent variables. A nonlinear PDE can be described as a PDE involving nonlinear terms.

The purpose of the present work is to study some classes of nonlinear partial differential and parabolic IDEs based on the well-known system of Maxwell equations.

The process of electromagnetic field penetration in the medium is described by the system of

Maxwell equations. In the quasistationary approximation this system has the form [47]:

$$\frac{\partial H}{\partial t} = -\nabla \times (\nu_m \nabla \times H), \tag{0.0.1}$$

$$\frac{\partial \theta}{\partial t} = \nu_m (\nabla \times H)^2, \tag{0.0.2}$$

where $H = (H_1, H_2, H_3)$ is a vector of magnetic field, θ is temperature, ν_m characterizes the electroconductivity of the medium. In system (0.0.1), (0.0.2), by $\nabla\times$ the usual operation of the field theory is designated. As a rule, the coefficient ν_m is a function of argument θ. Equations (0.0.1) describe the process of diffusion of the magnetic field, whereas equation (0.0.2) describes the change of temperature at the expense of the Joule heating.

Maxwell equations appeared first in Philosophical Transactions of the Royal Society of London under the title "A Dynamical Theory of the Electromagnetic Field", in 1865. James Clerk Maxwell wrote: "The agreement of the results seems to show that light and magnetism are affections of the same medium, and that light is an electromagnetic disturbance propagated through the field according to electromagnetic laws". With that knowledge, he has changed the world forever. In the span of 150 years since his celebrated paper, numerous scientific discoveries and technological innovations have originated from the Maxwell equations.

For a more thorough description of electromagnetic field propagation in a medium, it is desirable to take into consideration different physical effects, first of all, heat conductivity of the medium has to be taken into account. In this case, together with (0.0.1), instead of (0.0.2) the equation [47]

$$\frac{\partial \theta}{\partial t} = \nu_m (\nabla \times H)^2 + \operatorname{div}(\kappa_m \operatorname{grad} \theta) \tag{0.0.3}$$

is considered, where κ_m is a coefficient of heat conductivity. As a rule, this coefficient is a function of argument θ, as well. In (0.0.3), by div and grad the usual operations of the field theory are designated.

The literature on the questions of existence, uniqueness, regularity, asymptotic behavior of solutions and numerical resolutions of the initial-boundary value problems for one-, two- and three-dimensional cases to the models of (0.0.1), (0.0.2) and (0.0.1), (0.0.3) type is very rich (see, e.g., [2, 7, 9, 13, 18, 19, 21, 22, 25, 36, 40, 47, 49, 56, 60, 63, 68–70, 73] and the references therein).

Besides, the essential nonlinearity, complexities of the above-mentioned systems (0.0.1), (0.0.2) and (0.0.1), (0.0.3) are caused by its multi-dimensionality. It is well known that the general method for constructing economic algorithms for multi-dimensional problems of mathematical physics is the method of decomposition. This approach allows one to reduce multi-dimensional problems to a set of one-dimensional ones, whose numerical realizations obviously need less computer resources (see, e.g., [1, 14, 15, 57, 64, 72] and the references therein).

Complex nonlinearity dictates also to split along the physical process and investigate the basic model by splitted ones, where the first model considers the Joule law, whereas the second process deals with the heat conductivity.

Investigation of splitting-up along the physical processes in one-dimensional case is the natural starting point to study this problem.

Let us also note that the system of Maxwell equations can be written in terms of electric field and temperature. This and more general type systems are trated in Chapter 1. Some qualitative and quantitative properties of such type models are studied in the next parts of the book.

It is well-known that by the above-mentioned system of Maxwell equations many very important applied processes are described. For example, in one-dimensional case, system (0.0.1), (0.0.2) describes an adiabatic shearing flow [9, 13].

As an important case, let us consider the following system of nonlinear partial differential equations:

$$\begin{aligned} \frac{\partial U}{\partial t} &= \frac{\partial}{\partial x}\Big(A(V)\frac{\partial U}{\partial x}\Big), \\ \frac{\partial V}{\partial t} &= F\Big(V, \frac{\partial U}{\partial x}\Big), \end{aligned} \tag{0.0.4}$$

where A and F are the given functions of their arguments.

The numerous diffusion problems are reduced to the system of nonlinear differential equations (0.0.4). In particular, if

$$F\Big(V,\frac{\partial U}{\partial x}\Big)=B(V)\Big(\frac{\partial U}{\partial x}\Big)^2,$$

then as it is visible from (0.0.1), (0.0.2), system (0.0.4) meets upon modeling of the process of electromagnetic field penetration into a medium, a coefficient of electroconductivity of which depends on temperature, without taking into account heat conductivity. In this case, the above system allows one to describe an adiabatic shearing flow [9, 13]. This type of one-dimensional system with coefficients depending also on the space and time variables is a model for the behavior of nonhomogeneous, stratified, thermoviscoplastic materials exhibiting thermal softening and temperature dependent rate of plastic work converted into heat (see, e.g., [39, 69, 70] and the references therein). So, in this case we have the following system of nonlinear partial differential equations:

$$\frac{\partial U}{\partial t}=\frac{\partial}{\partial x}\Big(A(V)\frac{\partial U}{\partial x}\Big), \tag{0.0.5}$$

$$\frac{\partial V}{\partial t}=B(V)\Big(\frac{\partial U}{\partial x}\Big)^2. \tag{0.0.6}$$

This is a one-dimensional analogue of system (0.0.1), (0.0.2) with one-component magnetic field $H=(0,0,U)$ and temperature θ denoted by V.

If

$$A(V)\equiv V,\quad F\Big(V,\frac{\partial U}{\partial x}\Big)=-V+G\Big(V,\frac{\partial U}{\partial x}\Big), \tag{0.0.7}$$

where $0<g_o\le G(\xi)\le G_0$, g_o and G_0 are constants, and G is a smooth enough function, then system (0.0.4), (0.0.7) is a one-dimensional analogue of the two-dimensional system which arises in studying the process of vein formation in the young leaves of higher plants [59].

If

$$\begin{aligned}\frac{\partial U}{\partial t}&=\frac{\partial}{\partial x}\Big(A(V)\frac{\partial U}{\partial x}\Big),\\ \frac{\partial V}{\partial t}&=B(V)\Big(\frac{\partial U}{\partial x}\Big)^2+\frac{\partial}{\partial x}\Big(C(V)\frac{\partial V}{\partial x}\Big),\end{aligned} \tag{0.0.8}$$

then system (0.0.8) is again a one-dimensional analogue of system (0.0.1), (0.0.3) with one-component magnetic field, describing penetration of the electromagnetic field into a medium, with taking into account the heat conductivity.

The above system (0.0.8) can also be used as a model for an incompressible, unidirectional flow with temperature-dependent viscosity (see, e.g., references in [39]). There are two major difficulties for that system. The first one is that the system is coupled in the coefficient of the leading term. The second is that the growth order with respect to the gradient of the solution is critical. Therefore, the general regularity theory is not applicable.

Let us now give some descriptions of integro-differential models which will be discussed in the present work. One of the main purposes of the present work is to study two classes of the nonlinear PIDEs based on the system of Maxwell equations (0.0.1), (0.0.2) describing the process of electromagnetic field penetration into a medium:

$$\frac{\partial W}{\partial t}+\nabla\times\Big[a\Big(\int_0^t|\nabla\times W|^2\,d\tau\Big)\nabla\times W\Big]=0 \tag{0.0.9}$$

and

$$\frac{\partial W}{\partial t}-a\Big(\int_0^t\int_\Omega|\nabla\times W|^2\,dx\,d\tau\Big)\Delta W=0, \tag{0.0.10}$$

$W=(W_1,W_2,W_3)$ denotes a vector, which is connected with a vector of the magnetic field $H=(H_1,H_2,H_3)$.

The scalar analogues of systems (0.0.9) and (0.0.10) have, respectively, the following forms:

$$\frac{\partial U}{\partial t} = \nabla\left[a\left(\int_0^t |\nabla U|^2\,d\tau\right)\nabla U\right] \tag{0.0.11}$$

and

$$\frac{\partial U}{\partial t} = a\left(\int_0^t \int_\Omega |\nabla U|^2\,dx\,dt\right)\Delta U. \tag{0.0.12}$$

In equations (0.0.10)–(0.0.12), ∇ and Δ denote the usual differential operations.

Note that equations such as (0.0.9), (0.0.11) have arisen for the first time in [28]. In [16,17,20], the unique solvability of the initial-boundary value problems for equations (0.0.9) and (0.0.11) is given for rather general assumptions on the function $a = a(S)$ than in [28]. Based on the works [16,17,20,28], the models of type (0.0.10), (0.0.12) appeared in G. Laptev's investigation, and the author named those models as the averaged integro-differential equations (AIDEs).

The literature on the questions of the existence, uniqueness, regularity, asymptotic behavior of the solutions and numerical resolutions of the initial-boundary value problems for one-, two- and three-dimensional cases to the models of (0.0.9)–(0.0.12) type is very rich (see, e.g., [3,5,6,10,16,17,19,20, 23,24,28,30–33,35,37–39,44,46,49,50,52,53,55,65–67,74] and the references therein).

The models of (0.0.9)–(0.0.12) types are complex and have been intensively studied by many authors. The existence and uniqueness of global solutions of initial-boundary value problems for systems and equations of (0.0.9), (0.0.11) type were studied in [5, 6, 16, 17, 19, 20, 28, 32] and in a number of other works, as well. The existence theorems that are proved in [16, 17, 20, 28] are based on a priori estimates, modified version of Galerkin's method and on compactness arguments as in [54, 71] for nonlinear parabolic equations. The asymptotic behavior, as $t \to \infty$, of the solutions of such type models have been an object of intensive researches. In this direction the so-called average integro-differential models (0.0.10), (0.0.12) are intensively investigated as well (see, e.g., [3, 19, 23, 24, 31, 32, 35, 39, 46] and the references therein).

The numerous scientific works are devoted to the construction and justification of algorithms of the numerical resolution of initial-boundary value problems for the above-stated models (see, e.g., [10,18, 19,22,30,33,35,37–40,44,52,65–67,74] and the references therein). In those works, the construction and investigation of the semi-discrete schemes are given. The possibility of finite difference approximations, application of Galerkin's methods and the method of finite elements, as well as the algorithms for their realization are discussed.

The purpose of the present monograph is to continue our study and give a description of both the results obtained at the research and numerical resolution of partial differential and integro-differential models based on the systems of Maxwell equations and systems of Maxwell type equations and their generalizations.

The work consists of five chapters.

Chapter 1 gives mathematical modeling of the process of penetration of the electromagnetic field in a medium. The corresponding nonlinear partial differential model is based on the well-known system of Maxwell equations. On the basis of that system the general statement of the problem is given. Some mathematical features of the problems under investigation are given in the same chapter. In particular, the examples of one-dimensional nonlinear systems and their analytical solutions are presented which show that those systems generally do not have global solutions. Consequently, the case of the blow-up solution is considered. Linear stability of the stationary solution of the initial-boundary value problem for one nonlinear partial differential system is proved. Possibility of the occurrence of the Hopf-type bifurcation is established. Global asymptotic stability of the solution for one diffusion problem is also given. Discrete schemes and splitting analogues with respect to the physical processes are constructed and investigated for one-dimensional (0.0.8) type system, too. The reduction of the system of Maxwell equations to integro-differential models is considered. Consequently, two types of nonlinear partial integro-differential models are obtained. Both models in different physical assumptions describe the process of penetration of the electromagnetic field into a medium with the Joule law. First, the reduction of the system of Maxwell equations to the (0.0.9) and (0.0.11) type nonlinear Volterra-type parabolic integro-differential model is discussed. The average type (0.0.10) and (0.0.12) integro-differential model is constructed in the same chapter, as well.

Chapter 2 is devoted to the asymptotic behavior of some types of initial-boundary value problems for one-dimensional integro-differential (0.0.9) type models. Asymptotic behavior of solutions when time tends to infinity are studied for nonlinear parabolic integro-differential models with homogeneous data on the whole boundary. The nonhomogeneous data on a part of the boundary are discussed in Chapter 2, as well. Different properties of asymptotic behavior in those two cases are observed. Particularly, for the homogeneous boundary conditions the exponential and for a nonhomogeneous case the power-like stability are established.

In Chapter 3, the same questions are discussed for an average one-dimensional integro-differential (0.0.12) type models. Results of numerical experiments with appropriate table and graphical illustrations are given too. Results of numerical experiments fully agree with theoretical researches.

The well-posedness of the initial-boundary value problems for both (0.0.11) and (0.0.12) type multi-dimensional integro-differential models is discussed in Chapter 4. Asymptotic behavior of the solutions are also studied therein.

The decomposition methods for building an approximate solution for the nonlinear multi-dimensional (0.0.11) and (0.0.12) type integro-differential models are discussed in the final Chapter 5. In particular, the additive averaged Rothe-type scheme for those models is constructed and investigated. Results of numerical experiments with tables and graphical illustrations, as well as their analysis are given in the same Chapter 5.

At the end of the work, the list of the cited literature is given. The list of references is not intended to be a complete bibliography on the subject, but it is, nevertheless, detailed enough to enable further independent work.

The author hopes that the work will be valuable for a wide range of scientists interested in the investigation and numerical resolution of nonlinear differential and integro-differential models. In the author's opinion, the material stated here is acceptable for a variety of specialists engages in different fields of mathematical physics, problems of applied and numerical mathematics, as well as for MS and PhD students of appropriate specializations. Because of this, in some places, despite the similarity in the studies of the same initial-boundary value problems for different type models, they are given in sufficient detail. For example, this concerns the study of semi-discrete and finite difference schemes for Volterra-type IDEs and AIDEs. Relatively, in less portion so is done also for studying the asymptotic behavior of solutions for the above-mentioned models. But note that from the asymptotics point of view, with the same approach, the above problems were studied for different classes of nonlinearity.

Chapter 1

Nonlinear partial differential and integro-differential models based on the system of Maxwell equations

Chapter 1 consists of four sections which are devoted to the mathematical modeling of systems of nonlinear partial differential and integro-differential models. The models are based on the well-known system of Maxwell equations. Some mathematical features of those models are fixed. Maxwell equations represent one of the most elegant and concise way to state the fundamentals of electricity and magnetism. Here, the basic attention is given to the mathematical description of the process of penetration of an electromagnetic field into a medium whose coefficient of conductivity depends on temperature. General statement of the diffusion process is given on the basis of the system of Maxwell differential equations. Some mathematical features of the problems under investigation are given. In particular, the examples of nonlinear systems and their analytical solutions are given which show that those systems generally do not have global solutions. Consequently, the blow-up solution case is presented. Linear stability of the stationary solution of the initial-boundary value problem for one nonlinear partial differential system is proved. The opportunity of the occurrence of a Hopf-type bifurcation is established. Global asymptotic stability of the solution for one diffusion problem is given, too. Discrete schemes and splitting-up analogues with respect to the physical processes are constructed and investigated for one-dimensional case. The reduction of the Maxwell system to the integro-differential models is considered. Two types of nonlinear integto-differential models are obtained. First, the reduction of the system of Maxwell equations to a nonlinear Volterra-type parabolic integro-differential model is discussed. The average type integro-differential model is constructed in this chapter as well.

1.1 Some features of systems of Maxwell type equations

1.1.1 Process with the Joule law

Let us consider the phenomena that occur in a conducting medium placed in an external variable electromagnetic field. The conductor is bordered by a vacuum region (dielectric). It is required to determine the coordinated change of fields, both inside and outside of the conductive region.

Let an electromagnetic field and the currents satisfy the quasistationary conditions [47]. The system of Maxwell equations in this approach looks like

$$\operatorname{div} E = 4\pi\rho, \tag{1.1.1}$$

$$-\frac{1}{c}\frac{\partial \mu H}{\partial t} = \nabla \times E, \tag{1.1.2}$$

$$\operatorname{div}(\mu H) = 0, \tag{1.1.3}$$

$$\frac{4\pi}{c}\sigma E = \nabla \times H, \tag{1.1.4}$$

where $E = (E_1, E_2, E_3)$ and $H = (H_1, H_2, H_3)$ are the electric and magnetic vector fields, respectively, ρ is a given distribution of charges, μ is magnetic permeability, σ is electrical conductivity of the medium, c is velocity of light in vacuum.

In (1.1.4), following an assumption of quasistationarity, the term adequate a current of displacement (it is proportional to $\partial E/\partial t$) is omitted and the Ohm law is used, connecting the vector E with a vector of density of a current J by the relation

$$J = \sigma E. \tag{1.1.5}$$

As for the environment, where the diffusion process takes place, we assume that it is an isotropic medium with $\mu = 1$ and the coefficient of electroconductivity depends on temperature, $\sigma = \sigma(\theta)$. In applications, the form of σ is power-like, for example, for metals $\sigma \sim \theta^{-1}$, for homogeneous plasma $\sigma \sim \theta^{-3/2}$, etc.

For definition of temperature it is necessary to use the equation of a heat balance. First, we make the following assumption. Let characteristic time resistive diffusion be much less, than that of heat transfer. Then, neglecting the effect of heat conductivity, the change of temperature is defined only by the Joule heating and, taking into account (1.1.5), we have

$$c_\nu \frac{\partial \theta}{\partial t} = EJ = \sigma E^2, \tag{1.1.6}$$

where c_ν is a specific heat capacity of the medium. Thus, the coefficient of heat capacity can also depend on temperature.

Equations (1.1.2), (1.1.4), (1.1.6) form the closed system for finding an electromagnetic field and temperature under the appropriate initial and boundary conditions.

In the general case, the questions about how to state the boundary and initial conditions, and also the conditions of coincidence on the border of environments, are discussed in detail in [19,25] (see also the references therein).

For a diffusion system, it is possible to obtain an important energy identity. Multiplying (1.1.2) scalarly by $c/(4\pi)H$, (1.1.4) by $c/(4\pi)E$ and taking into account the differential identity

$$\operatorname{div}(A \times B) = (\nabla \times A) \cdot B - A \cdot (\nabla \times B),$$

where $A \times B$ denotes the vector product, for any area Ω, we deduce the following balance equality

$$\int\limits_{\Omega} \frac{\partial}{\partial t}\Big(\frac{H^2}{8\pi} + \sigma E^2\Big)\, dx = \int\limits_{\partial\Omega} P \cdot \nu d\gamma. \tag{1.1.7}$$

Here, $d\gamma$ is an infinitesimal element of a surface $\partial\Omega$, ν is outer normal to $\partial\Omega$, $P = c/(4\pi)E \times H$ is the Poynting vector. Taking into account an onward equation

$$\frac{\partial \varepsilon}{\partial t} = \sigma E^2, \tag{1.1.8}$$

where ε is specific internal energy of the environment, $d\varepsilon = c_\nu d\theta$, after integration over time, from (1.1.7) we get

$$\int\limits_{\Omega} \Big(\frac{H^2}{8\pi} + \varepsilon\Big)\, dx = \int\limits_{\Omega} \Big(\frac{H^2(x,0)}{8\pi} + \varepsilon(x,0)\Big)\, dx + \int\limits_0^t \int\limits_{\partial\Omega} P \times \nu \, d\gamma \, d\tau. \tag{1.1.9}$$

The expression $H^2/(8\pi) + \varepsilon$ under the first integral represents density of complete energy of the medium.

From the mathematical point of view, the complexity of the study of systems (1.1.2), (1.1.4), (1.1.8) is caused by the following two factors. Firstly, as it has been mentioned in Introduction, the dependence of the electroconductivity coefficient on the temperature (on ε) and, eventually, on a field is, substantially, of nonlinear character; secondly, under transition from the medium to the vacuum area the type of equations is changing. Indeed, excluding the vectors H from (1.1.2) and (1.1.4), we obtain the equation

$$\sigma\,\frac{\partial E}{\partial t}=-\nabla\times(\nabla\times E),$$

from which it follows that in a conductor ($\sigma>0$) it is necessary to solve the parabolic, and in a vacuum area ($\sigma=0$) the elliptic problems.

The questions of the existence and uniqueness of solutions of linear differential problems ($\sigma=\sigma(x)$), in a general enough statement, have been considered in many works. The change of environment is supposed to be within the framework of magnetic-hydrodynamical approach. In those works, as a rule, the transition from a classical statement to the generalized one is made. The requirement that the functions satisfy both equation (1.1.3) and the boundary conditions, is replaced by the requirement of their belonging to the special functional spaces. The problem for equations (1.1.2), (1.1.4) is formulated in the variational form, in particular, in terms of variational inequalities.

The study of system (1.1.1)–(1.1.4) can be also made on the basis of the equations which have been written down only for the vectors E or H.

1.1.2 Process with the Joule law and heat conductivity

As it has been already noted, the above considered equations take no account of many physical effects. For a more thorough description of electromagnetic field propagation in the medium, it is desirable to take into account different physical effects, first of all the heat conductivity of the medium has to be taken into consideration. In this case, the same process of penetration of magnetic field into a medium is described by the following system [47]:

$$\begin{aligned}\frac{\partial H}{\partial t}&=-\nabla\times(\nu_m\nabla\times H),\\ \frac{\partial\theta}{\partial t}&=\nu_m(\nabla\times H)^2+\operatorname{div}(\kappa\,\operatorname{grad}\theta),\end{aligned}\tag{1.1.10}$$

where κ is a coefficient of heat conductivity. As a rule, this coefficient is a function of the argument θ, as well.

The literature on the questions of the existence, uniqueness, regularity, asymptotic behavior of solutions and numerical resolution of the initial-boundary value problems to the (1.1.10) type models and models like them is very rich (see, e.g., [2, 19, 25, 26, 68] and the references therein).

Besides essential nonlinearity, complexities of the above-mentioned system (1.1.10) are caused by its multi-dimensionality. This circumstance complicates to get numerical results for concrete real problems. Naturally, there arises the possibility to reduce them to the suitable one-dimensional models.

It is well known that the general method for construction of economic algorithms for multi-dimensional problems of mathematical physics is a decomposition method. This method allows one to reduce multi-dimensional problems to a set of one-dimensional problems, whose numerical realizations need, obviously, lesser computer resources. Work in this direction began in the 50th of the past century and intensively continues nowadays (see, e.g., [1, 14, 15, 57, 64, 72] and the references therein).

Complex nonlinearity dictates also to split along physical processes and then investigate the basic model by their means. In particular, it is logical to split system (1.1.10) into the following two models:

$$\begin{aligned}\frac{\partial\widetilde H}{\partial t}&=-\nabla\times(\nu_m(\widetilde\theta)\nabla\times\widetilde H),\\ \frac{\partial\widetilde\theta}{\partial t}&=\nu_m(\widetilde\theta)(\nabla\times\widetilde H)^2\end{aligned}$$

and

$$\frac{\partial \tilde{\tilde{H}}}{\partial t} = -\nabla \times (\nu_m(\tilde{\tilde{\theta}})\nabla \times \tilde{\tilde{H}}),$$
$$\frac{\partial \tilde{\tilde{\theta}}}{\partial t} = \operatorname{div}(\kappa(\tilde{\tilde{\theta}}) \operatorname{grad} \tilde{\tilde{\theta}}).$$

In the first system of the above-mentioned two systems the Joule law is considered, whereas in the second, the process of heat conductivity.

Investigation of splitting-up along the physical processes in one-dimensional case is the natural starting point for studying this problem. In this direction the first step was made in [2].

This question and some other studies for one-dimensional variant of system (1.1.10) are discussed in Section 1.3.

1.1.3 One-dimensional processes for cylindrical case

In this subsection, the system of Maxwell equations for one-dimensional axi-symmetric problems is stated. The consideration is conducted similarly to that made, for example, in [25].

Let us write the initial equations in a dimensionless form and let the scales for length, time and density of a current (r^*, t^*, J^*) be chosen. In differential equations given below, the following normalizations are performed: $H^* = \frac{4\pi r^*}{c} J^*$ for magnetic field; $E^* = \frac{H^* r^*}{t^* c} = \frac{4\pi r^{*2}}{c^2 t^*} J^*$ for electrical field; $\sigma^* = \frac{J^*}{E^*}$ for electroconductivity; $\theta^* = \frac{H^{*2}}{4\pi}$ for internal energy. Thus, taking into account these normalizations, we obtain the system of equations:

$$\frac{\partial H}{\partial t} = -\nabla \times E, \tag{1.1.11}$$
$$\sigma E = \nabla \times H, \tag{1.1.12}$$
$$\operatorname{div} H = 0, \tag{1.1.13}$$
$$\frac{\partial \varepsilon}{\partial t} = \sigma E^2, \quad \sigma = \sigma(\varepsilon). \tag{1.1.14}$$

This system has no additional multipliers, and for the dimensionless values the initial designations are maintained.

When the coefficient of electroconductivity depends only on time and spatial variables, then system of the diffusion equations is linear. The fields are determined from equations (1.1.11)–(1.1.13), and equation (1.1.14) provides the distribution of temperature. From the mathematical point of view, such cases are interesting only in two- and three-dimensional statements.

However, if $\sigma = \sigma(\varepsilon)$, it is necessary to solve all equations of system (1.1.11)–(1.1.14) jointly. The case of a spatial variable is of importance, both in the sense of theoretical study and numerical analysis, and for a qualitative understanding of the real diffusion process.

Thus suppose that all unknowns are the functions of one spatial and time variables.

Assume that the medium is occupied by the field bounded by two infinite coaxial cylinders of radii r_a and r_b. Introduce the appropriate cylindrical system of coordinates (r, φ, z) and assume that the unknowns H, E and θ are the functions of arguments (r, t).

In our further investigations the necessary requirements of regularity for the considered functions are assumed to be fulfilled.

The system of equations (1.1.11)–(1.1.14) is solved in the area

$$\Omega = \Omega_1 \cup \Omega_2,$$

where

$$\Omega_1 = (r_0, r_a) \cup (r_b, r_c), \quad \Omega_2 = (r_a, r_b), \;\; r_0 > 0.$$

The conductivity σ is defined by the formula

$$\sigma = \begin{cases} \sigma(\varepsilon(r,t)) > 0, & r \in \Omega_2, \\ 0, & r \in \Omega_1. \end{cases}$$

On the intersection of two environments (points r_a, r_b) the conditions of conjugation are satisfied:

$$E_\varphi^{(1)} = E_\varphi^{(2)}, \quad E_z^{(1)} = E_z^{(2)}, \quad H^{(1)} = H^{(2)}. \tag{1.1.15}$$

The boundary conditions at the points r_0 and r_c, and the initial conditions will be specified later. Taking into account the above assumptions, system (1.1.11)–(1.1.14) can be written in the form:

$$\frac{\partial H_r}{\partial t} = 0, \quad \frac{\partial H_\varphi}{\partial t} = \frac{\partial E_z}{\partial r}\,, \quad \frac{\partial H_z}{\partial t} = -\frac{1}{r}\,\frac{\partial r E_\varphi}{\partial r}\,, \tag{1.1.16}$$

$$\sigma E_r = 0, \quad \sigma E_\varphi = -\frac{\partial H_z}{\partial r}\,, \quad \sigma E_z = \frac{1}{r}\,\frac{\partial r H_\varphi}{\partial r}\,, \tag{1.1.17}$$

$$\frac{\partial r H_r}{\partial r} = 0, \tag{1.1.18}$$

$$\frac{\partial \varepsilon}{\partial t} = \sigma(E_r^2 + E_z^2 + E_\varphi^2). \tag{1.1.19}$$

For the radial components, we find

$$H_r = \frac{C_r}{r}, \quad E_r = 0 \quad \text{if } r \in \Omega_2.$$

At the same time, if Ω contains the axis of symmetry, then $C_r = 0$ and $H_r = 0$.

Assume also that $E_r = 0$ in Ω_1.

In Ω_1, we have $\sigma \equiv 0$, and from (1.1.17) we get

$$H_\varphi = \frac{C_\varphi(t)}{r}\,, \quad H_z = -C_z(t), \tag{1.1.20}$$

where C_φ and C_z are some functions depending on t and, in general, different on the intervals (r_0, r_a) and (r_a, r_b). Substituting (1.1.20) into (1.1.16) and integrating with respect to t, if $r \in \Omega_1$, we obtain

$$r E_\varphi = C_z'(t)\,\frac{r^2}{2} + C_1(t), \quad E_z = C_\varphi'(t) \ln r + C_2(t), \tag{1.1.21}$$

where

$$C_z' = \frac{dC_z}{dt}\,, \quad C_\varphi' = \frac{dC_\varphi}{dt}\,.$$

Before we proceed to discussing the boundary and initial conditions, we notice that equations (1.1.20) and (1.1.21) allow us to reduce the diffusion problem to the system

$$\frac{\partial H_\varphi}{\partial t} = \frac{\partial E_z}{\partial r}\,, \quad \frac{\partial H_\varphi}{\partial t} = -\frac{1}{r}\,\frac{\partial r E_\varphi}{\partial r}\,, \tag{1.1.22}$$

$$\sigma E_\varphi = -\frac{\partial H_z}{\partial r}\,, \quad \sigma E_z = \frac{1}{r}\,\frac{\partial r H_\varphi}{\partial r}\,, \tag{1.1.23}$$

$$\frac{\partial \varepsilon}{\partial t} = \sigma(E_\varphi^2 + E_z^2) = \frac{1}{\sigma}\left[\Big(\frac{\partial H_z}{\partial r}\Big)^2 + \Big(\frac{1}{r}\,\frac{\partial r H_\varphi}{\partial r}\Big)^2\right] \tag{1.1.24}$$

in Ω_2. Consider now some boundary conditions.

(a) Let magnetic fields at points r_0 and r_c be given as follows:

$$H_\tau(r_0, t) = \psi^0(t), \quad H_\tau(r_c, t) = \psi^c(t),$$

where

$$H_\tau = (H_\varphi, H_z), \quad \psi^0 = (\psi_\varphi^0, \psi_z^0), \quad \psi^c = (\psi_\varphi^c, \psi_z^c).$$

Then by virtue of (1.1.20) in Ω_1, we have

$$H_\varphi^{(1)}(r,t) = \begin{cases} \dfrac{r_0}{r}\,\psi_\varphi^0(t), & r_0 \le r \le r_a, \\ \dfrac{r_c}{r}\,\psi_\varphi^c(t), & r_b \le r \le r_c, \end{cases}$$

$$H_z^{(1)}(r,t) = \begin{cases} \psi_z^0(t), & r_0 \le r \le r_a, \\ \psi_z^c(t), & r_b \le r \le r_c. \end{cases}$$

To define H_τ in Ω_2, it is necessary to solve the problem

$$\begin{gathered}\frac{\partial H_\varphi}{\partial t}=\frac{\partial}{\partial r}\Big(\frac{1}{r\sigma}\frac{\partial rH_\varphi}{\partial r}\Big),\quad \frac{\partial H_z}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}\Big(\frac{r}{\sigma}\frac{\partial H_z}{\partial r}\Big),\\ \frac{\partial\varepsilon}{\partial t}=\frac{1}{\sigma}\bigg[\Big(\frac{\partial H_z}{\partial r}\Big)^2+\Big(\frac{1}{r}\frac{\partial rH_\varphi}{\partial r}\Big)^2\bigg],\quad \sigma=\sigma(\varepsilon),\\ H_\tau(r_a,t)=H_\tau^{(1)}(r_a,t),\quad H_\tau(r_b,t)=H_\tau^{(1)}(r_b,t),\quad H_\tau(r,0)=H_{\tau0}(r),\quad \varepsilon(r,0)=\varepsilon_0(r),\end{gathered}\tag{1.1.25}$$

where $H_{\tau0}$ and ε_0 are initial distributions for H_τ and ε.

If problem (1.1.25) is solved, from (1.1.23) in Ω_2 we get

$$E_\tau=\frac{1}{\sigma}\,(\nabla\times H)_\tau.$$

Using the condition of interface on the border and (1.1.21), it is easy to find E_τ in Ω_1.

(b) Assume now that the following boundary conditions are given:

$$E_\tau(r_0,t)=\Phi^0(t),\quad E_\tau(r_c,t)=\Phi^c(t),$$

where

$$E_\tau=(E_\varphi,E_z),\quad \Phi^0=(\Phi^0_\varphi,\Phi^0_z),\quad \Phi^c=(\Phi^c_\varphi,\Phi^c_z).$$

Taking into account (1.1.21) in Ω_1, we have

$$rE_\varphi(r,t)=\begin{cases}\dfrac{dC_z^0}{dt}\,\dfrac{r^2-r_0^2}{2}+r_0\Phi^0_\varphi(t), & r_0\le r\le r_a,\\[2mm] \dfrac{dC_z^c}{dt}\,\dfrac{r^2-r_c^2}{2}+r_c\Phi^c_\varphi(t), & r_b\le r\le r_c,\end{cases}$$

$$E_z(r,t)=\begin{cases}\dfrac{dC_\varphi^0}{dt}\,\ln\dfrac{r}{r_0}+\Phi^0_z(t), & r_0\le r\le r_a,\\[2mm] \dfrac{dC_\varphi^c}{dt}\,\ln\dfrac{r}{r_c}+\Phi^c_\varphi(t), & r_b\le r\le r_c.\end{cases}$$

Noticing that

$$C'_\varphi(t)=r\,\frac{\partial E_z}{\partial r}\,,\quad C'_z(t)=\frac{1}{r}\,\frac{\partial rE_\varphi}{\partial r}\,,$$

in Ω_2 we pose the following problem:

$$\begin{gathered}\frac{\partial\sigma E_\varphi}{\partial t}=\frac{\partial}{\partial r}\Big(\frac{1}{r}\frac{\partial rE_\varphi}{\partial r}\Big),\quad \frac{\partial\sigma E_z}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}\Big(r\,\frac{\partial E_z}{\partial r}\Big),\\ \frac{\partial\varepsilon}{\partial t}=\sigma(E_\varphi^2+E_z^2),\quad \sigma=\sigma(\varepsilon),\\ \Big[rE_\varphi-\frac{1}{r}\,\frac{\partial rE_\varphi}{\partial r}\,\frac{r^2-r_0^2}{2}\Big]_{r=r_a}=r_0\Phi^0_\varphi(t),\\ \Big[rE_\varphi-\frac{1}{r}\,\frac{\partial rE_\varphi}{\partial r}\,\frac{r^2-r_c^2}{2}\Big]_{r=r_b}=r_c\Phi^c_\varphi(t),\\ \Big[E_z-r\,\frac{\partial E_z}{\partial r}\,\ln\frac{r}{r_0}\Big]_{r=r_a}=\Phi^0_z(t),\\ \Big[E_z-r\,\frac{\partial E_z}{\partial r}\,\ln\frac{r}{r_c}\Big]_{r=r_b}=\Phi^c_z(t),\\ E_\tau\big|_{t=0}=E_{\tau0},\quad \varepsilon\big|_{t=0}=\varepsilon_0(r).\end{gathered}\tag{1.1.26}$$

Consequently, solving problem (1.1.26) for E_τ together with the equation for ε, it is possible to define the derivative $\partial E_\tau/\partial r$ at the points r_a and r_b. Integration with respect to t gives C_φ and C_z and finally the vectors H_τ and E_τ we can found in Ω_1.

Note that if Ω_1 contains an axis of symmetry, it is necessary to modify the given formulations by taking into account the singularity at $r = 0$.

Suppose that there are no extraneous currents, $r_0 = 0$, $r_c = \infty$, the field inside the conductor is absent at the initial state, $\varepsilon = \varepsilon_0$ in Ω_2, $\sigma = \sigma_0\varepsilon^{-\alpha}$. External electromotive forces provide the total longitudinal current $I_z = I_z(t)$ and the linear density of the azimuth current $I_\varphi = I_\varphi(t)$.

Using the results of the preceding reasoning, we find

$$H_\varphi(r,t) = \begin{cases} 0, & 0 \le r \le r_a, \\ \dfrac{I_z(t)}{2\pi r}, & r \ge r_b, \end{cases}$$
$$H_z(r,t) = \begin{cases} I_\varphi(t), & 0 \le r \le r_a, \\ 0, & r \ge r_b. \end{cases}$$

Thus, in Ω_2, it is necessary to solve the following problem:

$$\begin{gathered} \frac{\partial H_\varphi}{\partial t} = \frac{\partial}{\partial r}\Big(\frac{v}{r}\,\frac{\partial r H_\varphi}{\partial r}\Big), \quad \frac{\partial H_z}{\partial t} = \frac{1}{r}\,\frac{\partial}{\partial r}\Big(vr\,\frac{\partial H_z}{\partial r}\Big), \\ \frac{\partial \varepsilon}{\partial t} = v\bigg[\Big(\frac{\partial H_z}{\partial r}\Big)^2 + \Big(\frac{1}{r}\,\frac{\partial r H_\varphi}{\partial r}\Big)^2\bigg], \quad \nu = \nu_0\varepsilon^\alpha, \\ H_\varphi(r_a,t) = 0, \quad H_z(r_a,t) = H_a(t), \\ H_\varphi(r_b,t) = H_b(t), \quad H_z(r_b,t) = 0, \\ H(r,0) = 0, \quad \varepsilon(r,0) = \varepsilon_0, \end{gathered} \tag{1.1.27}$$

where $H_b(t) = I(t)/2\pi r_b$, $H_a(t) = I_\varphi(t)$, $v_0 = 1/\sigma_0$.

Note that many scientific papers are devoted to the study of the problems of type (1.1.27) in the cylindrical system of coordinates (see, e.g., [25] and the references therein).

1.1.4 Some mathematical features of magnetic field penetration processes

In the present subsection we establish some features of those types of equations that are considered in Introduction and which are a one-dimensional analogue of the models described in the first two subsections of this chapter.

As a model, let us consider the system of nonlinear PDEs of the following kind:

$$\begin{aligned} \frac{\partial U}{\partial t} &= \frac{\partial}{\partial x}\Big(A(V)\,\frac{\partial U}{\partial x}\Big), \\ \frac{\partial V}{\partial t} &= \frac{\partial}{\partial x}\Big(C(V)\,\frac{\partial V}{\partial x}\Big) + F\Big(V, \frac{\partial U}{\partial x}\Big), \end{aligned} \tag{1.1.28}$$

where A, C and F are the given functions of their arguments.

The numerous diffusion problems are reduced to system (1.1.28) of differential equations. In particular, if

$$C(V) \equiv 0, \quad F\Big(V, \frac{\partial U}{\partial x}\Big) = B(V)\Big(\frac{\partial U}{\partial x}\Big)^2,$$

system (1.1.28), as it has been mentioned in Subsections 1.1.1–1.1.3, meets at the modeling of penetration of an electromagnetic field into a medium, whose coefficient of electroconductivity depends on temperature, without taking into account the heat conductivity.

If, for example,

$$A(V) \equiv V, \quad C(V) \equiv 0, \quad F\Big(V, \frac{\partial U}{\partial x}\Big) = -V + G\Big(V, \frac{\partial U}{\partial x}\Big), \tag{1.1.29}$$

where $0 < g_o \le G(\xi) \le G_0$, g_0 and G_0 are constants, and g is a smooth enough function, (1.1.28), (1.1.29) is a one-dimensional analogue of system which arises in studying the process of vein formation in young leaves of higher plants [59].

If

$$C(V) \neq 0, \quad F\Big(V, \frac{\partial U}{\partial x}\Big) = -D(V) + B(V)\Big(\frac{\partial U}{\partial x}\Big)^2, \tag{1.1.30}$$

then by system (1.1.28), (1.1.30) it is described penetration of an electromagnetic field into the medium, taking into account the heat conductivity.

1.1.5 On the blow-up solution

Let us consider the following initial-boundary value problem:

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\Big(V^\alpha \frac{\partial U}{\partial x}\Big), \tag{1.1.31}$$

$$\frac{\partial V}{\partial t} = V^\alpha \Big(\frac{\partial U}{\partial x}\Big)^2, \tag{1.1.32}$$

$$U(0,t) = 0, \quad U(1,t) = \psi, \tag{1.1.33}$$

$$U(x,0) = U_0(x), \quad V(x,0) = V_0(x), \tag{1.1.34}$$

where $\psi = const > 0$, and $U_0 = U_0(x)$ and $V_0 = V_0(x)$ are the given functions.

If $U(x,0) = \psi x$ and $V(x,0) = \delta_0 = const > 0$, it is easy to find that the pair of functions

$$U(x,t) = \psi x, \quad V(x,t) = \big[\delta_0^{1-\alpha} + (1-\alpha)\psi^2 t\big]^{\frac{1}{1-\alpha}} \tag{1.1.35}$$

is the solution of the initial-boundary value problem (1.1.31)–(1.1.34) for any $\alpha \neq 1$. However, if $\alpha > 1$, then for a finite time $t_0 = \delta_0^{1-\alpha}/(\psi^2(\alpha-1))$ the function V becomes unbounded. This example shows that the solutions of a system such as (1.1.31), (1.1.32) with smooth initial and boundary conditions can blow-up at a finite time.

Note that the functions U and V, determined by formulas (1.1.35), satisfy as well the system

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\Big(V^\alpha \frac{\partial U}{\partial x}\Big), \tag{1.1.36}$$

$$\frac{\partial V}{\partial t} = V^\alpha \Big(\frac{\partial U}{\partial x}\Big)^2 + \frac{\partial^2 V}{\partial x^2}, \tag{1.1.37}$$

with the initial and boundary conditions (1.1.33), (1.1.34) and adding to them the following boundary conditions:

$$\frac{\partial V}{\partial x}\Big|_{x=0} = \frac{\partial V}{\partial x}\Big|_{x=1} = 0. \tag{1.1.38}$$

From this we can conclude that if $\alpha > 1$, then for problem (1.1.33), (1.1.34), (1.1.36)–(1.1.38) the theorem on the existence of the global solution also does not hold.

1.1.6 Linear stability of the stationary solution

It is known that in electric circuits for cryogenic current systems, voltage and temperature oscillations often arise with the initially stationary distribution of parameters.

The question of the stability of the stationary solution for appropriate diffusion problems is interesting for a mathematical explanation of this phenomenon. In this connection we consider the following initial-boundary value problem:

$$\begin{gathered} \frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\Big(V^\alpha \frac{\partial U}{\partial x}\Big), \\ \frac{\partial V}{\partial t} = -V^\beta + V^\gamma \Big(\frac{\partial U}{\partial x}\Big)^2, \\ U(0,t) = 0, \quad V^\alpha \frac{\partial U}{\partial x}\Big|_{x=1} = \psi, \\ U(x,0) = U_0(x), \quad V(x,0) = V_0(x), \end{gathered} \tag{1.1.39}$$

where $\alpha \neq 0$, $2\alpha + \beta - \gamma \neq 0$.

It is easy to be convinced that the stationary solution of problem (1.1.39) has the form

$$\big(\psi^{\frac{\beta-\gamma}{2\alpha+\beta-\gamma}} x, \psi^{\frac{2}{2\alpha+\beta-\gamma}}\big).$$

Assume that there exists a smooth solution of problem (1.1.39) with $V(x,t) \geq const > 0$. Introducing the notation $W = V^\alpha \partial U/\partial x$, it is convenient to reduce the problem (1.1.39) to the equivalent form:

$$\begin{gathered}
\frac{\partial W}{\partial t} = V^\alpha \frac{\partial^2 W}{\partial x^2} + \alpha(V^{\gamma-2\alpha-1}W^2 - V^{\beta-1})W, \\
\frac{\partial V}{\partial t} = -V^\beta + V^{\gamma-2\alpha}W^2, \\
\frac{\partial W}{\partial x}\bigg|_{x=0} = 0, \quad (1,t) = \psi, \\
W(x,0) = V_0^\alpha(x)\,\frac{dU_0(x)}{dx}\,, \quad V(x,0) = V_0(x).
\end{gathered} \tag{1.1.40}$$

The stationary solution of problem (1.1.40) is a pair of constants $(\psi, \psi^{\frac{2}{2\alpha+\beta-\gamma}})$.

We linearize system (1.1.40) by assuming

$$W(x,t) = \psi + W_1(x)e^{\lambda t}, \quad V(x,t) = \psi^{\frac{2}{2\alpha+\beta-\gamma}} + V_1(x)e^{\lambda t}.$$

After simple transformations, we obtain the problem

$$\begin{gathered}
\lambda W_1 = \psi^{\frac{2\alpha}{2\alpha+\beta-\gamma}}\,\frac{d^2 W}{dx^2} + 2\alpha\psi^{\frac{2(\beta-1)}{2\alpha+\beta-\gamma}} - \alpha(2\alpha+\beta-\gamma)\psi^{\frac{2\alpha+3\beta-\gamma-4}{2\alpha+\beta-\gamma}}V_1, \\
\big[\lambda + (2\alpha+\beta-\gamma)\psi^{\frac{2(\beta-1)}{2\alpha+\beta-\gamma}}\big]V_1 = 2\psi^{\frac{\beta-2\alpha+\gamma}{2\alpha+\beta-\gamma}}W_1, \\
\frac{dW_1(x)}{dx}\bigg|_{x=0} = W_1(1) = 0,
\end{gathered}$$

which, in turn, is easily reduced to an eigenvalue problem

$$\frac{d^2 W_1}{dx^2} + \eta^2 W_1 = 0, \quad \frac{dW_1(x)}{dx}\bigg|_{x=0} = W_1(1) = 0, \tag{1.1.41}$$

where

$$\eta^2 = 2\alpha\psi^{\frac{2(\beta-\alpha-1)}{2\alpha+\beta-\gamma}} - 2\alpha(2\alpha+\beta-\gamma)\psi^{\frac{2(2\beta-\alpha-2)}{2\alpha+\beta-\gamma}}\big(\lambda + (2\alpha+\beta-\gamma)\psi^{\frac{2(\beta-1)}{2\alpha+\beta-\gamma}}\big)^{-1} - \lambda\psi^{\frac{-2\alpha}{2\alpha+\beta-\gamma}}.$$

It is obvious that problem (1.1.41) has no trivial solution only for

$$\eta^2 = \eta_n^2 = \Big(n + \frac{1}{2}\Big)^2 \pi^2, \ \ n \in \mathrm{Z}_0\,.$$

For an appropriate $\lambda = \lambda_n$, we have

$$\lambda_n^2 + \bigg[\Big(n+\frac{1}{2}\Big)^2\pi^2\psi^{\frac{2\alpha}{2\alpha+\beta-\gamma}} + (\beta-\gamma)\psi^{\frac{2(\beta-1)}{2\alpha+\beta-\gamma}}\bigg]\lambda_n + \Big(n+\frac{1}{2}\Big)^2\pi^2(2\alpha+\beta-\gamma)\psi^{\frac{2(\alpha=\beta-1)}{2\alpha+\beta-\gamma}} = 0,$$

or

$$\lambda_n^2 - P_n(\psi,\alpha,\beta,\gamma) + L_n(\psi,\alpha,\beta,\gamma) = 0, \tag{1.1.42}$$

where we have used the following notations:

$$\begin{gathered}
P_n(\psi,\alpha,\beta,\gamma) = (\alpha-\beta)\psi^{\frac{2(\beta-1)}{2\alpha+\beta-\gamma}} - \Big(n+\frac{1}{2}\Big)^2\pi^2\psi^{\frac{2\alpha}{2\alpha+\beta-\gamma}}, \\
L_n(\psi,\alpha,\beta,\gamma) = \Big(n+\frac{1}{2}\Big)^2\pi^2(2\alpha+\beta-\gamma)\psi^{\frac{2(\alpha+\beta-1)}{2\alpha+\beta-\gamma}}.
\end{gathered} \tag{1.1.43}$$

The stationary solution of problem (1.1.40) is linearly stable only if $\mathrm{Re}(\lambda_n) < 0$, for any n, and unstable, if there exists m such that $\mathrm{Re}(\lambda_m) > 0$.

From (1.1.42) and (1.1.43) it is obvious that if $2\alpha + \beta - \gamma > 0$, stationary solution of problem (1.1.40) is linearly stable if and only if the inequality

$$P_n(\psi, \alpha, \beta, \gamma) < 0, \quad n \in Z_0,$$

is fulfilled, or in an expanded form,

$$(\gamma - \beta)\psi^{\frac{2(\beta-\alpha-1)}{2\alpha+\beta-\gamma}} < \left(n + \frac{1}{2}\right)^2 \pi^2, \quad n \in Z_0.$$

Hence, eventually, we have

$$(\gamma - \beta)\psi^{\frac{2(\beta-\alpha-1)}{2\alpha+\beta-\gamma}} < \frac{\pi^2}{4}\,. \tag{1.1.44}$$

Thus, the following statement is true.

Theorem 1.1.1. If $2\alpha + \beta - \gamma > 0$, then the stationary solution of problem (1.1.40)

$$(\psi, \psi^{\frac{2}{2\alpha+\beta-\gamma}})$$

is linearly stable if and only if condition (1.1.44) takes place.

Remark 1.1.1. If $\gamma - \beta \le 0$, then the stationary solution of problem (1.1.40) is always linearly stable.

1.1.7 Hopf-type bifurcation

Let

$$\gamma - \beta > 0, \quad \beta - \alpha - 1 \neq 0,$$

and consider the quantity

$$\psi_c = \left[\frac{\pi^2}{4(\gamma - \beta)}\right]^{\frac{2\alpha+\beta-\gamma}{2(\beta-\alpha-1)}}.$$

We have

$$P_0(\psi_c, \alpha, \beta, \gamma) = 0, \quad P_n(\psi_c, \alpha, \beta, \gamma) < 0, \quad n \in N.$$

In addition, let $\beta - \alpha - 1 > 0$. It is clear that if $0 < \psi < \psi_c$, then $P_n(\psi, \alpha, \beta, \gamma) < 0$ for any $n \in \mathrm{Z}_0$.

Thus, if $0 < \psi < \psi_c$, the stationary solution of problem (1.1.40) is linearly stable, and if $\psi > \psi_c$ it becomes unstable. For $\psi = \psi_c$, we have $\mathrm{Re}(\lambda_0) = 0$ and $\mathrm{Im}(\lambda_0) \neq 0$, i.e., there is the possibility of occurrence of the Hopf-type bifurcation [58]. Small perturbations of the stationary solution can be transformed into a periodic in time self-oscillation. Based on [21], where the issues of Subsections 1.1.6–1.1.8 are studied, the analogous investigations for more general models are given in [36, 40, 45].

1.1.8 Global stability of the stationary solution

Let us now prove the global stability of a solution of problem (1.1.39) for one particular case. Consider the following problem:

$$\begin{gathered}
\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left(V \frac{\partial U}{\partial x}\right), \\
\frac{\partial V}{\partial t} = -V + \left(\frac{\partial U}{\partial x}\right)^2, \\
U(0,t) = 0, \quad V \frac{\partial U}{\partial x}\bigg|_{x=1} = \psi, \\
U(x,0) = U_0(x), \quad V(x,0) = V_0(x).
\end{gathered} \tag{1.1.45}$$

It is obvious that the stationary solution of problem (1.1.45) looks like $(\psi^{\frac{1}{3}}x, \psi^{\frac{2}{3}})$.

Introduce the notations:

$$y(x,t)=U(x,t)-\psi^{\frac{1}{3}}x,\quad z(x,t)=V(x,t)-\psi^{\frac{2}{3}},$$

where (U,V) is a solution of problem (1.1.45). We have

$$\begin{gathered}\frac{\partial y}{\partial t}=\frac{\partial U}{\partial t}=\frac{\partial}{\partial x}\Big(V\frac{\partial U}{\partial x}\Big),\\ \frac{\partial z}{\partial t}=\frac{\partial V}{\partial t}=-z-\psi^{\frac{2}{3}}+\Big(\frac{\partial U}{\partial x}\Big)^2.\end{gathered}\tag{1.1.46}$$

Multiplying the first equation (1.1.46) scalarly by y, integrating the obtained identity by parts and taking into account the boundary conditions, we get

$$\frac{1}{2}\,\frac{d}{dt}\int\limits_0^1 y^2\,dx=\int\limits_0^1 y\,\frac{\partial}{\partial x}\Big(V\frac{\partial U}{\partial x}\Big)\,dx,$$

$$\begin{aligned}\frac{1}{2}\,\frac{d}{dt}\int\limits_0^1 y^2\,dx&=\psi y(1,t)-\int\limits_0^1 V\,\frac{\partial U}{\partial x}\,\frac{\partial y}{\partial x}\,dx=\psi\int\limits_0^1\frac{\partial y}{\partial x}\,dx-\int\limits_0^1 V\Big(\frac{\partial y}{\partial x}+\psi^{\frac{1}{3}}\Big)\,\frac{\partial y}{\partial x}\,dx\\ &=\int\limits_0^1\Big[-V\Big(\frac{\partial y}{\partial x}\Big)^2+(\psi-V\psi^{\frac{1}{3}})\,\frac{\partial y}{\partial x}\Big]\,dx.\end{aligned}$$

Note that

$$\psi-\psi^{\frac{1}{3}}V=\psi-(z+\psi^{\frac{2}{3}})\psi^{\frac{1}{3}}=-\psi^{\frac{1}{3}}z,$$

so,

$$\frac{1}{2}\,\frac{d}{dt}\int\limits_0^1 y^2\,dx=\int\limits_0^1\Big(-V\Big(\frac{\partial y}{\partial x}\Big)^2-\psi^{\frac{1}{3}}z\,\frac{\partial y}{\partial x}\Big)\,dx.\tag{1.1.47}$$

From the second equation of (1.1.46), we have

$$\begin{aligned}\frac{1}{2}\,\frac{d}{dt}\int\limits_0^1 z^2\,dx&=\int\limits_0^1\Big(-z^2-\psi^{\frac{2}{3}}z+z\Big(\frac{\partial U}{\partial x}\Big)^2\Big)\,dx=\int\limits_0^1\Big\{-z^2-\Big[\psi^{\frac{2}{3}}-\Big(\frac{\partial y}{\partial x}+\psi^{\frac{1}{3}}\Big)^2\Big]z\Big\}\,dx\\ &=\int\limits_0^1\Big(-z^2+V\Big(\frac{\partial y}{\partial x}\Big)^2-\psi^{\frac{2}{3}}\Big(\frac{\partial y}{\partial x}\Big)^2+2\psi^{\frac{1}{3}}z\,\frac{\partial y}{\partial x}\Big)\,dx.\end{aligned}\tag{1.1.48}$$

Adding equalities (1.1.47) and (1.1.48) and using the inequality $ab\le a^2/2+b^2/2$, we have

$$\begin{aligned}\frac{1}{2}\,\frac{d}{dt}\int\limits_0^1(y^2+z^2)\,dx&=\int\limits_0^1\Big(-z^2-\psi^{\frac{2}{3}}\Big(\frac{\partial y}{\partial x}\Big)^2+\psi^{\frac{1}{3}}z\,\frac{\partial y}{\partial x}\Big)\,dx\\ &\le\int\limits_0^1\Big(-z^2-\psi^{\frac{2}{3}}\Big(\frac{\partial y}{\partial x}\Big)^2+\frac{1}{2}\,z^2+\frac{1}{2}\,\psi^{\frac{2}{3}}\Big(\frac{\partial y}{\partial x}\Big)^2\Big)\,dx=-\frac{1}{2}\int\limits_0^1\Big(z^2+\psi^{\frac{2}{3}}\Big(\frac{\partial y}{\partial x}\Big)^2\Big)\,dx.\end{aligned}$$

Applying the Poincaré inequality

$$\int\limits_0^1 y^2\,dx\le\frac{1}{2}\int\limits_0^1\Big(\frac{\partial y}{\partial x}\Big)^2\,dx,$$

we finally arrive at

$$\frac{d}{dt}\int_0^1 (y^2+z^2)\,dx \le -C\int_0^1 (y^2+z^2)\,dx, \tag{1.1.49}$$

where C is a positive constant.

From (1.1.49) we deduce

$$\int_0^1 \left[y^2(x,t)+z^2(x,t)\right]dx \le e^{-Ct}\int_0^1 \left\{\left[U_0(x)-\psi^{\frac{1}{3}}x\right]^2+\left[V_0(x)-\psi^{\frac{2}{3}}x\right]^2\right\}dx.$$

Thus, the following statement is true.

Theorem 1.1.2. For the stationary solution of problem (1.1.45)

$$(\psi^{\frac{1}{3}}x, \psi^{\frac{2}{3}})$$

there takes place the global and monotone stability in $L_2(0,1)$.

Note that it is not difficult to get a certain generalization of the results considered in this subsection for the diffusion model, where the process of heat conductivity is taken into account.

Note also that some results regarding the asymptotic behavior of solutions of the corresponding integro-differential models with different kinds of boundary conditions will be studied in the subsequent parts of the monograph.

1.2 Finite difference schemes for one-dimensional problems with the Joule law

1.2.1 Discretization

The significant quantity of works are devoted to the problems of numerical integration of linear and nonlinear electrodynamic problems in the quasistationary approximation. Such a variety is dictated, on the one hand, by the desire to utilize fully the specific features of a particular applied problem, and on the other hand, by choosing the main variables (magnetic or electric field, vector potential, etc.) and the approach to constructing a numerical model. Most of the computational algorithms are based on the finite element method. The corresponding theoretical and methodological references are available in many works. It should also be mentioned that of importance is the construction of a difference scheme in which the principles analogous to those of the conservatism and fully conservatism are widely used. According to those principles, the discrete model, in addition to natural requirements (approximation, stability, convergence), should be subordinated to specific physical conditions. In the case of diffusion problems, this means that for an approximate model the analogues of electromagnetic induction laws and the magnetic field circulation, as well as the energy identity (1.1.45) must be satisfied (see Section 1.1).

In this section, following the technique as e.g. in [64], the discrete model is constructed for problem (1.1.27) and the question of the convergence of the difference scheme is studied.

Let us divide the areas $\Omega=[r_a,r_b]$ and $[0,T]$ uniformly by M and N points, respectively, and introduce the notations:

$$h=\frac{r_b-r_a}{M},\quad \tau=\frac{T}{N},\quad x_i=ih,\quad t_j=j\tau,\quad u_i^j=u(x_i,t_j),$$
$$\omega_h=\big\{x_i,\ i=1,2,\dots,M-1\big\},\quad \overline{\omega}_h=\omega_h\cup\big\{x_0=r_a,\ x_M=r_b\big\},$$
$$u_i=u_i^{j+1},\quad u_t=\frac{u_i-u_i^j}{\tau},\quad u_x=\frac{u_{i+1}^{j+1}-u_i^{j+1}}{h},\quad u_{\overline{x}}=\frac{u_i^{j+1}-u_{i-1}^{j+1}}{h},$$
$$(u,v)=\sum_{i=1}^{M-1}u_iv_ih,\quad (u,v]=\sum_{i=1}^{M}u_iv_ih,$$
$$\|u\|_{L_2(\omega_h)}=(u,u)^{\frac12},\quad \|u]|_{L_2(\omega_h)}=(u,u]^{\frac12}.$$

Using an integral-interpolation approach (see, e.g., [25,64]), for problem (1.1.27) we construct the following differential-difference scheme:

$$\frac{1}{r}\,\frac{dH\varphi}{dt}=\Big(\frac{\nu}{r}H\varphi_{\overline{x}}\Big)_x,\quad r\,\frac{dHZ}{dt}=(\nu RHZ_{\overline{x}})_x, \tag{1.2.1}$$
$$\frac{de}{dt}=\frac{\nu}{R^2}\,(H\varphi_{\overline{x}})^2+\nu(HZ_{\overline{x}})^2, \tag{1.2.2}$$
$$H\varphi_0(t)=0,\quad H\varphi_M(t)=H_b(t),$$
$$HZ_0(t)=H_a(t),\quad HZ_M(t)=0,$$
$$H\varphi_i(0)=0,\quad HZ_i(0)=0,$$
$$e_i(0)=\varepsilon_{0i},\ \ i=1,2,\dots,M-1.$$

Here, for the grid functions we use the customary notations:

$$H\varphi\approx(rH_\varphi)_i,\quad HZ\approx(H_Z)_i,\quad e\approx(\varepsilon)_{i+\frac12},$$
$$r=x_i,\quad R=\frac{r_{i+1}+r_i}{2},\quad \nu=\nu(e).$$

The difference analogue of the Ohm law looks like

$$EZ=\nu JZ=\frac{\nu}{R}H\varphi_x,$$
$$E\varphi=\nu J\varphi=-\nu RHZ_x,$$

where

$$EZ\approx(E_Z)_{i+\frac12},\quad E\varphi\approx\Big(\frac{E_\varphi}{r}\Big)_{i+\frac12},$$
$$JZ\approx(J_Z)_{i+\frac12},\quad J\varphi\approx(J_\varphi)_{i+\frac12}.$$

For the system of difference equations (1.2.1), (1.2.2) it is possible to deduce energetic equality similar to (1.1.9). Towards this end, we multiply the first equation of (1.2.1) by $H\varphi_ih$, and the second one by HZ_ih. Summing-up the resulting equations with respect to i together with (1.2.2) multiplied by R_ih, we obtain

$$\frac{d}{dt}\bigg[\sum_{i=1}^{M-1}\Big(\frac{H\varphi_i^2}{2r_i}+\frac{r_iHZ_i^2}{2}\Big)h+\sum_{i=0}^{M-1}R_ie_ih\bigg]$$
$$=\sum_{i=1}^{M-1}(EZ_{i\overline{x}}H\varphi_i-E\varphi_{i\overline{x}}HZ_i)h+\sum_{i=0}^{M-1}(EZ_iH\varphi_{ix}-E\varphi_iHZ_{ix})h.$$

Applying in the right-hand side the discrete analogue of the formula of integration by parts, after simple transformations we get

$$\frac{d}{dt}\bigg[\sum_{i=1}^{M-1}\Big(\frac{H\varphi_i^2}{2r_i}+\frac{r_iHZ_i^2}{2}\Big)h+\sum_{i=0}^{M-1}R_ie_ih\bigg]=EZ_{M-1}H\varphi_M-E\varphi_0HZ_0. \tag{1.2.3}$$

Integrating equality (1.2.3) in time and taking into account the boundary and initial conditions, we find the analogue of the complete energy change law

$$\sum_{i=1}^{M-1}\Big(\frac{H\varphi_i^2}{2r_i}+\frac{r_iHZ_i^2}{2}\Big)h+\sum_{i=0}^{M-1}R_ie_ih=\sum_{i=0}^{M-1}R_i\varepsilon_{0i}h+\int\limits_0^t\big[EZ_{M-1}H_b(t')-E\varphi_0H(t')\big]\,dt'.$$

The construction of the difference equations is carried out in a usual way, replacing time derivative by the difference relations and build the two-layer schemes. Following [64] and making the time domain discretization in a standard way, we get a family of difference schemes for problem (1.1.27):

$$\begin{gathered}\frac{1}{r}\,H\varphi_t=\Big(\frac{\nu}{R}\,H\varphi_{\overline{x}}\Big)_x^{(\sigma_1)},\quad r_iHZ_t=(\nu RHZ_{\overline{x}})_x^{(\sigma_1)},\\ \mathrm{Re}_t=\Big(\frac{\nu}{R}\,H\varphi_{\overline{x}}\Big)^{(\sigma_2)}(H\varphi_{\overline{x}})^{(\sigma_3)}+(\nu RHZ_{\overline{x}})^{(\sigma_2)}(HZ_{\overline{x}})^{(\sigma_3)},\end{gathered}\tag{1.2.4}$$

where $0\le\sigma_k\le 1$, $k=1,2,3$ and the known notation $y^{(\sigma)}=\sigma y^{j+1}+(1-\sigma)y^j$ is also used.

It is possible to show that for the weights $\sigma_1=\sigma_2=\sigma$, $\sigma_3=0,5$, the scheme (1.2.4) has the property of a complete conservation (the difference analogues of the conservation laws of fluxes of a magnetic field and total energy are fulfilled).

Obviously, the study of a question on the convergence of difference scheme for a rather general dependence $\nu=\nu(\varepsilon)$, is a difficult problem.

1.2.2 Convergence of the finite difference schemes

Let us study the convergence of difference schemes for the case $|\alpha|\le 1/2$. To simplify the calculations, we assume that the magnetic field vector is represented by one component and consider the Cartesian system of coordinates. Thus, we consider the following problem:

$$\begin{gathered}\frac{\partial U}{\partial t}=\frac{\partial}{\partial x}\Big(V^\alpha\,\frac{\partial U}{\partial x}\Big),\\ \frac{\partial V}{\partial t}=V^\alpha\Big(\frac{\partial U}{\partial x}\Big)^2,\end{gathered}\tag{1.2.5}$$

$$\begin{gathered}U(0,t)=U(1,t)=0,\quad U(x,0)=U_0(x),\\ V(x,0)=V_0(x)\ge\delta_0=const>0.\end{gathered}\tag{1.2.6}$$

Using the relation $V_0(x,t)\ge\delta_0$, equations (1.2.5) can be rewritten in the form

$$\frac{\partial U}{\partial t}=\frac{\partial}{\partial x}\Big(W^\gamma\,\frac{\partial U}{\partial x}\Big),\quad \frac{\partial W}{\partial t}=\frac{1}{2}\,W^{\gamma-1}\Big(\frac{\partial U}{\partial x}\Big)^2,\tag{1.2.7}$$

where $\gamma=2\alpha$, $W=V^{1/2}$ and the initial condition for V will be converted accordingly.

The grid-function $u=\{u_i\}$ corresponding to U is considered in $\overline{\omega}_h$, whereas the function $w=\{w_i\}$ approximating W is considered at the centers of grid points.

Let us put the following implicit difference scheme into correspondence with problem (1.2.6), (1.2.7),

$$\begin{gathered}u_t=(wu_{\overline{x}})_x,\quad w_t=\frac{1}{2}\,w^{\gamma-1}u_{\overline{x}}^2,\\ u_0^j=u_M^j=0,\quad u_i^0=U_0(x_i),\quad w_i^0=V_0^{\frac{1}{2}}(x_{i+\frac{1}{2}}).\end{gathered}\tag{1.2.8}$$

The order of approximation accuracy of equation (1.2.8) on the smooth solution of the initial differential equations is $O(\tau+h^2)$.

The difference scheme (1.2.8), representing the system of the nonlinear algebraic equations, has the unique solution. To be convinced of the solvability, it is enough to use an a priori estimation which follows after multiplication of equations (1.2.8) by u and w, respectively, and apply the Brouwer fixed-point lemma (see, e.g., [54], or Section 4.1). Note that applying the same technique as that we will use in proving the convergence below, it is not difficult to prove the uniqueness of the solution and the stability of scheme (1.2.8).

For the errors $z = u - U$, $s = w - W$, we have the equations

$$z_t = (w^\gamma u_{\overline{x}})_x - (W^\gamma U_{\overline{x}})_x + \psi_1, \quad s_t = \frac{1}{2}\,(w^{\gamma-1}u_x^2 - W^{\gamma-1}U_x^2) + \psi_2, \tag{1.2.9}$$

where

$$\psi_1 = O(\tau + h^2), \quad \psi_2 = O(\tau + h^2).$$

Multiplying equation (1.2.9) scalarly by $2\tau z$ and $2\tau s$, respectively, we have

$$\begin{aligned} \|z\|_h^2 - \|z^j\|_h^2 + \tau^2\|z_t\|_h^2 &= -2\tau\big(w^\gamma u_x - W^\gamma U_x, u_x - U_x\big] + 2\tau(\psi_1, z), \\ \|s\|_h^2 - \|s^j\|_h^2 + \tau^2\|s_t\|_h^2 &= 2\tau\big(w^{\gamma-1}u_x^2 - W^{\gamma-1}U_x^2, s\big] + 2\tau(\psi_2, s). \end{aligned} \tag{1.2.10}$$

Here we use the difference analogue of Green's formula and the relation

$$2\tau(r_t, r) = \|r\|_h^2 - \|r^j\|_h^2 + \tau^2\|r_t\|_h^2,$$

where r is an arbitrary grid-function.

Summing-up relations (1.2.10) and applying some simple transformations, we get

$$\begin{aligned} &\|z\|_h^2 + \|s\|_h^2 + \tau^2\|z_t\|_h^2 + \tau^2\|s_t\|_h^2 - \|z^j\|_h^2 - \|s^j\|_h^2 \\ &\quad = -\Big(\frac{w^\gamma + w^{\gamma-1}W}{2}\,u_{\overline{x}}^2 + \frac{W^\gamma + W^{\gamma-1}w}{2}\,U_{\overline{x}}^2, 1\Big] + \big(w^\gamma + W^\gamma, u_{\overline{x}}U_{\overline{x}}\big] + 2\tau(\psi_l, z) + 2\tau(\psi_2, s) \\ &\quad \le \Big(\Big[(W^\gamma + W^{\gamma-1}w)(w^\gamma + w^{\gamma-1}W)\Big]^{\frac{1}{2}} + W^\gamma - w^\gamma, |u_{\overline{x}}|\,|U_{\overline{x}}|\Big] + 2\tau(\psi_1, z) + 2\tau(\psi_2, s). \end{aligned}$$

Considering the inequality

$$\begin{aligned} &(W^\gamma + W^{\gamma-1}w)(w^\gamma + w^{\gamma-1}W) \\ &\qquad = (W^\gamma + w^\gamma)^2 - (W^{\gamma-1} - w^{\gamma-1})(W^{\gamma+1} - w^{\gamma+1}) \ge (W^\gamma + w^\gamma)^2, \end{aligned}$$

which, in turn, for $|\gamma| \le 1$ follows from an obvious inequality

$$-(W^{\gamma-1} - w^{\gamma-1})(W^{\gamma+1} - w^{\gamma+1}) \ge 0,$$

we arrive at

$$\|z\|_h^2 + \|s\|_h^2 + \tau^2\|z_t\|_h^2 + \tau^2\|s_t\|_h^2 \le \|z^j\|_h^2 + \|s^j\|_h^2 + 2\tau(\psi_1, z) + 2\tau(\psi_2, s).$$

Applying usual methodology, from the above inequality we get

$$\|z^{k+1}\|_h + \|s^{k+1}\|_h = O(\tau + h^2).$$

Thus, the convergence of the difference scheme (1.2.8) is proved.

Note that in [22] the convergence of the following two-parameterized difference scheme is proved:

$$\begin{aligned} u_t + \beta\tau u_{tt} &= \big[(w^{(\alpha)})^\gamma u_x^{(\alpha)}\big]_x, \\ w_t + \beta\tau w_{tt} &= \frac{1}{2}\,(w^{(\alpha)})^{\gamma-1}(u_x^{(\alpha)})^2, \\ u(0,t) = u(1,t) &= 0, \quad u(x,0) = U_0(x), \\ u(x,\tau) &= U_0(x) + \tau\big[(W^{(\alpha)})^\gamma U_x^{(\alpha)}\big]_{t=0}, \\ W(x,0) = W_0(x), \quad W(x,\tau) &= W_0(x) + \frac{1}{2}\,\tau\big[(W^{(\alpha)})^{\gamma-1}(U_x^{(\alpha)})^2\big]_{t=0}. \end{aligned} \tag{1.2.11}$$

Here,

$$v^{(\alpha)} = \alpha v^{j+1} + (1-\alpha)v^j.$$

The scheme (1.2.11) has the following order of approximation:

$$O(\tau^2 + h^2 + (\alpha - 0,5 - \beta)\tau).$$

The convergence of this difference scheme is established under the condition $\alpha - 0.5 - \beta \ge 0$. Thus, if $\alpha = 1/2$, $\beta = 0$, the two-layer difference scheme with accuracy of order $O(\tau^2 + h^2)$ is constructed. The same accuracy takes place if $\alpha = 1$, $\beta = 1/2$. In this case, (1.2.11) is the three-layer scheme.

1.3 Splitting methods and difference schemes for the system with the Joule law and heat conductivity

1.3.1 Some preliminary remarks

Let us consider the questions of the approximate integration of problems, which are connected with the following system of nonlinear equations:

$$\begin{aligned}\frac{\partial U}{\partial t} &= \frac{\partial}{\partial x}\Big(V^\alpha \frac{\partial U}{\partial x}\Big),\\ \frac{\partial V}{\partial t} &= V^\alpha\Big(\frac{\partial U}{\partial x}\Big)^2 + A(V),\end{aligned} \tag{1.3.1}$$

where $A(V)$ is the second order differential operator of an elliptic type.

It is easy to see that system (1.3.1) includes a diffusion system considered in Section 1.1 as a partial case, if under the functions U and V respectively are given the components of the magnetic field and temperature, and the differential operator $A(V)$ has the form

$$A(V) = \frac{\partial}{\partial x}\Big(V^\beta \frac{\partial V}{\partial x}\Big).$$

Some questions concerning systems (1.3.1) are also considered in [2]. In particular, the uniqueness of the solution of the initial-boundary value problem for fairly general operators $A(V)$ is studied. Various discrete analogues are constructed and their mathematical investigations are carried out.

In the numerical integration of equations (1.3.1) it is natural to begin with the standard finite difference models. Accordingly, in this section the difference scheme is constructed and its convergence is proved for equations (1.3.1). However, besides the usual approaches, it is interesting to develop additive models, which are, as is known, successfully applied to numerous problems of mathematical physics (see, e.g., [1, 2, 14, 15, 57, 64, 72] and the references therein). As we already have mentioned in Subsection 1.1.2, it is possible to build various additive models if system (1.3.1) is divided into two groups of equations: the first group describes diffusion process with regard for only the Joule law of heating and the second group is used for the description of a heat conductivity process.

The present section consists of three subsections. In the first subsection we establish the uniqueness of the stated problem. In the second and third subsections we construct and investigate the approximate analogues of the following problem:

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\Big(V^\alpha \frac{\partial U}{\partial x}\Big), \tag{1.3.2}$$

$$\frac{\partial V}{\partial t} = V^\alpha\Big(\frac{\partial U}{\partial x}\Big)^2 + \frac{\partial^2 V}{\partial x^2}, \tag{1.3.3}$$

$$U(0,t) = U(1,t) = \frac{\partial V}{\partial x}\Big|_{x=0} = \frac{\partial V}{\partial x}\Big|_{x=1} = 0, \tag{1.3.4}$$

$$U(x,0) = U_0(x), \quad V(x,0) = V_0(x) \ge \delta_0 = const > 0, \tag{1.3.5}$$

where $|\alpha| \le 1/2$; $U_0(x)$ and $V_0(x)$ are the given functions on $[0,1]$.

Following [2] and using the notations $V^{1/2} = W$, $2\alpha = \gamma$, we can rewrite equations (1.3.2), (1.3.3) in the form:

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\Big(W^\gamma \frac{\partial U}{\partial x}\Big), \tag{1.3.6}$$

$$\frac{\partial W}{\partial t} = \frac{1}{2}W^{\gamma-1}\Big(\frac{\partial U}{\partial x}\Big)^2 + \frac{\partial^2 W}{\partial x^2} + \frac{1}{W}\Big(\frac{\partial W}{\partial x}\Big)^2, \tag{1.3.7}$$

for which the boundary and initial conditions will be transformed accordingly, and $|\gamma| \le 1$.

Assume that the solution of problem (1.3.4)–(1.3.7) has a required smoothness which will be necessary in our further discussions.

1.3.2 On the uniqueness of a solution

Let U_1, W_1, U_2, W_2 be two pairs of solutions of system (1.3.6), (1.3.7) under the appropriate initial-boundary conditions.

For $Z = U_1 - U_2$ and $S = W_1 - W_2$, we easily get the following relations:

$$\begin{aligned}\frac{1}{2}\,\frac{d}{dt}\,\|Z\|^2 &= -\Big(W_1^\gamma\,\frac{\partial U_1}{\partial x} - W_2^\gamma\,\frac{\partial U_2}{\partial x}\,,\frac{\partial Z}{\partial x}\Big)\\ &= -\bigg(W_1^\gamma,\Big[\frac{\partial U_1}{\partial x}\Big]^2\bigg) - \bigg(W_2^\gamma,\Big[\frac{\partial U_2}{\partial x}\Big]^2\bigg) + \Big(W_1^\gamma + W_2^\gamma,\,\frac{\partial U_1}{\partial x}\,\frac{\partial U_2}{\partial x}\Big),\end{aligned} \tag{1.3.8}$$

$$\begin{aligned}\frac{1}{2}\,\frac{d}{dt}\,\|S\|^2 &= \frac{1}{2}\,\bigg(W_1^{\gamma-1}\Big[\frac{\partial U_1}{\partial x}\Big]^2 - W_2^{\gamma-1}\Big[\frac{\partial U_2}{\partial x}\Big]^2,S\bigg) - \Big\|\frac{\partial S}{\partial x}\Big\|^2\\ &\quad+\bigg(\frac{1}{W_1}\,\Big[\frac{\partial W_1}{\partial x}\Big]^2 - \frac{1}{W_2}\,\Big[\frac{\partial W_2}{\partial x}\Big]^2,S\bigg),\end{aligned} \tag{1.3.9}$$

where $(\,\cdot\,,\,\cdot\,)$ and $\|\,\cdot\,\|$ denote the scalar product and the norm in $L_2(0,\,1)$, respectively.

Designating the last term in (1.3.9) through J, we can transform it as follows:

$$\begin{aligned}J &= \bigg(\Big[\frac{\partial W_1}{\partial x}\Big]^2 + \Big[\frac{\partial W_2}{\partial x}\Big]^2,1\bigg) - \bigg(\frac{W_2}{W_1}\,\Big[\frac{\partial W_1}{\partial x}\Big]^2 + \frac{W_1}{W_2}\,\Big[\frac{\partial W_2}{\partial x}\Big]^2,1\bigg)\\ &= \Big\|\frac{\partial S}{\partial x}\Big\|^2 - \bigg\|\Big(\frac{W_2}{W_1}\Big)^{\frac{1}{2}}\,\frac{\partial W_1}{\partial x} - \Big(\frac{W_1}{W_2}\Big)^{\frac{1}{2}}\,\frac{\partial W_2}{\partial x}\bigg\|^2.\end{aligned}$$

Using the last identity from (1.3.9), we obtain the inequality

$$\frac{1}{2}\,\frac{d}{dt}\,\|S\|^2 \le \frac{1}{2}\,\bigg(W_1^\gamma\Big[\frac{\partial U_1}{\partial x}\Big]^2 + W_2^\gamma\Big[\frac{\partial U_2}{\partial x}\Big]^2,1\bigg) - \frac{1}{2}\,\bigg(W_1^{\gamma-1}W_2\Big[\frac{\partial U_1}{\partial x}\Big]^2 - W_2^{\gamma-1}W_1\Big[\frac{\partial U_2}{\partial x}\Big]^2,1\bigg),$$

by adding of which to (1.3.8) we find that

$$\begin{aligned}&\frac{1}{2}\,\frac{d}{dt}\,\big(\|S\|^2 + \|Z\|^2\big)\\ &\quad\le -\bigg(\frac{W_1^\gamma + W_1^{\gamma-1}W_2}{2}\,\Big[\frac{\partial U_1}{\partial x}\Big]^2 + \frac{W_2^\gamma + W_2^{\gamma-1}W_1}{2}\,\Big[\frac{\partial U_2}{\partial x}\Big]^2,1\bigg) + \Big(W_2^\gamma + W_1^\gamma,\,\frac{\partial U_1}{\partial x}\,\frac{\partial U_2}{\partial x}\Big)\\ &\quad\le \int\limits_\Omega \bigg\{\Big[(W_1^\gamma + W_1^{\gamma-1}W_2)(W_2^\gamma + W_2^{\gamma-1}W_1)\Big]^{\frac{1}{2}} - W_1^\gamma - W_2^\gamma\bigg\}\,\Big|\frac{\partial U_2}{\partial x}\Big|\,\Big|\frac{\partial U_2}{\partial x}\Big|\,dx.\end{aligned} \tag{1.3.10}$$

Note that if $-1 \le \gamma \le 1$, then $-(W_1^{\gamma-1} - W_2^{\gamma-1})(W_1^{\gamma+1} - W_2^{\gamma+1}) \ge 0$, and the validity of the estimation

$$\begin{aligned}(W_1^\gamma + W_1^{\gamma-1}W_2)(W_2^\gamma + W_2^{\gamma-1}W_1) &= 2W_1^\gamma W_2^\gamma + W_1^{\gamma-1}W_2^{\gamma+1} + W_2^{\gamma-1}W_1^{\gamma+1}\\ &= (W_1^\gamma + W_2^\gamma)^2 - (W_1^{\gamma-1} - W_2^{\gamma-1})(W_1^{\gamma+1} - W_2^{\gamma+1})\\ &\ge (W_1^\gamma + W_2^\gamma)^2\end{aligned}$$

becomes obvious.

By virtue of the above inequality, from (1.3.10) it follows that

$$\frac{d}{dt}\,\big(\|S\|^2 + \|Z\|^2\big) \le 0$$

and, therefore, $Z \equiv 0$, $S \equiv 0$.

Thus, the uniqueness of the solution of problem (1.3.4)–(1.3.7) is proved.

Remark 1.3.1. It is easy to be convinced that the proof given above can likewise be applied to the case, where instead of the second equation of system (1.3.2), (1.3.3) we consider the equation

$$\frac{\partial V}{\partial t} = V^\alpha\Big(\frac{\partial U}{\partial x}\Big)^2 + A(V),$$

where $A(V)$ is the differential operator such that

$$\Big(V_1^{-\frac{1}{2}}A(V_1) - V_2^{-\frac{1}{2}}A(V_2), V_1^{\frac{1}{2}} - V_2^{\frac{1}{2}}\Big) \le 0.$$

Now, let us take the initial system of equations as follows:

$$\begin{aligned} \frac{\partial U}{\partial t} &= \frac{\partial}{\partial x}\Big(V^{\beta}\,\frac{\partial U}{\partial x}\Big), \\ \frac{\partial V}{\partial t} &= V^{\alpha}\Big(\frac{\partial U}{\partial x}\Big)^2 + A(V). \end{aligned} \tag{1.3.11}$$

Here, $A(V)$ is the elliptic operator satisfying some analogue of the requirements of monotonicity

$$\big(V_1^{-\alpha}A(V_1) - V_2^{-\alpha}A(V_2), V_1^{\beta} - V_2^{\beta}\big) \le 0. \tag{1.3.12}$$

The functions U and V satisfy conditions (1.3.4), (1.3.5). The parameters satisfy the following restrictions: $\alpha + \beta = 1$, $\alpha < 1$.

Rewrite the second equation of (1.3.11) as

$$\frac{1}{\beta}\,\frac{\partial V^{\beta}}{\partial t} = \Big(\frac{\partial U}{\partial x}\Big)^2 + V^{-\alpha}A(V).$$

Consider now the differences $Z = U_1 - U_2$, $S = V_1^{\beta} - V_2^{\beta}$, where U_1, V_1 and U_2, V_2 are again two pairs of solutions of the problem under investigation. After simple transformations, we get the following relations:

$$\begin{aligned} \frac{1}{2}\,\frac{d}{dt}\,\|Z\|^2 &= -\bigg(\frac{V_1^{\beta} - V_2^{\beta}}{2}, \Big[\frac{\partial U_1}{\partial x}\Big]^2 - \Big[\frac{\partial U_2}{\partial x}\Big]^2\bigg) - \bigg(\frac{V_1^{\beta} + V_2^{\beta}}{2}, \Big[\frac{\partial Z}{\partial x}\Big]^2\bigg), \\ \frac{1}{4\beta}\,\frac{d}{dt}\,\|S\|^2 &\le \bigg(\frac{V_1^{\beta} + V_2^{\beta}}{2}, \Big[\frac{\partial U_1}{\partial x}\Big]^2 - \Big[\frac{\partial U_2}{\partial x}\Big]^2\bigg). \end{aligned}$$

To derive the last inequality, we use condition (1.3.12). Further, it is easy to see that

$$\frac{d}{dt}\Big(\|Z\|^2 + \frac{1}{2\beta}\,\|S\|^2\Big) \le 0,$$

whence we obtain $U_1 \equiv U_2$, $V_1 \equiv V_2$.

1.3.3 Splitting-up methods with respect to physical processes

It is possible to build various additive models if system (1.3.1) is divided into two groups of equations: the first group describes diffusion process with regard for only the Joule law of heating and looks like

$$\begin{aligned} \frac{\partial U_1}{\partial t} &= \frac{\partial}{\partial x}\Big(V_1^{\alpha}\,\frac{\partial U_1}{\partial x}\Big), \\ \frac{\partial V_1}{\partial t} &= V_1^{\alpha}\Big(\frac{\partial U_1}{\partial x}\Big)^2, \end{aligned}$$

and the second group is used to describe the heat conductivity process. The similar splitting-up of an initial problem is named as a division according to physical processes. It allows one to transfer the results available in connection with equations (1.2.5). On the other hand, to the appropriate heat equation (in the absence of the nonlinear term of the Joule heating) the well-advanced numerical methods can be applied.

According to the standard approach, the sub-problems are solved sequentially, taking the solution of a first step as the initial data for the next one. The possibility of constructing the additive model with parallel count algorithm is also considered. For this purpose the semi-discrete average additive model is constructed on the basis of the approach suggested for the linear problems and advanced ones in a nonlinear case (see, e.g., [2] and the references therein). For the cases, where the uniqueness of the solution of an initial problem takes place, it is possible, as usual, to establish the convergence of the approximate solutions to the exact one. A multi-dimensional case for equations (1.3.11) can be investigated according to the similar scheme, if splitting-up regarding the spatial variables is performed in advance.

1.3.4 Convergence of the semi-discrete additive models

In this subsection, following the technique developed in [57, 64, 72], the semi-discrete scheme (the Rothe-type scheme) of sum-approximation is constructed. The split problems are solved in parallel and the question of the convergence is studied.

We divide the domain $[0,T]$ uniformly into N parts and introduce the following notations:

$$\tau=\frac{T}{N}\,,\quad \omega_\tau=\big\{t_j:\ t_0=0,\ t_{j+1}=t_j+\tau,\ j=0,1,\dots,N-1\big\},$$
$$y_\alpha(t_{j-1})=y_\alpha^{j-1},\quad y_\alpha(t_{j+1})=y_\alpha^{j+1}=y_\alpha,\quad y_\alpha(t_j)=y_\alpha^j,$$
$$y_t=\frac{y^{j+1}-y^j}{\tau}\,,\quad y_{\alpha t}=\frac{y_\alpha-y^j}{\tau}\,,$$
$$\eta_1+\eta_2=1,\quad \eta_1>0,\quad \eta_2>0,\quad y^j=\eta_1y_1^j+\eta_2y_2^j.$$

Let to problem (1.3.4)–(1.3.7) be assigned the semi-discrete additive scheme.

Let
$$u=\eta_1u_1+\eta_2u_2\ \text{ and }\ w=\eta_1w_1+\eta_2w_2,$$
where u_α, w_α, $\alpha=1,2$, are the solutions of the following systems of ordinary differential equations:

$$u_{1t}=\frac{d}{dx}\Big[w_1^\gamma\,\frac{du_1}{dx}\Big],\quad \eta_1w_{1t}=\frac12\,w_1^{\gamma-1}\Big[\frac{du_1}{dx}\Big]^2, \tag{1.3.13}$$
$$u_{2t}=\frac{d}{dx}\Big[w_2^\gamma\,\frac{du_2}{dx}\Big],\quad \eta_2w_{2t}=\frac{d^2\,w_2}{dx^2}+\frac{1}{w_2}\,\Big[\frac{dw_2}{dx}\Big]^2, \tag{1.3.14}$$
$$y_\alpha^0=y^0=U_0(x),\quad w_\alpha^0=w^0=W_0(x),$$

where $\eta_1+\eta_2=1$, $\eta_1>0$, $\eta_2>0$ and η_1, η_2 are the constants.

Define the errors
$$z_\alpha=U_\alpha-u_\alpha,\quad s_\alpha=W_\alpha-w_\alpha,\ \ \alpha=1,2.$$

Obviously,
$$z=\eta_1z_1+\eta_2z_2,\quad s=\eta_1s_1+\eta_2s_2.$$

It is easy to be convinced that these errors satisfy the following system:

$$\begin{aligned}
&\eta_1z_{1t}=\frac{d}{dx}\Big[W^\gamma\,\frac{\partial U}{\partial x}-w_1^\gamma\,\frac{du_1}{dx}\Big]+\psi_1,\\
&\eta_1s_{1t}=\frac12\Big[W^{\gamma-1}\Big(\frac{\partial U}{\partial x}\Big)^2-w_1^{\gamma-1}\Big(\frac{du_1}{dx}\Big)^2\Big]+\psi_2,\\
&\eta_2z_{2t}=\eta_2\,\frac{d}{dx}\Big[W^\gamma\,\frac{\partial U}{\partial x}-w_2^\gamma\,\frac{du_2}{dx}\Big]+\psi_3,\\
&\eta_2s_{2t}=\frac{\partial^2s_2}{\partial x^2}+\frac{1}{W}\,\Big(\frac{\partial W}{\partial x}\Big)^2-\frac{1}{w_2}\,\Big(\frac{dw_2}{dx}\Big)^2+\psi_4,
\end{aligned} \tag{1.3.15}$$

where

$$\begin{aligned}
&\psi_1=\eta_1\Big(U_t-\frac{dU}{dt}\Big)=O(\tau),\\
&\psi_2=-\eta_2\,\frac{W^{\gamma-1}}{2}\,\Big(\frac{dU}{dx}\Big)^2+\eta_1\Big(\frac{d^2W}{dx^2}+\frac{1}{W}\,\Big(\frac{dW}{dx}\Big)^2\Big)+\eta_1\Big(W_t-\frac{dW}{dt}\Big),\\
&\psi_3=\eta_2\Big(U_t-\frac{dU}{dt}\Big)=O(\tau),\\
&\psi_4=\eta_2\,\frac{W^{\gamma-1}}{2}\,\Big(\frac{dU}{dx}\Big)^2-\eta_1\Big(\frac{d^2W}{dx^2}+\frac{1}{W}\Big(\frac{dW}{dx}\Big)^2\Big)+\eta_2\Big(W_t-\frac{dW}{dt}\Big).
\end{aligned}$$

Thus, the condition of sum-approximation is satisfied as
$$\psi_2+\psi_4=O(\tau).$$

After multiplication of equations (1.3.15) scalarly by the appropriate errors, we have

$$\begin{aligned} 2\eta_2\tau(z_{1t},z_1) &= 2\tau(\psi_1,z_1)+J_1, \quad 2\eta_1^2\tau(s_{1t},s_1)=2\tau\eta_1(\psi_2,s_1)+J_2,\\ 2\eta_2\tau(z_{2t},z_2) &= 2\tau(\psi_3,z_2)+J_3, \quad 2\eta_1\eta_2\tau(s_{2t},s_1)=2\tau\eta_1(\psi_4,s_1)+J_4, \end{aligned} \tag{1.3.16}$$

where

$$\begin{aligned} J_1 &= -2\tau\eta_1\Big(W^\gamma\frac{dU}{dx}-w_1^\gamma\frac{dy_1}{dx},\frac{dz_1}{dx}\Big),\\ J_2 &= \tau\eta_1\bigg(W^{\gamma-1}\Big(\frac{dU}{dx}\Big)^2-w_1^{\gamma-1}\Big(\frac{dy_1}{dx}\Big)^2,s_1\bigg),\\ J_3 &= -2\tau\eta_2\Big(W^\gamma\frac{dU}{dx}-w_2^\gamma\frac{dy_2}{dx},\frac{dz_2}{dx}\Big),\\ J_4 &= 2\tau\eta_1\Big|\frac{ds_1}{dx}\Big|^2+2\tau\eta_1\bigg(\frac{1}{W}\Big(\frac{dW}{dx}\Big)^2-\frac{1}{w_2}\Big(\frac{dw_2}{dx}\Big)^2,s_2\bigg). \end{aligned}$$

Using the same transformations as are carried out in proving the uniqueness of the solution of the initial-boundary value problem for system (1.3.6), (1.3.7) (see also [2]), we deduce

$$J_1+J_2\le 0, \quad J_4\le 0.$$

Estimate the scalar product J_3,

$$\begin{aligned} J_3 &= -2\tau\eta_2\bigg[\Big(W^\gamma-w_2^\gamma,\frac{dz_2}{dx},\frac{dU}{dx}\Big)+\Big(w_2^\gamma,\Big(\frac{dz_2}{dx}\Big)^2\Big)\bigg]\\ &\le \frac{\tau\eta_2}{2}\bigg(\frac{\big(W^\gamma-w_2^\gamma\big)^2}{w_2^\gamma},\Big(\frac{dU}{dx}\Big)^2\bigg)\le C\tau\Big(s_2^2,\Big(\frac{dU}{dx}\Big)^2\Big). \end{aligned}$$

Here, the constant $C=\eta_2\gamma\delta_0^{2-\gamma}/2$ comes from the inequality

$$\frac{|W^\gamma-w_2^\gamma|}{w_2^{\frac{\gamma}{2}}}\le|\gamma|\delta_0^{1-\frac{\gamma}{2}}|W^\gamma-w_2|,$$

the validity of which is easily deduced by using the following known inequality [29]

$$kr(x-y)x^{k-1}<x^k-y^k<k(x-y)y^{k-1}, \ \ 0<k<1,$$

and taking

$$W\ge\delta_0, \quad w_2\ge\delta_0.$$

Adding equalities (1.3.16) and using the relations

$$r^{j+1}=r^j+\tau r_t, \quad 2\tau(r_t,r)=\|r^{j+1}\|^2-\|r^j\|^2+\tau^2\|r_t\|^2,$$

where r is an arbitrary grid-function, we find that

$$\begin{aligned} \eta_1\|z_1\|^2+\eta_1\|s_1\|^2+\eta_2\|z_2\|^2+\eta_1\eta_2\|s_2\|^2+\tau^2\big(\eta_1\|z_{1t}\|^2+\eta_2\|z_{2t}\|^2\big)+\tau^2\eta_1\big(\eta_1\|s_{1t}\|^2+\eta_2\|s_{2t}\|^2\big)\\ \le\eta_1\|s^j\|^2+\|z^j\|^2+2\tau\big[(\psi_1+\psi_3,\,z^j)+\eta_1(\psi_2+\psi_4,s^j)\big]+C\tau\Big(s_2^2,\Big(\frac{\partial U}{\partial x}\Big)^2\Big)\\ +2\tau^2\Big[(\psi_1,z_{1t})+(\psi_3,z_{2t})+\eta_1(\psi_2,s_{1t})+\eta_1(\psi_4,s_{2t})\Big], \end{aligned}$$

from which, using the ε-inequality, it follows that

$$\begin{aligned} &\eta_1\|z_1\|^2+\eta_1\big(\eta_1\|s_1\|^2+\eta_2\|s_2\|^2\big)+\eta_2\|z_2\|^2\\ &\qquad\le\eta_1\|s^j\|^2+\|z^j\|^2+\tau^2\|\psi\|^2+2\tau\big(O(\tau),|z^j+s^j|\big)+C_1\eta_1\tau\|s_2\|^2, \end{aligned}$$

where

$$\|\psi\|^2 = \eta_1\|\psi_1\|^2 + \eta_2\|\psi_3\|^2 + \|\psi_2\|^2 + \frac{\eta_1}{\eta_2}\,\|\psi_4\|^2.$$

Let us strengthen the last inequality

$$(1-\tau C_1)\Big[\eta_1\|z_1\|^2 + \eta_2\|z_2\|^2 + \eta_1\big(\eta_1\|s_1\|^2 + \eta_2\|s_2\|^2\big)\Big] \le \tau^2\|\psi\|^2 + (1+\tau)\big(\|z^j\|^2 + \eta_1\|s^j\|^2\big).$$

Here, the term of the rate $O(\tau^3)$ is ignored. Further, using the property of convexity of the norm

$$\eta_1\|z_1\|^2 + \eta_2\|z_2\|^2 \ge \|\eta_1 z_1\|^2 + \|\eta_2 z_2\|^2 = \|z\|^2, \quad \eta_1\|s_1\|^2 + \eta_2\|s_2\|^2 \ge \|s\|^2,$$

we finally obtain

$$\|\Phi^{j+1}\|^2 \le \|\Phi^j\|^2 + \tau C_2\|\Phi^j\|^2 + \frac{\tau^2}{1-\tau C_1}\,\|\psi\|^2,$$

where

$$\|\Phi^j\|^2 = \|z^j\|^2 + \eta_1\|s^j\|^2, \quad C_2 = \frac{1+C_1}{1-\tau C_1}\,.$$

Summing-up by j and taking into account the relation $\Phi^0 = 0$ (under obvious restriction of a step τ), we find

$$\|\Phi^n\|^2 \le \tau C_1 \sum_{j=1}^{n-1} \|\Phi^j\|^2 + \tau C_3\|\psi\|^2, \quad C_3 = \frac{1}{1-\tau C_1}\,.$$

Finally, using a discrete analogue of Gronwall's inequality, the validity of the estimation

$$\|u^j - U(\,\cdot\,,t_j)\| + \|w^j - W(\,\cdot\,,t_j)\| = O(\tau^{\frac{1}{2}})$$

is proved.

Remark 1.3.2. The following average semi-discrete model can be investigated in a similar way:

$$u_t = \frac{d}{dx}\Big[(\eta_1 w_1^\gamma + \eta_2 w_2^\gamma)\,\frac{du}{dx}\Big], \quad \eta_1 w_{1t} = \frac{1}{2}\,w_1^{\gamma-1}\Big[\frac{du}{dx}\Big]^2, \quad \eta_2 w_{2t} = \frac{d^2 w_2}{dx^2} + \frac{1}{w_2}\,\Big[\frac{dw_2}{dx}\Big]^2,$$
$$w_1^j = w_2^j = w^j = \eta_1 w_1(t_j) + \eta_2 w_2(t_j).$$

1.3.5 Convergence of the finite difference scheme

Let us study the convergence of the difference scheme. For the sake of convenience, we consider the case with $\gamma = 1$. Thus, we consider the following problem:

$$\begin{aligned} \frac{\partial U}{\partial t} &= \frac{\partial}{\partial x}\Big(W\,\frac{\partial U}{\partial x}\Big), \\ \frac{\partial W}{\partial t} &= \frac{1}{2}\,\Big(\frac{\partial U}{\partial x}\Big)^2 + \frac{\partial^2 W}{\partial x^2} + \frac{1}{W}\,\Big(\frac{\partial W}{\partial x}\Big)^2, \end{aligned} \tag{1.3.17}$$

$$\begin{gathered} U(0,t) = U(1,t) = W(0,t) = W(1,t) = 0, \\ U(x,0) = U_0(x), \quad W(x,0) = W_0(x) \ge \delta_0 > 0. \end{gathered} \tag{1.3.18}$$

On $\Omega = [0,1]$, once again, we introduce a uniform grid ω_h, where h designates, as usual, a step on a spatial variable. The grid-function $u = \{u_i\}$, corresponding to U, is considered in $\overline{\omega}_h$, whereas the function $w = \{w_i\}$, approximating W, is considered at the centers of grid points.

Let to problem (1.3.17), (1.3.18) be assigned the following implicit difference scheme:

$$\begin{aligned} &u_t = (w u_{\overline{x}})_x, \\ &w_t = \frac{1}{2}\,u_{\overline{x}}^2 + w_{\overline{x}x} + \frac{1}{2w}\,(w_{\overline{x}}^2 + w_x^2), \end{aligned} \tag{1.3.19}$$

$$\begin{aligned} &u_0^j = u_M^j = 0, \quad w_{\frac{1}{2}}^j = w_{M+\frac{1}{2}}^j = 0, \\ &u_i^0 = U_0(x_i), \qquad w_i^0 = W_0(x_{i+\frac{1}{2}}). \end{aligned} \tag{1.3.20}$$

It is not difficult to find that difference equations (1.3.19) approximate the initial differential equations on the smooth solutions with accuracy of order $O(\tau + h^2)$.

The above-mentioned difference scheme represents a system of nonlinear algebraic equations which has the unique solution. The proof of the solvability can be carried out by using a priori estimations which are obtained by multiplication of (1.3.19) scalarly by u and w, respectively, and applying the Brouwer fixed-point lemma (see, e.g., [54], or Section 4.1). Note that applying the technique as in proving the convergence below, it is not difficult to prove the uniqueness of the solution and the stability of scheme (1.3.19)–(1.3.20).

The equations for an error look as follows:

$$\begin{aligned} z_t &= (wu_{\overline{x}})_x - (WU_{\overline{x}})_x + \psi_1, \\ s_t &= 0.5(u_{\overline{x}}^2 - U_{\overline{x}}^2) + s_{\overline{x}x} + \frac{w_{\overline{x}}^2 + w_x^2}{2w} + \frac{W_{\overline{x}}^2 + W_x^2}{2W} + \psi_2, \end{aligned} \tag{1.3.21}$$

where

$$z = u - U, \quad s = w - W,$$

and the errors of approximation will be

$$\psi_1 = O(\tau + h^2), \quad \psi_2 = O(\tau + h^2).$$

Multiplying the first equation of scheme (1.3.21) scalarly by $2\tau(u-U) = 2\tau z$ and applying Green's difference formula, we get

$$\|z\|_h^2 - \|z^j\|_h^2 + \tau^2\|z\|_h^2 = -2\tau\big(wu_{\overline{x}} - WU_{\overline{x}}, u_{\overline{x}} - U_{\overline{x}}\big]_h + 2\tau(\psi_1, z)_h, \tag{1.3.22}$$

here, $\|\cdot\|_h$, $(\cdot\,,\cdot)_h$, $(\cdot\,,\cdot]_h$ denote discrete analogues of norm and scalar products in $L_2(\omega_h)$.

Taking into account the relation $z = z^j + \tau z_t$ and identity

$$(wu_{\overline{x}} - WU_{\overline{x}})(u_{\overline{x}} - U_{\overline{x}}) = \frac{w+W}{2}\, z_{\overline{x}}^2 + \frac{s}{2}\,(u_{\overline{x}}^2 - U_{\overline{x}}^2),$$

from (1.3.22) we get the inequality

$$\|z\|_h^2 - \|z^j\|_h^2 + \tau^2\|z_t\|_h^2 + \big(u_{\overline{x}}^2 - U_{\overline{x}}^2, s\big]_h \le 2\tau(\psi_1, z^j)_h + \tau^2\|z_t\|_h^2 + \tau^2\|\psi_1\|_h^2,$$

whence

$$\|z\|_h^2 + \tau\big(u_{\overline{x}}^2 - U_{\overline{x}}^2, s\big]_h \le \|z^j\|_h^2 + 2\tau(\psi_1, z^j)_h + \tau^2\|\psi_1\|_h^2. \tag{1.3.23}$$

Multiplying the second equation of the scheme (1.3.21) scalarly by $2\tau s$ and acting similarly, we get

$$\|s\|_h^2 - \tau\big(u_x^2 - U_x^2, s\big]_h \le \|s^j\|_h^2 + 2\tau(\psi_2, s^j)_h + \tau^2\|\psi_2\|_h^2. \tag{1.3.24}$$

Adding inequalities (1.3.23) and (1.3.24), we deduce

$$\|z\|_h^2 + \|s\|_h^2 \le \|z^j\|_h^2 + \|s^j\|_h^2 + \tau^2\|\psi\|_h^2 + 2\tau\big[(\psi_1, z^j)_h - (\psi_2, s^j)_h\big], \tag{1.3.25}$$

where

$$\|\psi\|_h^2 = \|\psi_1\|_h^2 + \|\psi_2\|_h^2.$$

Summing-up (1.3.25) over j from zero to k, we arrive at

$$\|z^{k+1}\|_h^2 + \|s^{k+1}\|_h^2 \le \tau \sum_{j=1}^{k} \big(\|z^j\|_h^2 + \|s^j\|_h^2 + \|\psi^j\|_h^2\big).$$

Thus, we have obtained the convergence of the finite difference scheme (1.3.19), (1.3.20) and the validity of the estimation

$$\|z^{k+1}\|_h + \|s^{k+1}\|_h = \|u^{k+1} - U^{k+1}\|_h + \|w^{k+1} - W^{k+1}\|_h = O(\tau + h^2).$$

1.4 Reduction of the system of Maxwell equations to the nonlinear parabolic Volterra-type and averaged integro-differential models

1.4.1 Volterra-type integro-differential model

The IDEs arising in mathematics and physics often contain derivatives with respect to several variables; therefore, these equations are referred to as PIDEs. Numerous publications deal with the study of IDEs of various kinds (see, e.g., [4, 8, 11, 12, 27, 39, 41, 48, 51, 61] and the references therein). Many scientific papers are dedicated to the investigation of nonlinear integro-differential equations of parabolic type. Such an integro-differential model appears, for example, in the mathematical modeling of penetration of electromagnetic field into a medium whose electric conductivity substantially depends on temperature. On the basis of the system of Maxwell differential equations [47], in [28] the above-mentioned diffusion problem was reduced to an integro-differential model. The corresponding initial-boundary value problems were posed. The uniqueness and existence of their global solutions were considered.

Let us briefly describe reduction to the above-mentioned integro-differential model. Assume that the massive body is placed in a variable magnetic field. It is necessary to describe the field distribution inside the body. According to [47], consider the following system of Maxwell equations describing the interaction of electromagnetic field with the medium:

$$-\frac{1}{c}\frac{\partial H}{\partial t} = \nabla \times E, \tag{1.4.1}$$

$$\operatorname{div} H = 0, \tag{1.4.2}$$

$$\frac{4\pi}{c}\, J = \nabla \times H, \tag{1.4.3}$$

$$J = \sigma E. \tag{1.4.4}$$

Introducing the resistance $\rho = 1/\sigma$, from the last two equations, we can express electric field as follows:

$$E = \rho\,\frac{c}{4\pi}\,\nabla \times H.$$

Substituting this expression into (1.4.1), we get

$$\frac{\partial H}{\partial t} + \frac{c^2}{4\pi}\,\nabla \times (\rho \nabla \times H) = 0. \tag{1.4.5}$$

When penetrating into a medium, a variable magnetic field induces a variable electric field which causes appearance of currents. These currents lead to heating of a medium and raising its temperature θ which affects the resistance ρ. According to the reasoning cited in [47], it follows that the power-like change of temperature leads to a change of the resistance ρ in several orders. Therefore, for large variations of temperature the dependence $\rho = \rho(\theta)$ must be taken into account. The last significant restriction that should be made is due to the assumption that the change of temperature in the medium under the influence of the current J obeys the Joule law which has the form

$$c_v\,\frac{\partial \theta}{\partial t} = \rho J^2. \tag{1.4.6}$$

Here, ρ is the medium density and c_v is the heat capacity. In general, they are also dependent on the temperature θ. Equation (1.4.6) does not consider the process of heat transfer by heat conductivity and radiation. A number of other physical effects are not considered either. However, in this form, system (1.4.5), (1.4.6) is quite complicated from a mathematical point of view. Many scientific papers are devoted to the investigation and numerical solution of a system of Maxwell equations and to models like those equations (see, e.g., [19, 25, 39, 56, 60, 63, 73] and the references therein). It should also be noted that (1.4.5), (1.4.6) type models arise in mathematical modeling of many other processes (see, e.g., [9, 13, 39, 69, 70] and the references therein).

Let us begin with a reduction of system (1.4.5), (1.4.6) to the system of nonlinear integro-differential equations.

We rewrite equation (1.4.6) in the form

$$\frac{c_v}{\rho(\theta)}\frac{\partial \theta}{\partial t} = J^2.$$

Introducing the function

$$S(\theta) = \int_{\theta_0}^{\theta} \frac{c_v}{\rho(\xi)}\, d\xi,$$

we have

$$\frac{\partial S}{\partial t} = J^2.$$

Suppose that the process starts for $t = 0$, which corresponds to a constant temperature θ_0 over the medium. Integrating this equation over the interval $[0, t]$, we obtain

$$S(\theta(x,t)) - S(\theta_0) = \int_0^t J^2\, d\tau.$$

The functions c_v and ρ are positive due to physical sense; therefore, $S(\theta)$ is a monotonically increasing function. Thus, there exists a uniquely defined inverse function $\theta = \varphi(S)$, related to the function $S(\theta)$ by the relation $\varphi(S(\theta)) = \theta$. So, we can write

$$\theta(x,t) = \varphi\Big(\int_0^t J^2\, d\tau\Big).$$

From equation (1.4.3) we have

$$J = \frac{c}{4\pi}\nabla \times H,$$

so,

$$\theta(x,t) = \varphi\Big(\int_0^t \Big|\frac{c}{4\pi}\nabla \times H\Big|^2 d\tau\Big).$$

Substituting this expression into (1.4.5) as an argument of the function $\rho = \rho(\theta)$, we get

$$\frac{\partial H}{\partial t} + \frac{c^2}{4\pi}\nabla \times \Big[\rho\Big(\varphi\Big(\int_0^t \Big|\frac{c}{4\pi}\nabla \times H\Big|^2 d\tau\Big)\Big)\nabla \times H\Big] = 0,$$

$$\operatorname{div} H = 0.$$

Introduce the notations

$$a(S) = \frac{c^2}{4\pi}\rho(\varphi(S)), \quad W = \frac{c}{4\pi}H,$$

and rewrite the above system in the form

$$\frac{\partial W}{\partial t} + \nabla \times \Big[a\Big(\int_0^t |\nabla \times W|^2\, d\tau\Big)\nabla \times W\Big] = 0, \qquad \operatorname{div} W = 0. \tag{1.4.7}$$

If the magnetic field has the form $W = (0, U, V)$, where the function $U = U(x, y, t)$ depends on two spatial variables, then we have

$$\nabla \times U = \Big(\frac{\partial U}{\partial y}, -\frac{\partial U}{\partial x}, 0\Big)$$

and equation (1.4.7) takes the form

$$\frac{\partial U}{\partial t} = \nabla\left[a\left(\int_0^t |\nabla U|^2\,d\tau\right)\nabla U\right]. \tag{1.4.8}$$

It should be noted that model (1.4.7) was appeared first in [28]. This work, together with the other issues, provide us with the uniqueness of solutions of the initial-boundary value problems for (1.4.7) under fairly general assumptions on the function $a = a(S)$. Forms of models (1.4.7) and (1.4.8) naturally give an answer to the question why they are called Volterra-type.

Assume that c_v is constant. Below, we present some examples of functions $a = a(S)$ and inducing them functions $\rho(\theta)$ (see, e.g., [19, 28, 49, 50]).

If the function $\rho(\theta)$ is uniformly bounded, that is, $0 < \rho_0 \le \rho(\theta) \le \rho_1$, where ρ_0 and ρ_1 are some constants, then $a(S)$ possesses an analogous property

$$\frac{c^2}{4\pi}\,\rho_0 \le a(S) \le \frac{c^2}{4\pi}\,\rho_1.$$

If $\rho(\theta) = \theta^\alpha$, $\alpha > 1$, then

$$a(S) = C_1(C_0 - S)^p,$$

where C_0 and C_1 are some positive constants and $p = -\frac{\alpha}{\alpha-1} < -1$. For example, if $\rho(\theta) = 1 + \theta^2$, then

$$a(S) = C_0\,\mathrm{tg}(C_1 + c_\nu^{-2} S).$$

Thus, the polynomial growth of $\rho(\theta)$ leads to a determination of $a = a(S)$ only on a finite interval. Note that there are no physical media with such a property.

If $\rho(\theta) = \theta$, then

$$a(S) = C_1 e^{C_0 S}$$

with positive constants C_0 and C_1, i.e., the linear growth of the function $\rho(\theta)$ leads to an exponential function $a = a(S)$. The linear growth of the resistance with the growth of temperature is characteristic for metals.

If $\rho(\theta) = \theta^\alpha$, $0 < \alpha < 1$, then

$$a(S) = C_1(C_0 + S)^p,$$

where C_0 and C_1 are some positive constants, and $p = \frac{\alpha}{1-\alpha} > 0$. Thus, under-linear growth of the resistance $\rho(\theta)$ leads to a polynomial growth of the function $a = a(S)$.

If $\rho(\theta) = \theta^\alpha$, $\alpha < 0$, then

$$a(S) = C_1(C_0 + S)^p,$$

where C_0 and C_1 are some positive constants and $p = \frac{\alpha}{1-\alpha} > 0$, $-1 < p < 0$. Thus, a decreasing function $\rho(\theta)$ induces a decreasing function $a = a(S)$. This is a general rule if c_v is constant. Indeed, suppose that the function $\rho(\theta)$ is differentiable for $\theta \ge \theta_0$. Then, according to the definition,

$$\frac{da}{dS} = \frac{c^2}{4\pi}\,\frac{d\rho}{d\theta}\,\frac{d\theta}{dS} = \frac{c^2}{4\pi}\,\frac{d\rho/d\theta}{dS/d\theta} = \frac{c^2}{4\pi\rho c_\nu}\,\rho(\theta)\rho'(\theta).$$

Hence, it is obvious that the function $a = a(S)$ increases or decreases simultaneously with the function $\rho(\theta)$. Note that the decrease of the resistance $\rho(\theta)$ with increasing temperature is characteristic for semiconductors in the solid phase, for gases and also for plastic, for which the formula $\rho(\theta) = K\theta^{-3/2}$ takes place.

1.4.2 On the averaged integro-differential model

A certain generalization of equations of type (1.4.7) is proposed by G. Laptev. In particular, assuming the temperature of the considered body to be constant throughout the material, i.e., depending on time but independent of the spatial coordinates, the same process of penetration of the magnetic field into the material is modeled by the so-called averaged integro-differential model.

Let us begin with a procedure of derivation of the above-mentioned average equation describing again the process of penetration of an electromagnetic field into a medium [47].

Following [47], as in Subsection 1.4.1, let us consider a system of Maxwell equations (1.4.1)–(1.4.4) describing the interaction of an electromagnetic field into a medium.

The Joule law (1.4.6) represents the law of localization of heat allocation [47],

$$dQ = \int\limits_{\Omega} JE \, dx \, dt. \tag{1.4.9}$$

Here, dQ is an inflow of a thermal energy gained by a body in an electromagnetic field within time dt in all mass of the medium, occupying the area $\Omega \subset R^3$. The Joule law does not take into account the process of heat transfer inside a body, that is valid if temperature of a body is considered to be constant along the medium, i.e., dependent on time, but independent of spatial coordinates. Thus, in this case it is possible to admit that $\theta = \theta(t)$. Under such an assumption, $dQ = mc_\nu d\theta$, where m is a mass of the medium, and then equation (1.4.1) takes the form

$$mc_v(\theta) \frac{d\theta}{dt} = \int\limits_{\Omega} JE \, dx.$$

Using the Ohm law, $E = \rho(\theta)J$, and repeating the process of substitution, we shall obtain the following analogue of system (1.4.7) in the same notations:

$$\frac{\partial W}{\partial t} + \nabla \times \left[a\left(\frac{1}{|\Omega|} \int\limits_0^t \int\limits_{\Omega} |\nabla \times W|^2 \, dx \, dt \right) \nabla \times W \right] = 0,$$
$$\operatorname{div} W = 0.$$

Here, $|\Omega|$ is the medium volume. Due to averaging through the area, the coefficient of this equation depends only on a time variable t and, consequently, the equation can be rewritten in the form

$$\frac{\partial W}{\partial t} = a\left(\int\limits_0^t \int\limits_{\Omega} |\nabla \times W|^2 \, dx \, d\tau \right) \Delta W. \tag{1.4.10}$$

Here, we have used the known formula of the field theory,

$$\nabla \times (\nabla \times W) = -\Delta W + \nabla \operatorname{div} W.$$

For a plane field $W = (0, 0, U)$, where $U = U(x, y, t)$ is a function of two spatial variables, equation (1.4.10) takes the form

$$\frac{\partial U}{\partial t} = a\left(\int\limits_0^t \int\limits_{\Omega} |\nabla U|^2 \, dx \, dt \right) \Delta U. \tag{1.4.11}$$

G. Laptev mentioned that investigation of models (1.4.10) and (1.4.11) type requires somewhat different approach, than that of the Volterra-type models.

Chapter 2

Investigation and numerical solution of one-dimensional Volterra-type nonlinear integro-differential equation

Chapter 2 studies asymptotics of a solution and a finite difference approximation of the nonlinear Volterra-type IDEs associated with the penetration of a magnetic field into a medium. Asymptotic properties of solutions for the initial-boundary value problem with homogeneous as well as nonhomogeneous Dirichlet boundary conditions are considered. The corresponding finite difference scheme is studied. The convergence of this scheme is proven.

Chapter 2 is organized as follows. In Section 2.1 we state the problem and consider the large time behavior of solutions of the first type initial-boundary value problems for the Volterra-type nonlinear IDE with the homogeneous boundary conditions and nonhomogeneous boundary condition on one side of lateral boundary. In Section 2.2 the stability and convergence of semi-discrete and finite difference schemes are discussed. Section 2.2 we finish with some conclusions and remarks on numerical implementations.

2.1 Large time behavior of solutions

2.1.1 Introduction

Many practical problems are described by integro-differential models. One type integro-differential nonlinear parabolic model is obtained at mathematical simulation of processes of electromagnetic field penetration into the medium. Based on the system of Maxwell equations [47], as we have mentioned in Chapter 1, this model appeared first in [28]. Many other processes are described by means of integro-differential models (see, e.g., [4, 8, 12, 27, 39, 42, 48, 51, 61] and the references therein). A lot of scientific works are dedicated to the investigation and numerical resolution of the initial-boundary value problems for models obtained in [28] and considered in Chapter 1 (see, e.g., [39] and the references therein). The existence, uniqueness and asymptotic behavior of a solution for such type of equations and systems are studied in [3, 5, 6, 16, 17, 19, 20, 23, 24, 28, 31, 32, 35, 37–39, 46, 49, 50, 53, 55] and in a number of other works, as well (for more detailed citations see, e.g., [39] and the references therein).

The present section is devoted to the investigation of asymptotic properties of a solution of the initial-boundary value problem with the first type boundary conditions for the model whose one-dimensional scalar analogue has the following form [28]:

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left[a\left(\int_0^t \left(\frac{\partial U}{\partial x}\right)^2 d\tau\right)\frac{\partial U}{\partial x}\right], \tag{2.1.1}$$

where $a = a(S)$ is a given function defined for $S \in [0,\infty)$.

Principal characteristic peculiarity of equation (2.1.1) is connected with the appearance of a nonlinear term depending on the time integral in the coefficient of a higher order derivative.

Note that the integro-differential equation of type (2.1.1) is complex and only special cases were investigated (see, e.g., [39] and the references therein).

The existence and uniqueness of the solutions of the initial-boundary value problems for the equations of type (2.1.1) are studied in [5,6,16,17,19,20,32,49,50,53,55] and in a number of other works as well. The existence theorems proved in [16,17,19,20,32] are based on Galerkin's method and compactness arguments as in [54,71] for nonlinear parabolic problems.

Asymptotic behavior of solutions, as $t \to \infty$, of the initial-boundary value problem for equation (2.1.1) are given in many works (see, e.g., [3,39] and the references therein).

2.1.2 Asymptotic behavior of solutions with homogeneous boundary conditions

Consider the following initial-boundary value problem:

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left[a(S)\frac{\partial U}{\partial x}\right], \quad (x,t) \in Q = (0,1)\times(0,\infty), \tag{2.1.2}$$

$$U(0,t) = U(1,t) = 0, \quad t \ge 0, \tag{2.1.3}$$

$$U(x,0) = U_0(x), \quad x \in [0,1], \tag{2.1.4}$$

where

$$S(x,t) = \int_0^t \left(\frac{\partial U}{\partial x}\right)^2 d\tau, \tag{2.1.5}$$

and $U_0 = U_0(x)$ is a given function.

For the earlier work on the asymptotic behavior in time of solutions of problem (2.1.2)–(2.1.5) see, e.g., [31].

It is easy to verify the following statement [32].

Lemma 2.1.1. If $a(S) \ge a_0 = const > 0$ and $U_0 \in L_2(0,1)$, then for the solution of problem (2.1.2)–(2.1.5) the estimate

$$\|U\| \le C\exp(-a_0 t)$$

holds.

Here and below, C denote positive constants independent of t.

Note that Lemma 2.1.1 gives exponential stabilization of the solution of problem (2.1.2)–(2.1.5) in the norm of the space $L_2(0,1)$. In [31], using the scheme of investigation presented in [43], we study an asymptotic behavior of a solution in the norm of the space $W_2^1(0,1)$ without rate of convergence. In this direction see Section 4.4, where more general (2.1.2) type multi-dimensional equations are treated.

Here and below, we use the usual Sobolev spaces $W_2^k(0,1)$ and $\overset{\circ}{W}{}_2^k(0,1)$.

The purpose of this section is to show that for some cases the stabilization is likewise achieved in the norm of the space $C^1(0,1)$. First, we formulate the result of the stabilization in the space $W_2^1(0,1)$ [23]. Indeed,

Theorem 2.1.1. If $a(S)=(1+S)^p$, $0<p\le 1$, $U_0\in W_2^2(0,1)\cap \overset{\circ}{W}{}_2^1(0,1)$, then for the solution of problem (2.1.2)–(2.1.5) the estimate

$$\Big\|\frac{\partial U}{\partial x}\Big\|+\Big\|\frac{\partial U}{\partial t}\Big\|\le C\exp\Big(-\frac{t}{2}\Big)$$

is true.

Let us now prove the following main statement of this section.

Theorem 2.1.2. If $a(S)=(1+S)^p$, $0<p\le 1$, $U_0\in W_2^2(0,1)\cap \overset{\circ}{W}{}_2^1(0,1)$, then for the solution of problem (2.1.2)–(2.1.5) the relation

$$\Big|\frac{\partial U(x,t)}{\partial x}\Big|\le C\exp\Big(-\frac{t}{2}\Big)$$

holds.

In order to prove Theorem 2.1.2, we present some auxiliary estimates.

Lemma 2.1.2. For the function S the estimates

$$c\varphi^{\frac{1}{1+2p}}(t)\le 1+S(x,t)\le C\varphi^{\frac{1}{1+2p}}(t)$$

are true, where

$$\varphi(t)=1+\int_0^t\int_0^1\sigma^2\,dx\,d\tau \tag{2.1.6}$$

and $\sigma=(1+S)^p\,\partial U/\partial x$.

Proof. From definition of the function S it follows that

$$\frac{\partial S}{\partial t}=\Big(\frac{\partial U}{\partial x}\Big)^2,\quad S(x,0)=0. \tag{2.1.7}$$

We multiply (2.1.7) by $(1+S)^{2p}$,

$$\frac{1}{1+2p}\,\frac{\partial(1+S)^{1+2p}}{\partial t}=\Big(\frac{\partial U}{\partial x}\Big)^2(1+S)^{2p}.$$

Note that equation (2.1.2) can be rewritten as

$$\frac{\partial U}{\partial t}=\frac{\partial\sigma}{\partial x}\,. \tag{2.1.8}$$

We have

$$\frac{1}{1+2p}\,\frac{\partial(1+S)^{1+2p}}{\partial t}=\sigma^2, \tag{2.1.9}$$

$$\sigma^2(x,t)=\int_0^1\sigma^2(y,t)\,dy+2\int_0^1\int_y^x\sigma(\xi,t)\,\frac{\partial U(\xi,t)}{\partial t}\,d\xi\,dy. \tag{2.1.10}$$

From Theorem 2.1.1 and relations (2.1.6), (2.1.9), (2.1.10), we get

$$\begin{aligned}
\frac{1}{1+2p}\,(1+S)^{1+2p} &= \int_0^t \sigma^2\,d\tau + \frac{1}{1+2p}\\
&= \int_0^t\!\!\int_0^1 \sigma^2(y,\tau)\,dy\,d\tau + 2\int_0^t\!\!\int_0^1\!\!\int_y^x \sigma(\xi,\tau)\,\frac{\partial U(\xi,\tau)}{\partial\tau}\,d\xi\,dy\,d\tau + \frac{1}{1+2p}\\
&\le 2\int_0^t\!\!\int_0^1 \sigma^2(y,\tau)\,dy\,d\tau + \int_0^t\!\!\int_0^1 \Big(\frac{\partial U(x,\tau)}{\partial\tau}\Big)^2\,dx\,d\tau + \frac{1}{1+2p}\\
&\le 2\int_0^t\!\!\int_0^1 \sigma^2(y,\tau)\,dy\,d\tau + C_1\int_0^t \exp(-\tau)\,d\tau + \frac{1}{1+2p} \le C_2\varphi(t),
\end{aligned}$$

i.e.,

$$1+S(x,t) \le C\varphi^{\frac{1}{1+2p}}(t). \tag{2.1.11}$$

Analogously,

$$\begin{aligned}
\frac{1}{1+2p}(1+S)^{1+2p} &= \int_0^t\!\!\int_0^1 \sigma^2(y,\tau)\,dy\,d\tau + 2\int_0^t\!\!\int_0^1\!\!\int_y^x \sigma(\xi,\tau)\frac{\partial U(\xi,\tau)}{\partial\tau}\,d\xi\,dy\,d\tau + \frac{1}{1+2p}\\
&\ge \frac{1}{2}\int_0^t\!\!\int_0^1 \sigma^2(y,\tau)\,dy\,d\tau - C_2 = \frac{1}{2}\,\varphi(t) - C_3.
\end{aligned} \tag{2.1.12}$$

We have

$$C_3(1+S)^{1+2p} \ge C_3. \tag{2.1.13}$$

From (2.1.10) and (2.1.11), we get

$$\Big(\frac{1}{1+2p} + C_3\Big)(1+S)^{1+2p} \ge \frac{1}{2}\,\varphi(t),$$

or

$$1+S(x,t) \ge c\varphi^{\frac{1}{1+2p}}(t). \tag{2.1.14}$$

Finally, from (2.1.11) and (2.1.14) follows Lemma 2.1.2.

Taking into account relation (2.1.6), Lemma 2.1.2 and Theorem 2.1.1, we have

$$\frac{d\varphi(t)}{dt} = \int_0^1 (1+S)^{2p}\Big(\frac{\partial U}{\partial x}\Big)^2\,dx \le C\varphi^{\frac{2p}{1+2p}}(t)\exp(-t),$$

or

$$\frac{d}{dt}(\varphi^{\frac{1}{1+2p}}(t)) \le C\exp(-t).$$

After integrating from 0 to t, keeping in mind definition (2.1.6), we get

$$1 \le \varphi(t) \le C,$$

whence, using Lemma 2.1.2, we obtain

$$1 \le 1+S(x,t) \le C. \tag{2.1.15}$$

Using (2.1.15) and Theorem 2.1.1, equality (2.1.10) yields

$$\sigma^2(x,t) \le 2\int_0^1 (1+S)^{2p}\Big(\frac{\partial U}{\partial x}\Big)^2\,dx + \int_0^1 \Big(\frac{\partial U}{\partial t}\Big)^2\,dx \le C\exp(-t),$$

or

$$|\sigma(x,t)| \le C\exp\Big(-\frac{t}{2}\Big).$$

This estimate, taking into account (2.1.15) and the relation $\sigma = (1+S)^p\partial U/\partial x$, completes the proof of Theorem 2.1.2. □

2.1.3 Asymptotic behavior of solutions with nonhomogeneous condition on a part of the boundary

In the domain Q, let us consider the following initial-boundary value problem:

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\Big[a(S)\,\frac{\partial U}{\partial x}\Big],\quad (x,t)\in Q, \tag{2.1.16}$$

$$U(0,t)=0,\quad U(1,t)=\psi,\quad t\ge 0, \tag{2.1.17}$$

$$U(x,0)=U_0(x),\quad x\in[0,1], \tag{2.1.18}$$

where

$$S(x,t) = \int_0^t \Big(\frac{\partial U}{\partial x}\Big)^2\,d\tau, \tag{2.1.19}$$

$a(S)=(1+S)^p$, $0<p\le 1$, $\psi = Const > 0$; $U_0 = U_0(x)$ is a given function.

The main purpose of this subsection is to prove the following statement.

Theorem 2.1.3. If $a(S)=(1+S)^p$, $0<p\le 1$, $U_0\in W_2^2(0,1)$, $U_0(0)=0$, $U_0(1)=\psi$, then for the solution of problem (2.1.16)–(2.1.19) the estimate

$$\Big|\frac{\partial U(x,t)}{\partial x} - \psi\Big| \le Ct^{-1-p},\quad t\ge 1,$$

is true.

In this subsection, C, C_i and c denote the positive constants dependent on ψ, U_0 and independent of t.

The proof of Theorem 2.1.3 is based on a priori estimates which will be obtained below.

Lemma 2.1.3. For the solution of problem (2.1.16)–(2.1.19) the following estimate takes place

$$\int_0^t\int_0^1 \Big(\frac{\partial U}{\partial \tau}\Big)^2\,dx\,d\tau \le C.$$

Proof. Let us differentiate equation (2.1.16) with respect to t,

$$\frac{\partial^2 U}{\partial t^2} - \frac{\partial}{\partial x}\Big[\frac{\partial(1+S)^p}{\partial t}\,\frac{\partial U}{\partial x} + (1+S)^p\,\frac{\partial^2 U}{\partial t\partial x}\Big] = 0,$$

and multiply the result scalarly by $\partial U/\partial t$. Using the formula of integrating by parts and boundary conditions (2.1.17), we get

$$\frac{1}{2}\,\frac{d}{dt}\int_0^1 \Big(\frac{\partial U}{\partial t}\Big)^2\,dx + \int_0^1 (1+S)^p\Big(\frac{\partial^2 U}{\partial t\partial x}\Big)^2\,dx + p\int_0^1 (1+S)^{p-1}\Big(\frac{\partial U}{\partial x}\Big)^3\,\frac{\partial^2 U}{\partial t\partial x}\,dx = 0. \tag{2.1.20}$$

From (2.1.20), taking into account the Poincaré inequality, we have

$$\frac{d}{dt}\int_0^1 \Big(\frac{\partial U}{\partial t}\Big)^2 dx + 2\int_0^1 \Big(\frac{\partial U}{\partial t}\Big)^2 dx + \frac{p}{2}\int_0^1 (1+S)^{p-1}\,\frac{\partial}{\partial t}\Big(\frac{\partial U}{\partial x}\Big)^4 dx \le 0. \tag{2.1.21}$$

Let us integrate relation (2.1.21) from 0 to t

$$\int_0^1 \Big(\frac{\partial U}{\partial t}\Big)^2 dx + 2\int_0^t\!\!\int_0^1 \Big(\frac{\partial U}{\partial \tau}\Big)^2 dx\,d\tau + \frac{p}{2}\int_0^t\!\!\int_0^1 (1+S)^{p-1}\,\frac{\partial}{\partial \tau}\Big(\frac{\partial U}{\partial x}\Big)^4 dx\,d\tau \le C.$$

An integration by parts gives

$$\int_0^1 \Big(\frac{\partial U}{\partial t}\Big)^2 dx + 2\int_0^t\!\!\int_0^1 \Big(\frac{\partial U}{\partial \tau}\Big)^2 dx\,d\tau \le C. \tag{2.1.22}$$

Therefore, Lemma 2.1.3 is proved. □

Note that from Lemma 2.1.3, according to the scheme applied in the second subsection, we get the validity of Lemma 2.1.2 for problem (2.1.16)–(2.1.19), as well.

Lemma 2.1.4. The following estimates are true

$$c\varphi^{\frac{2p}{1+2p}}(t) \le \int_0^1 \sigma^2(x,t)\,dx \le C\varphi^{\frac{2p}{1+2p}}(t).$$

Proof. Taking into account Lemma 2.1.2, we get

$$\int_0^1 \sigma^2\,dx = \int_0^1 (1+S)^{2p}\Big(\frac{\partial U}{\partial x}\Big)^2 dx \ge c\varphi^{\frac{2p}{1+2p}}(t)\int_0^1 \Big(\frac{\partial U}{\partial x}\Big)^2 dx$$
$$\ge c\varphi^{\frac{2p}{1+2p}}(t)\Big(\int_0^1 \frac{\partial U}{\partial x}\,dx\Big)^2 = \psi^2 c\varphi^{\frac{2p}{1+2p}}(t),$$

or

$$\int_0^1 \sigma^2(x,t)\,dx \ge c\varphi^{\frac{2p}{1+2p}}(t). \tag{2.1.23}$$

From (2.1.22), it follows that

$$\int_0^1 \Big(\frac{\partial U}{\partial t}\Big)^2 dx \le C. \tag{2.1.24}$$

Let us multiply equation (2.1.16) scalarly by U. Using the boundary conditions (2.1.17), we have

$$\int_0^1 U\,\frac{\partial U}{\partial t}\,dx + \int_0^1 (1+S)^p\Big(\frac{\partial U}{\partial x}\Big)^2 dx = \psi\sigma(1,t).$$

Using this equality, Lemma 2.1.3, relations (2.1.8), (2.1.10), (2.1.23), (2.1.24) and the maximum principle

$$|U(x,t)| \le \max_{0\le y\le 1}|U_0(y)|,\ \ 0\le x\le 1,\ \ t\ge 0,$$

we get

$$\Big\{\int_0^1 \sigma^2(x,t)\,dx\Big\}^2 \le C_1\varphi^{\frac{2p}{1+2p}}(t)\Big[\int_0^1 (1+S)^p\Big(\frac{\partial U}{\partial x}\Big)^2 dx\Big]^2$$
$$\le 2C_1\varphi^{\frac{2p}{1+2p}}(t)\bigg[(\psi\sigma(1,t))^2+\bigg(\int_0^1 U\,\frac{\partial U}{\partial t}\,dx\bigg)^2\bigg]$$
$$\le 2C_1\varphi^{\frac{2p}{1+2p}}(t)\bigg[2\psi^2\int_0^1\sigma^2\,dx+\psi^2\int_0^1\Big(\frac{\partial\sigma}{\partial x}\Big)^2 dx+\int_0^1 U^2\,dx\int_0^1\Big(\frac{\partial U}{\partial t}\Big)^2 dx\bigg]$$
$$\le 2C_1\varphi^{\frac{2p}{1+2p}}(t)\bigg[2\psi^2\int_0^1\sigma^2\,dx+\psi^2\int_0^1\Big(\frac{\partial U}{\partial t}\Big)^2 dx+\Big(\max_{0\le y\le 1}|U_0(y)|\Big)^2\int_0^1\Big(\frac{\partial U}{\partial t}\Big)^2 dx\bigg]$$
$$\le 2C_1\varphi^{\frac{2p}{1+2p}}(t)\bigg(C_2\int_0^1\sigma^2\,dx+\frac{C_3}{\varphi^{\frac{2p}{1+2p}}(t)}\int_0^1\sigma^2\,dx\bigg).$$

From this, taking into account the relation $\varphi(t)\ge 1$, we have

$$\int_0^1 \sigma^2(x,t)\,dx \le C\varphi^{\frac{2p}{1+2p}}(t).$$

Thus Lemma 2.1.4 is proved. □

From Lemma 2.1.4 and (2.1.6), we have the estimates

$$c\varphi^{\frac{2p}{1+2p}}(t)\le\frac{d\varphi(t)}{dt}\le C\varphi^{\frac{2p}{1+2p}}(t). \tag{2.1.25}$$

Lemma 2.1.5. The derivative $\partial U/\partial t$ satisfies the inequality

$$\int_0^1\Big(\frac{\partial U}{\partial t}\Big)^2 dx\le C\varphi^{-\frac{2}{1+2p}}(t).$$

Proof. Equality (2.1.20) yields

$$\frac{d}{dt}\int_0^1\Big(\frac{\partial U}{\partial t}\Big)^2 dx+\int_0^1(1+S)^p\Big(\frac{\partial^2 U}{\partial t\partial x}\Big)^2 dx\le p^2\int_0^1(1+S)^{p-2}\Big(\frac{\partial U}{\partial x}\Big)^6 dx. \tag{2.1.26}$$

Using now Lemmas 2.1.3, 2.1.4, the relation $\sigma=(1+S)^p\,\partial U/\partial x$ and identity

$$\int_0^1\Big(\frac{\partial\sigma}{\partial x}\Big)^2 dx=-\int_0^1\sigma\,\frac{\partial^2\sigma}{\partial x^2}\,dx,$$

from (2.1.26) we get

$$\frac{d}{dt}\int_0^1\Big(\frac{\partial U}{\partial t}\Big)^2 dx+c\varphi^{\frac{p}{1+2p}}(t)\int_0^1\Big(\frac{\partial^2 U}{\partial t\partial x}\Big)^2 dx$$
$$\le C_1\varphi^{-\frac{5p+2}{1+2p}}(t)\int_0^1\sigma^6\,dx\le C_1\varphi^{-\frac{5p+2}{1+2p}}(t)\int_0^1\sigma^2(x,t)\,dx\Big[\max_{0\le x\le 1}\sigma^2(x,t)\Big]^2$$

$$\le C_2\varphi^{-\frac{3p+2}{1+2p}}(t)\Big\{\int\limits_0^1 \sigma^2\,dx + 2\Big[\int\limits_0^1 \sigma^2\,dx\Big]^{\frac12}\Big[\int\limits_0^1 \Big(\frac{\partial\sigma}{\partial x}\Big)^2 dx\Big]^{\frac12}\Big\}^2$$

$$\le C_2\varphi^{-\frac{3p+2}{1+2p}}(t)\Big\{\int\limits_0^1 \sigma^2\,dx + 2\Big[\int\limits_0^1 \sigma^2\,dx\Big]^{\frac34}\Big[\int\limits_0^1 \Big(\frac{\partial^2\sigma}{\partial x^2}\Big)^2 dx\Big]^{\frac14}\Big\}^2$$

$$\le C_3\varphi^{\frac{p-2}{1+2p}}(t) + C_4\varphi^{-\frac{3p+2}{1+2p}}(t)\varphi^{\frac{3p}{1+2p}}(t)\Big[\int\limits_0^1 \Big(\frac{\partial^2 U}{\partial t\partial x}\Big)^2 dx\Big]^{\frac12}$$

$$\le C_3\varphi^{\frac{p-2}{1+2p}}(t) + C_5\varphi^{-\frac{p+4}{1+2p}}(t) + \frac{c}{2}\varphi^{\frac{p}{1+2p}}(t)\int\limits_0^1 \Big(\frac{\partial^2 U}{\partial t\partial x}\Big)^2 dx.$$

Note that in our case $p-2 > -p-4$. So, the last relation, by using the Poincaré inequality, results in

$$\frac{d}{dt}\int\limits_0^1 \Big(\frac{\partial U}{\partial t}\Big)^2 dx + \frac{c}{2}\varphi^{\frac{p}{1+2p}}(t)\int\limits_0^1 \Big(\frac{\partial U}{\partial t}\Big)^2 dx \le C\varphi^{\frac{p-2}{1+2p}}(t).$$

From Gronwall's inequality, we get

$$\frac{d}{dt}\int\limits_0^1 \Big(\frac{\partial U}{\partial t}\Big)^2 dx \le \exp\Big(-\frac{c}{2}\int\limits_0^t \varphi^{\frac{p}{1+2p}}(\tau)\,d\tau\Big)$$
$$\times\Big[\int\limits_0^1 \Big(\frac{\partial U}{\partial t}\Big)^2 dx\Big|_{t=0} + C\int\limits_0^t \exp\Big(\frac{c}{2}\int\limits_0^\tau \varphi^{\frac{p}{1+2p}}(\xi)\,d\xi\Big)\varphi^{\frac{p-2}{1+2p}}(\tau)\,d\tau\Big]. \qquad (2.1.27)$$

Noting that $\varphi(t)\ge 1$, applying L'Hopital rule and estimate (2.1.25), we have

$$\lim_{t\to\infty}\frac{\int\limits_0^t \exp\Big(\frac{c}{2}\int\limits_0^\tau \varphi^{\frac{p}{1+2p}}(\xi)\,d\xi\Big)\varphi^{\frac{p-2}{1+2p}}(\tau)\,d\tau}{\exp\Big(\frac{c}{2}\int\limits_0^t \varphi^{\frac{p}{1+2p}}(\tau)\,d\tau\Big)\varphi^{-\frac{2}{1+2p}}(t)}$$

$$= \lim_{t\to\infty}\frac{\exp\Big(\frac{c}{2}\int\limits_0^t \varphi^{\frac{p}{1+2p}}(\tau)\,d\tau\Big)\varphi^{\frac{p-2}{1+2p}}(t)}{\exp\Big(\frac{c}{2}\int\limits_0^t \varphi^{\frac{p}{1+2p}}(\tau)\,d\tau\Big)\Big(\frac{c}{2}\varphi^{\frac{p-2}{1+2p}}(t) - \frac{2}{1+2p}\,\varphi up^{\frac{-3-2p}{1+2p}}(t)\,\frac{d\varphi}{dt}\Big)}$$

$$\le \lim_{t\to\infty}\frac{1}{\frac{c}{2} - \frac{2C}{1+2p}\varphi^{-\frac{p+1}{1+2p}}(t)} \le C. \qquad (2.1.28)$$

Therefore, the validity of Lemma 2.1.5 follows from (2.1.27) and (2.1.28). □

Let us now estimate $\partial S/\partial x$ in $L_1(0,1)$.

Lemma 2.1.6. For $\partial S/\partial x$, the following estimate is true:

$$\int\limits_0^1 \Big|\frac{\partial S}{\partial x}\Big|\,dx \le C\varphi^{-\frac{p}{1+2p}}(t).$$

Proof. Let us differentiate (2.1.7) with respect to x,

$$\frac{\partial}{\partial t}\Big[(1+S)^{2p}\frac{\partial S}{\partial x}\Big] = 2\sigma\,\frac{\partial\sigma}{\partial x}. \qquad (2.1.29)$$

From Lemmas 2.1.4 and 2.1.5 we obtain

$$\int_0^1 \Big|\sigma\,\frac{\partial U}{\partial t}\Big|\,dx \le C\varphi^{\frac{p}{1+2p}}(t)\varphi^{-\frac{1}{1+2p}}(t) = C\varphi^{\frac{p-1}{1+2p}}(t). \tag{2.1.30}$$

Finally, from Lemma 2.1.3 and relations (2.1.8), (2.1.25), (2.1.29), (2.1.30), we have

$$(1+S)^{2p}\,\frac{\partial S}{\partial x} = \int_0^t 2\sigma\,\frac{\partial U}{\partial \tau}\,d\tau,$$

$$\int_0^1 \Big|\frac{\partial S}{\partial x}\Big|\,dx \le C\varphi^{-\frac{2p}{1+2p}}(t)\int_0^t \varphi^{\frac{p-1}{1+2p}}(\tau)\,d\tau \le C_1\varphi^{-\frac{2p}{1+2p}}(t)\int_0^t \varphi^{-\frac{p+1}{1+2p}}\,d\varphi$$
$$= C_2\varphi^{-\frac{2p}{1+2p}}(t)\int_0^t d\varphi^{\frac{p}{1+2p}} = C_2\varphi^{-\frac{2p}{1+2p}}(t)(\varphi^{\frac{p}{1+2p}}(t)-1) \le C\varphi^{-\frac{p}{1+2p}}(t).$$

Thus, Lemma 2.1.6 is proved. □

Proof of Theorem 2.1.3. We are ready to prove Theorem 2.1.3. Using (2.1.10) and Lemmas 2.1.4 and 2.1.5, we arrive at

$$\sigma^2(x,t) \le \int_0^1 \sigma^2(y,t)\,dy + 2\int_0^1 \Big|\sigma(y,t)\,\frac{\partial U(y,t)}{\partial t}\Big|\,dy \le C_1\varphi^{\frac{2p}{1+2p}}(t) + C_2\varphi^{\frac{p-1}{1+2p}}(t).$$

From this, we get

$$|\sigma(x,t)| \le C\varphi^{\frac{p}{1+2p}}(t).$$

Now, taking into account Lemmas 2.1.2, 2.1.5, 2.1.6, equality (2.1.8), definition of σ and the latter estimate, we derive

$$\int_0^1 \Big|\frac{\partial^2 U(x,t)}{\partial x^2}\Big|\,dx \le \int_0^1 \Big|\frac{\partial U}{\partial t}\,(1+S)^{-p}\Big|\,dx + p\int_0^1 \Big|\sigma(1+S)^{-p-1}\,\frac{\partial S}{\partial x}\Big|\,dx$$
$$\le \Big[\int_0^1 (1+S)^{-2p}\,dx\Big]^{\frac12}\Big[\int_0^1 \Big|\frac{\partial U}{\partial t}\Big|^2 dx\Big]^{\frac12} + p\int_0^1 \Big|\sigma(1+S)^{-p-1}\,\frac{\partial S}{\partial x}\Big|\,dx$$
$$\le C_1\varphi^{-\frac{p}{1+2p}}(t)\varphi^{-\frac{1}{1+2p}}(t) + C_2\varphi^{-\frac{p+1}{1+2p}}(t)\varphi^{\frac{p}{1+2p}}(t)\int_0^1 \Big|\frac{\partial S}{\partial x}\Big|\,dx$$
$$\le C_3\varphi^{-\frac{p+1}{1+2p}}(t).$$

Hence, we have

$$\int_0^1 \Big|\frac{\partial^2 U(x,t)}{\partial x^2}\Big|\,dx \le C\varphi^{-\frac{p+1}{1+2p}}(t).$$

From this estimate, taking into account the relation

$$\frac{\partial U(x,t)}{\partial x} = \int_0^1 \frac{\partial U(y,t)}{\partial y}\,dy + \int_0^1\!\!\int_y^x \frac{\partial^2 U(\xi,t)}{\partial \xi^2}\,d\xi\,dy,$$

we arrive at

$$\Big|\frac{\partial U(x,t)}{\partial x}-\psi\Big| = \Big|\int_0^1\int_y^x \frac{\partial^2 U(\xi,t)}{\partial \xi^2}\,d\xi\,dy\Big| \le \int_0^1 \Big|\frac{\partial^2 U(y,t)}{\partial y^2}\Big|\,dy \le C\varphi^{-\frac{p+1}{1+2p}}(t). \tag{2.1.31}$$

From (2.1.25), it is easy to show that

$$ct^{1+2p} \le \varphi(t) \le Ct^{1+2p}, \;\; t \ge 1.$$

From this, taking into account estimate (2.1.31), we get the validity of Theorem 2.1.3. □

Remark 2.1.1. The large time behavior to the solutions of the initial-boundary value problems (2.1.16)–(2.1.19) for the case -1/2 < p < 0 is studied in [32].

Note that in this section we have used the scheme similar to that of work [13] in which the adiabatic shearing of incompressible fluids with temperature-dependent viscosity is studied.

Remark 2.1.2. The existence of globally defined solutions of problems (2.1.2)–(2.1.5) and (2.1.16)–(2.1.19) can now be reobtained by a routine procedure, proving first the existence of local solutions at a maximal time interval and then, using the derived a priori estimates, showing that these solutions cannot escape at a finite time.

Results of Theorems 2.1.1, 2.1.2 and 2.1.3 show the difference between stabilization character of solutions with homogeneous and nonhomogeneous boundary conditions.

2.2 Stability and convergence of semi-discrete and finite difference schemes

2.2.1 Space-discretization of the problem

Many authors study the semi-discrete and finite difference approximations for a integro-differential models (see, e.g., [4, 8, 10, 11, 18, 19, 30, 33, 35, 37–39, 41, 42, 44, 51, 52, 61, 65–67, 74]).

In the present section we strengthen the results of stability and convergence of a semi-discrete scheme for the solution of first type initial-boundary value problem for equation (2.1.1) for the case $a(S) = (1+S)^p$, $0 < p \le 1$. We also discuss a finite difference scheme for equation (2.1.1) for the same case. Asymptotic behavior of a solution as $t \to \infty$ and a numerical solution of the initial-boundary value problem for equation (2.1.1) can be found in many works (see, e.g., [19,35,39] and the references therein).

Let us consider the problem

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left[\left(1+\int_0^t \left(\frac{\partial U}{\partial x}\right)^2 d\tau\right)^p \frac{\partial U}{\partial x}\right], \tag{2.2.1}$$

$$U(0,t) = U(1,t) = 0, \tag{2.2.2}$$

$$U(x,0) = U_0(x), \tag{2.2.3}$$

where $0 < p \le 1$ and U_0 is a given function.

On $[0,1]$, let us introduce a net with mesh points denoted by $x_i = ih$, $i = 0,1,\dots,M$, with $h = 1/M$. The boundaries are specified by $i = 0$ and $i = M$. In this subsection, the semi-discrete approximation for (x_i,t) is designed by $u_i = u_i(t)$. The exact solution to the problem for (x_i,t) is denoted by $U_i = U_i(t)$. At the points $i = 1,2,\dots,M-1$, the IDE will be replaced by approximation of the space derivatives by forward and backward differences. We will use the following known notations [64]:

$$r_{x,i}(t) = \frac{r_{i+1}(t)-r_i(t)}{h}\,, \quad r_{\overline{x},i}(t) = \frac{r_i(t)-r_{i-1}(t)}{h}\,.$$

Let to problem (2.2.1)–(2.2.3) be assigned the following semi-discrete scheme:

$$\frac{du_i}{dt}=\left\{\left(1+\int_0^t (u_{\overline{x},i})^2\,d\tau\right)^p u_{\overline{x},i}\right\}_x,\quad i=1,2,\dots,M-1, \tag{2.2.4}$$

$$u_0(t)=u_M(t)=0, \tag{2.2.5}$$

$$u_i(0)=U_{0,i},\quad i=0,1,\dots,M. \tag{2.2.6}$$

Thus, we have obtained the Cauchy problem (2.2.4)–(2.2.6) for the nonlinear system of OIDEs.

For the earlier work on the discretization in time or space, or both, of equations such as (2.2.4) for the case $p=1$, see [33].

Introduce usual inner products and norms:

$$(r,g)=h\sum_{i=1}^{M-1} r_i g_i,\quad (r,g]=h\sum_{i=1}^{M} r_i g_i,$$

$$\|r\|=(r,r)^{\frac12},\quad \|r]|=(r,r]^{\frac12}.$$

Multiplying equations (2.2.4) scalarly by $u(t)=(u_1(t),u_2(t),\dots,u_{M-1}(t))$, after simple transformations we get

$$\frac{d}{dt}\,\|u(t)\|^2+h\sum_{i=1}^{M}\left(1+\int_0^t (u_{\overline{x},i})^2\,d\tau\right)^p (u_{\overline{x},i})^2=0,$$

whence we obtain the inequality

$$\|u(t)\|^2+\int_0^t \|u_{\overline{x}}]|^2\,d\tau\le C; \tag{2.2.7}$$

here and below in this section C denotes a positive constant which does not depend on h.

Remark 2.2.1. The a priori estimate (2.2.7) guarantee the global solvability of scheme (2.2.4)–(2.2.6). Note that applying the same technique as when proving the convergence theorem below, it is not difficult to prove the uniqueness of the solution and stability of scheme (2.2.4)–(2.2.6), too.

The principal aim of the present subsection is to prove the following statement.

Theorem 2.2.1. If $0<p\le 1$ and problem (2.2.1)–(2.2.3) has a sufficiently smooth solution $U=U(x,t)$, then the solution $u=u(t)=(u_1(t),u_2(t),\dots,u_{M-1}(t))$ of problem (2.2.4)–(2.2.6) tends to the solution of the continuous problem $U=U(t)=(U_1(t),U_2(t),\dots,U_{M-1}(t))$ as $h\to 0$ and the estimate

$$\|u(t)-U(t)\|\le Ch \tag{2.2.8}$$

holds.

Proof. For $U=U(x,t)$, we have

$$\frac{dU_i}{dt}-\left\{\left(1+\int_0^t (U_{\overline{x},i})^2\,d\tau\right)^p U_{\overline{x},i}\right\}_x=\psi_i(t),\quad i=1,2,\dots,M-1, \tag{2.2.9}$$

$$U_0(t)=U_M(t)=0, \tag{2.2.10}$$

$$U_i(0)=U_{0,i},\quad i=0,1,\dots,M, \tag{2.2.11}$$

where

$$\psi_i(t)=O(h).$$

Let $z_i(t) = u_i(t) - U_i(t)$. From (2.2.4)–(2.2.6) and (2.2.9)–(2.2.11), we have

$$\frac{dz_i}{dt} - \left\{ \left(1 + \int\limits_0^t (u_{\overline{x},i})^2 \, d\tau\right)^p u_{\overline{x},i} - \left(1 + \int\limits_0^t (U_{\overline{x},i})^2 \, d\tau\right)^p U_{\overline{x},i} \right\}_x = -\psi_i(t), \tag{2.2.12}$$
$$z_0(t) = z_M(t) = 0,$$
$$z_i(0) = 0.$$

Multiplying equation (2.2.12) scalarly by $z(t) = (z_1(t), z_2(t), \dots, z_{M-1}(t))$ and using the discrete analogue of the formula of integration by parts, we get

$$\frac{1}{2}\frac{d}{dt}\|z\|^2 + \sum_{i=1}^{M} \left\{ \left(1 + \int\limits_0^t (u_{\overline{x},i})^2 \, d\tau\right)^p u_{\overline{x},i} - \left(1 + \int\limits_0^t (U_{\overline{x},i})^2 \, d\tau\right)^p U_{\overline{x},i} \right\} z_{\overline{x},i} h = -h \sum_{i=1}^{M-1} \psi_i z_i. \tag{2.2.13}$$

Note that

$$\begin{aligned}
&\left\{ \left(1 + \int\limits_0^t (u_{\overline{x},i})^2 \, d\tau\right)^p u_{\overline{x},i} - \left(1 + \int\limits_0^t (U_{\overline{x},i})^2 \, d\tau\right)^p U_{\overline{x},i} \right\} (u_{\overline{x},i} - U_{\overline{x},i}) \\
&\quad = p \int\limits_0^1 \left(1 + \int\limits_0^t \Big[U_{\overline{x},i} + \xi(u_{\overline{x},i} - U_{\overline{x},i})\Big]^2 d\tau\right)^{p-1} \\
&\qquad\qquad \times \frac{d}{dt}\left(\int\limits_0^t \Big[U_{\overline{x},i} + \xi(u_{\overline{x},i} - U_{\overline{x},i})\Big] (u_{\overline{x},i} - U_{\overline{x},i}) \, d\tau \right)^2 d\xi \\
&\qquad + \int\limits_0^1 \left(1 + \int\limits_0^t \Big[U_{\overline{x},i} + \xi(u_{\overline{x},i} - U_{\overline{x},i})\Big]^2 d\tau\right)^p d\xi \, (u_{\overline{x},i} - U_{\overline{x},i})^2.
\end{aligned}$$

After substituting this equality in (2.2.13), integrating the obtained equality on $(0, t)$ and using the formula of integration by parts, we get

$$\begin{aligned}
&\|z\|^2 + 2h \sum_{i=1}^{M} \int\limits_0^t \int\limits_0^1 \left(1 + \int\limits_0^{t'} \Big[U_{\overline{x},i} + \xi(u_{\overline{x},i} - U_{\overline{x},i})\Big]^2 d\tau'\right)^p (u_{\overline{x},i} - U_{\overline{x},i})^2 \, d\xi \, d\tau \\
&\quad + ph \sum_{i=1}^{M} \int\limits_0^1 \left(1 + \int\limits_0^t \Big[U_{\overline{x},i} + \xi(u_{\overline{x},i} - U_{\overline{x},i})\Big]^2 d\tau\right)^{p-1} \\
&\qquad \times \left(\int\limits_0^t \Big[U_{\overline{x},i} + \xi(u_{\overline{x},i} - U_{\overline{x},i})\Big] (u_{\overline{x},i} - U_{\overline{x},i}) \, d\tau \right)^2 d\xi \\
&\quad - p(p-1)h \sum_{i=1}^{M} \int\limits_0^1\!\int\limits_0^t \left(1 + \int\limits_0^{t'} \Big[U_{\overline{x},i} + \xi(u_{\overline{x},i} - U_{\overline{x},i})\Big]^2 d\tau'\right)^{p-2} \Big[U_{\overline{x},i} + \xi(u_{\overline{x},i} - U_{\overline{x},i})\Big]^2 \\
&\qquad \times \left(\int\limits_0^{t'} \Big[U_{\overline{x},i} + \xi(u_{\overline{x},i} - U_{\overline{x},i})\Big] (u_{\overline{x},i} - U_{\overline{x},i}) \, d\tau' \right)^2 d\xi \, d\tau \\
&\quad = -2h \sum_{i=1}^{M-1} \psi_i z_i.
\end{aligned}$$

Taking into account the restriction $0 < p \le 1$, the last equality yields

$$\|z(t)\|^2 \le \int_0^t \|z(\tau)\|^2\,d\tau + \int_0^t \|\psi_i\|^2\,d\tau. \tag{2.2.14}$$

From (2.2.14) we get (2.2.8), and thus Theorem 2.2.1 is proved. □

2.2.2 Finite difference scheme

In $[0,1]\times[0,T]$, let us consider again problem (2.2.1)–(2.2.3) and introduce a net on it with mesh points denoted by $(x_i,t_j)=(ih,j\tau)$, where $i=0,1,\dots,M$; $j=0,1,\dots,N$ with $h=1/M$, $\tau=T/N$. The initial line is denoted by $j=0$. The discrete approximation for (x_i,t_j) is designed by u_i^j and the exact solution to problem (2.2.1)–(2.2.3) by U_i^j. One again can use the following known notations [64]:

$$u_{x,i}^j = \frac{u_{i+1}^j - u_i^j}{h}\,, \quad u_{\overline{x},i}^j = \frac{u_i^j - u_{i-1}^j}{h}\,, \quad u_{t,i}^j = \frac{u_i^{j+1}-u_i^j}{\tau}\,.$$

For problem (2.2.1)–(2.2.3), let us consider the following finite difference scheme:

$$\frac{u_i^{j+1}-u_i^j}{\tau} - \left\{\Big(1+\tau\sum_{k=1}^{j+1}(u_{\overline{x},i}^k)^2\Big)^p u_{\overline{x},i}^{j+1}\right\}_x = f_i^j, \;\; i=1,2,\dots,M-1;\;\; j=0,1,\dots,N-1, \tag{2.2.15}$$

$$u_0^j = u_M^j = 0, \;\; j=0,1,\dots,N, \tag{2.2.16}$$

$$u_i^0 = U_{0,i}, \;\; i=0,1,\dots,M. \tag{2.2.17}$$

Multiplying equation (2.2.15) scalarly by u_i^{j+1}, it is not difficult to get the inequality

$$\|u^n\|^2 + \sum_{j=1}^n \|u_{\overline{x}}^j]|^2\tau < C, \;\; n=1,2,\dots,N, \tag{2.2.18}$$

where C is a positive constant independent of τ and h.

Remark 2.2.2. The a priori estimate (2.2.18) guarantees the solvability of scheme (2.2.15)–(2.2.17) by using the Brouwer fixed-point lemma (see, e.g., [54] or Section 4.1). Note that applying the same technique as when proving the convergence theorem below, it is not difficult to prove the uniqueness of the solution and stability of the scheme (2.2.15)–(2.2.17), as well.

The main statement of the present subsection can be stated as follows.

Theorem 2.2.2. If $0 < p \le 1$ and problem (2.2.1)–(2.2.3) has a sufficiently smooth solution $U(x,t)$, then the solution $u^j=(u_1^j,u_2^j,\dots,u_{M-1}^j)$, $j=1,2,\dots,N$ of the difference scheme (2.2.15)–(2.2.17) tends to the solution of the continuous problem (2.2.1)–(2.2.3) $U^j=(U_1^j,U_2^j,\dots,U_{M-1}^j)$, $j=1,2,\dots,N$ as $\tau\to 0$, $h\to 0$, and the estimate

$$\|u^j - U^j\| \le C(\tau+h) \tag{2.2.19}$$

is true.

Proof. To prove Theorem 2.2.2, we introduce the difference $z_i^j = u_i^j - U_i^j$. We have

$$z_{t,i}^{j+1} - \left\{\Big(1+\tau\sum_{k=1}^{j+1}(u_{\overline{x},i}^k)^2\Big)^p u_{\overline{x},i}^{j+1} - \Big(1+\tau\sum_{k=1}^{j+1}(U_{\overline{x},i}^k)^2\Big)^p U_{\overline{x},i}^{j+1}\right\}_x = -\psi_i^j, \tag{2.2.20}$$

$$z_0^j = z_M^j = 0,$$

$$z_i^0 = 0,$$

where

$$\psi_i^j = O(\tau + h).$$

Multiplying (2.2.20) scalarly by $\tau z^{j+1} = \tau(z_1^{j+1}, z_2^{j+1}, \dots, z_{M-1}^{j+1})$ and using the discrete analogue of the formula of integration by parts, we get

$$\|z^{j+1}\|^2 - (z^{j+1}, z^j) + \tau h \sum_{i=1}^{M} \left\{ \left(1 + \tau \sum_{k=1}^{j+1} (u_{\bar{x},i}^k)^2\right)^p u_{\bar{x},i}^{j+1} - \left(1 + \tau \sum_{k=1}^{j+1} (U_{\bar{x},i}^k)^2\right)^p U_{\bar{x},i}^{j+1} \right\} z_{\bar{x},i}^{j+1} = -\tau(\psi^j, z^{j+1}). \quad (2.2.21)$$

Note that

$$\left\{ \left(1 + \tau \sum_{k=1}^{j+1} (u_{\bar{x},i}^k)^2\right)^p u_{\bar{x},i}^{j+1} - \left(1 + \tau \sum_{k=1}^{j+1} (U_{\bar{x},i}^k)^2\right)^p U_{\bar{x},i}^{j+1} \right\} (u_{\bar{x},i}^{j+1} - U_{\bar{x},i}^{j+1})$$
$$= \int_0^1 \frac{d}{d\mu} \left\{ \left(1 + \tau \sum_{k=1}^{j+1} \left[U_{\bar{x},i}^k + \mu(u_{\bar{x},i}^k - U_{\bar{x},i}^k)\right]^2\right)^p \left[U_{\bar{x},i}^{j+1} + \mu(u_{\bar{x},i}^{j+1} - U_{\bar{x},i}^{j+1})\right] \right\} d\mu\, (u_{\bar{x},i}^{j+1} - U_{\bar{x},i}^{j+1})$$
$$= 2p \int_0^1 \left(1 + \tau \sum_{k=1}^{j+1} \left[U_{\bar{x},i}^k + \mu(u_{\bar{x},i}^k - U_{\bar{x},i}^k)\right]^2\right)^{p-1}$$
$$\times \tau \sum_{k=1}^{j+1} \left[U_{\bar{x},i}^k + \mu(u_{\bar{x},i}^k - U_{\bar{x},i}^k)\right] (u_{\bar{x},i}^k - U_{\bar{x},i}^k) \left[U_{\bar{x},i}^{j+1} + \mu(u_{\bar{x},i}^{j+1} - U_{\bar{x},i}^{j+1})\right] d\mu\, (u_{\bar{x},i}^{j+1} - U_{\bar{x},i}^{j+1})$$
$$+ \int_0^1 \left(1 + \tau \sum_{k=1}^{j+1} \left[U_{\bar{x},i}^k + \mu(u_{\bar{x},i}^k - U_{\bar{x},i}^k)\right]^2\right)^p (u_{\bar{x},i}^{j+1} - U_{\bar{x},i}^{j+1})\, d\mu\, (u_{\bar{x},i}^{j+1} - U_{\bar{x},i}^{j+1})$$
$$= 2p \int_0^1 \left(1 + \tau \sum_{k=1}^{j+1} \left[U_{\bar{x},i}^k + \mu(u_{\bar{x},i}^k - U_{\bar{x},i}^k)\right]^2\right)^{p-1}$$
$$\times \tau \sum_{k=1}^{j+1} \left[U_{\bar{x},i}^k + \mu(u_{\bar{x},i}^k - U_{\bar{x},i}^k)\right] (u_{\bar{x},i}^k - U_{\bar{x},i}^k) \left[U_{\bar{x},i}^{j+1} + \mu(u_{\bar{x},i}^{j+1} - U_{\bar{x},i}^{j+1})\right] (u_{\bar{x},i}^{j+1} - U_{\bar{x},i}^{j+1})\, d\mu$$
$$+ \int_0^1 \left(1 + \tau \sum_{k=1}^{j+1} \left[U_{\bar{x},i}^k + \mu(u_{\bar{x},i}^k - U_{\bar{x},i}^k)\right]^2\right)^p (u_{\bar{x},i}^{j+1} - U_{\bar{x},i}^{j+1})^2\, d\mu$$
$$= 2p \int_0^1 \left(1 + \tau \sum_{k=1}^{j+1} \left[U_{\bar{x},i}^k + \mu(u_{\bar{x},i}^k - U_{\bar{x},i}^k)\right]^2\right)^{p-1} \xi_i^{j+1}(\mu) \xi_{t,i}^j(\mu)\, d\mu$$
$$+ \int_0^1 \left(1 + \tau \sum_{k=1}^{j+1} \left[U_{\bar{x},i}^k + \mu(u_{\bar{x},i}^k - U_{\bar{x},i}^k)\right]^2\right)^p (u_{\bar{x},i}^{j+1} - U_{\bar{x},i}^{j+1})^2\, d\mu,$$

where

$$\xi_i^{j+1}(\mu) = \tau \sum_{k=1}^{j+1} \left[U_{\bar{x},i}^k + \mu(u_{\bar{x},i}^k - U_{\bar{x},i}^k)\right] (u_{\bar{x},i}^k - U_{\bar{x},i}^k),$$
$$\xi_i^0(\mu) = 0,$$

and therefore,

$$\xi_{t,i}^j(\mu) = \left[U_{\bar{x},i}^{j+1} + \mu(u_{\bar{x},i}^{j+1} - U_{\bar{x},i}^{j+1})\right] (u_{\bar{x},i}^{j+1} - U_{\bar{x},i}^{j+1}).$$

Introducing the notation

$$s_i^{j+1}(\mu)=\tau\sum_{k=1}^{j+1}\left[U_{\overline{x},i}^k+\mu(u_{\overline{x},i}^k-U_{\overline{x},i}^k)\right]^2,$$

from the previous equality we have

$$\begin{aligned}\bigg\{\Big(1+\tau\sum_{k=1}^{j+1}(u_{\overline{x},i}^k)^2\Big)^p u_{\overline{x},i}^{j+1}-\Big(1+\tau\sum_{k=1}^{j+1}(U_{\overline{x},i}^k)^2\Big)^p U_{\overline{x},i}^{j+1}\bigg\}(u_{\overline{x},i}^{j+1}-U_{\overline{x},i}^{j+1})\\ =2p\int\limits_0^1(1+s_i^{j+1}(\mu))^{p-1}\xi_i^{j+1}\xi_{t,i}^j\,d\mu+\int\limits_0^1(1+s_i^{j+1}(\mu))^p\big(u_{\overline{x},i}^{j+1}-U_{\overline{x},i}^{j+1}\big)^2\,d\mu.\end{aligned}$$

After substituting this equality into (2.2.21), we get

$$\begin{aligned}\|z^{j+1}\|^2-(z^{j+1},z^j)+2\tau hp\sum_{i=1}^M\int\limits_0^1(1+s_i^{j+1}(\mu))^{p-1}\xi_i^{j+1}\xi_{t,i}^j\,d\mu\\ +\tau h\sum_{i=1}^M\int\limits_0^1(1+s_i^{j+1}(\mu))^p(u_{\overline{x},i}^{j+1}-U_{\overline{x},i}^{j+1})^2\,d\mu=-\tau(\psi^j,z^{j+1}).\end{aligned}\tag{2.2.22}$$

Taking into account the restriction $p>0$ and relations

$$\begin{gathered}s_i^{j+1}(\mu)\ge 0,\\ (z^{j+1},z^j)=\frac12\,\|z^{j+1}\|^2+\frac12\,\|z^j\|^2-\frac12\,\|z^{j+1}-z^j\|^2,\\ \tau\xi_i^{j+1}\xi_{t,i}^j=\frac12\,(\xi_i^{j+1})^2-\frac12\,(\xi_i^j)^2+\frac{\tau^2}{2}\,(\xi_{t,i}^j)^2,\end{gathered}$$

from (2.2.22) we obtain

$$\begin{aligned}\|z^{j+1}\|^2-\frac12\,\|z^{j+1}\|^2-\frac12\,\|z^j\|^2+\frac12\,\|z^{j+1}-z^j\|^2\\ +hp\sum_{i=1}^M\int\limits_0^1(1+s_i^{j+1}(\mu))^{p-1}\left[(\xi_i^{j+1})^2-(\xi_i^j)^2\right]d\mu\\ +\tau^2hp\sum_{i=1}^M\int\limits_0^1(1+s_i^{j+1}(\mu))^{p-1}(\xi_{t,i}^j)^2\,d\mu+\tau h\sum_{i=1}^M(u_{\overline{x},i}^{j+1}-U_{\overline{x},i}^{j+1})^2\le-\tau(\psi^j,z^{j+1}).\end{aligned}\tag{2.2.23}$$

From (2.2.23), we arrive at

$$\begin{aligned}\frac12\,\|z^{j+1}\|^2-\frac12\,\|z^j\|^2+\frac{\tau^2}{2}\,\|z_t^j\|^2\\ +hp\sum_{i=1}^M\int\limits_0^1(1+s_i^{j+1}(\mu))^{p-1}\left[(\xi_i^{j+1})^2-(\xi_i^j)^2\right]d\mu+\tau\|z_{\overline{x}}^{j+1}]|^2\le\frac{\tau}{2}\,\|\psi^j\|^2+\frac{\tau}{2}\,\|z^{j+1}\|^2.\end{aligned}\tag{2.2.24}$$

Using a discrete analogue of the Poincaré inequality [64]

$$\|z^{j+1}\|^2\le\|z_{\overline{x}}^{j+1}]|^2,$$

from (2.2.24) we get

$$\|z^{j+1}\|^2 - \|z^j\|^2 + \tau^2\|z_t^j\|^2 + 2hp\sum_{i=1}^{M}\int_0^1 (1+s_i^{j+1}(\mu))^{p-1}\left[(\xi_i^{j+1})^2 - (\xi_i^j)^2\right] d\mu + \tau\|z_{\overline{x}}^{j+1}]|^2 \le \tau\|\psi^j\|^2. \quad (2.2.25)$$

Summing-up (2.2.24) from $j = 0$ to $j = n-1$, we arrive at

$$\|z^n\|^2 + \tau^2\sum_{j=0}^{n-1}\|z_t^j\|^2 + 2hp\sum_{j=0}^{n-1}\sum_{i=1}^{M}\int_0^1 (1+s_i^{j+1}(\mu))^{p-1}\left[(\xi_i^{j+1})^2 - (\xi_i^j)^2\right] d\mu + \tau\sum_{j=0}^{n-1}\|z_{\overline{x}}^{j+1}]|^2 \le \tau\sum_{j=0}^{n-1}\|\psi^j\|^2. \quad (2.2.26)$$

Note that since $s_i^{j+1}(\mu) \ge s_i^j(\mu)$ and $p \le 1$, for the second line of the last formula, we have

$$\begin{aligned}
&\sum_{j=0}^{n-1}(1+s_i^{j+1}(\mu))^{p-1}\left[(\xi_i^{j+1})^2 - (\xi_i^j)^2\right] \\
&\quad = (1+s_i^1(\mu))^{p-1}(\xi_i^1)^2 - (1+s_i^1(\mu))^{p-1}(\xi_i^0)^2 + (1+s_i^2(\mu))^{p-1}(\xi_i^2)^2 \\
&\qquad - (1+s_i^2(\mu))^{p-1}(\xi_i^1)^2 + \cdots + (1+s_i^n(\mu))^{p-1}(\xi_i^n)^2 - (1+s_i^n(\mu))^{p-1}(\xi_i^{n-1})^2 \\
&\quad = (1+s_i^n(\mu))^{p-1}(\xi_i^n)^2 + \sum_{j=1}^{n-1}\left[(1+s_i^j(\mu))^{p-1} - (1+s_i^{j+1}(\mu))^{p-1}\right](\xi_i^j)^2 \ge 0.
\end{aligned}$$

Taking into account the latter relation and (2.2.26), one can deduce

$$\|z^n\|^2 + \tau^2\sum_{j=0}^{n-1}\|z_t^j\|^2 + \tau\sum_{j=0}^{n-1}\|z_{\overline{x}}^{j+1}]|^2 \le \tau\sum_{j=0}^{n-1}\|\psi^j\|^2. \quad (2.2.27)$$

From (2.2.27) we get (2.2.19), and thus Theorem 2.2.2 is proved. □

2.2.3 Numerical implementation remarks

In Section 2.2, the finite difference scheme (2.2.15)–(2.2.17) is constructed and investigated for problem (2.2.1)–(2.2.3). It should be noted that the stability and convergence of the semi-discrete scheme for problem (2.2.15)–(2.2.17) for the case $0 < p \le 1$ is proved, as well. The fully discrete analogues for the case $p = 1$ for this type models are studied in [33] and in a number of other works for different type boundary conditions (see, e.g., [39] and the references therein). As is noted in [39], it is important to construct and investigate fully discrete finite difference schemes and finite element analogues for more general type nonlinearities and for multi-dimensional cases, as well. So, in the present section the finite difference scheme is investigated for the case of the nonlinearity such as $a(S) = (1+S)^p$, $0 < p \le 1$. Using the numerical implementation remarks that are given in the above-mentioned works, the numerical algorithms for solving nonlinear systems of algebraic equations based on the difference scheme (2.2.15)–(2.2.17) are constructed. The results of numerical experiments fully agree with theoretical investigations. Experimental calculations for the AIDEs in more details are given in Subsection 3.2.3.

Chapter 3

Investigation and numerical solution of one-dimensional averaged nonlinear integro-differential equation

Chapter 3 studies the large-time behavior of solutions and finite difference approximations of the nonlinear AIDE associated with the penetration of a magnetic field into a medium. Asymptotic properties of solutions for the initial-boundary value problem with the Dirichlet boundary conditions are considered. The rates of convergence are given, as well. The convergence of the semi-discrete and finite difference schemes is proved. Numerical experiments are carried out.

Chapter 3 is organized as follows. In Section 3.1 we formulate the problem and consider a large time behavior of solutions of the first type initial-boundary value problems with homogeneous conditions on the whole boundary and nonhomogeneous conditions on one side of the lateral boundary for nonlinear AIDE. The stability and convergence of semi-discrete and finite difference schemes are discussed in Section 3.2. Some remarks on numerical implementations conclude this section.

3.1 Large time behavior of solutions

3.1.1 Introduction

As it has been mentioned several times, a great deal of applied problems are being modeled by nonlinear IDEs or systems. Such systems arise, for instance, in the mathematical modeling of the process of penetration of an electromagnetic field into a medium.

G. Laptev proposed some generalizations of the Volterra-type system considered in Chapter 2. In particular, assuming the temperature of a body to be constant all along the material, i.e., depending on time, but independent of the spatial coordinates, the process of penetration of the magnetic field into a material is modeled by the so-called averaged integro-differential system. One-dimensional variant of this model has the form

$$\frac{\partial U}{\partial t} = a\left(\int_0^t \int_0^1 \left(\frac{\partial U}{\partial x} \right)^2 dx\, d\tau \right) \frac{\partial^2 U}{\partial x^2} \,. \tag{3.1.1}$$

Note that the AIDEs of type (3.1.1) are complex and that is why only special cases were investigated. The existence and uniqueness of the solutions of the initial-boundary value problems for the equations of type (3.1.1) were first studied in [32]. For more information about (3.1.1) type models see also [39] and the references therein.

The purpose of this section is to study an asymptotic behavior of the solutions for equation (3.1.1). Our objective is to give a large-time asymptotic behavior (as $t \to \infty$) of solutions of the initial-boundary value problem with the homogeneous Dirichlet boundary conditions. Asymptotic behavior of the solutions with nonhomogeneous conditions on a part of the boundary is also studied in this section. Note that investigations of difference schemes for these models can be found in [10, 19, 30, 38, 39, 44, 52, 65–67, 74]. Difference schemes for a certain nonlinear parabolic integro-differential model similar to (3.1.1) with the same boundary conditions were studied in a number of other works, as well.

3.1.2 Asymptotic behavior of solutions with the homogeneous boundary conditions

Consider the following initial-boundary value problem:

$$\frac{\partial U}{\partial t} = a(S)\,\frac{\partial^2 U}{\partial x^2}\,, \quad (x,t) \in Q = (0,1) \times (0,\infty), \tag{3.1.2}$$

$$U(0,t) = U(1,t) = 0, \quad t \ge 0, \tag{3.1.3}$$

$$U(x,0) = U_0(x), \quad x \in [0,1], \tag{3.1.4}$$

where

$$S(t) = \int_0^t\!\!\int_0^1 \Big(\frac{\partial U}{\partial x}\Big)^2 dx\, d\tau, \tag{3.1.5}$$

$a(S) = (1+S)^p$, $p > 0$; $U_0 = U_0(x)$ is a given function.

For earlier work on the unique solvability and asymptotic behavior in time of solutions for (3.1.2)–(3.1.5) type problems, see [32].

The main purpose of this subsection is to prove the following statement.

Theorem 3.1.1. If $a(S) = (1+S)^p$, $p > 0$; $U_0 \in W_2^2(0,1) \cap \overset{\circ}{W}{}_2^1(0,1)$, then for the solution of problem (3.1.2)–(3.1.5) the following estimates hold:

$$\Big|\frac{\partial U(x,t)}{\partial x}\Big| \le C\Big(-\frac{t}{2}\Big), \quad \Big|\frac{\partial U(x,t)}{\partial t}\Big| \le C\Big(-\frac{t}{2}\Big).$$

Now we intend to get a priori estimates for a solution of problem (3.1.2)–(3.1.5).

To prove the main Theorem 3.1.1, we need some auxiliary statements.

Theorem 3.1.2. If $a(S) = (1+S)^p$, $p > 0$; $U_0 \in \overset{\circ}{W}{}_2^1(0,1)$, then for the solution of problem (3.1.2)–(3.1.5) the estimate

$$\|U\| + \Big\|\frac{\partial U}{\partial x}\Big\| \le C\Big(-\frac{t}{2}\Big)$$

is true.

Proof. We multiply equation (3.1.2) by U and integrate over $(0,1)$. After integrating by parts and using the boundary conditions (3.1.3), we get

$$\frac{1}{2}\,\frac{d}{dt}\,\|U\|^2 + \int_0^1 (1+S)^p \Big(\frac{\partial U}{\partial x}\Big)^2 dx = 0,$$

whence, taking into account the relation $(1+S)^p \ge 1$ and the Poincaré inequality, we obtain

$$\frac{1}{2}\,\frac{d}{dt}\,\|U\|^2 + \Big\|\frac{\partial U}{\partial x}\Big\|^2 \le 0, \quad \frac{1}{2}\,\frac{d}{dt}\,\|U\|^2 + \|U\|^2 \le 0. \tag{3.1.6}$$

Multiplying equation (3.1.2) scalarly by $\partial^2 U/\partial x^2$, using again formula of integrating by parts and the boundary conditions (3.1.3), we get

$$\frac{\partial U}{\partial t}\,\frac{\partial U}{\partial x}\bigg|_0^1 - \int_0^1 \frac{\partial^2 U}{\partial x \partial t}\,\frac{\partial U}{\partial x}\,dx = \int_0^1 (1+S)^p \Big(\frac{\partial^2 U}{\partial x^2}\Big)^2\,dx,$$

$$\frac{1}{2}\,\frac{d}{dt}\,\Big\|\frac{\partial U}{\partial x}\Big\|^2 + (1+S)^p \Big\|\frac{\partial^2 U}{\partial x^2}\Big\|^2 = 0,$$

or

$$\frac{d}{dt}\,\Big\|\frac{\partial U}{\partial x}\Big\|^2 \le 0. \tag{3.1.7}$$

Combining (3.1.6) and (3.1.7), we deduce

$$\frac{d}{dt}\,\|U\|^2 + \|U\|^2 + \frac{d}{dt}\,\Big\|\frac{\partial U}{\partial x}\Big\|^2 + \Big\|\frac{\partial U}{\partial x}\Big\|^2 \le 0.$$

After multiplying by the function $\exp(t)$, the last inequality yields

$$\frac{d}{dt}\,\big(\exp(t)\|U\|^2\big) + \frac{d}{dt}\,\Big(\exp(t)\Big\|\frac{\partial U}{\partial x}\Big\|^2\Big) \le 0,$$

or

$$\frac{d}{dt}\,\Big[\exp(t)\Big(\|U\|^2 + \Big\|\frac{\partial U}{\partial x}\Big\|^2\Big)\Big] \le 0.$$

This inequality immediately proves Theorem 3.1.2. □

Note that Theorem 3.1.2 provides us with the exponential stabilization of the solution of problem (3.1.2)–(3.1.5) in the norm of the space $W_2^1(0,1)$. Let us show that stabilization is also achieved in the norm of the space $C^1(0,1)$.

First of all, let us show that the following statement is valid.

Theorem 3.1.3. For the solution of problem (3.1.2)–(3.1.5), the following estimate holds

$$\Big\|\frac{\partial U(x,t)}{\partial t}\Big\| \le C\exp\Big(-\frac{t}{2}\Big).$$

Proof. We differentiate equation (3.1.2) with respect to t,

$$\frac{\partial^2 U}{\partial t^2} = (1+S)^p\,\frac{\partial^3 U}{\partial x^2 \partial t} + p(1+S)^{p-1}\int_0^1 \Big(\frac{\partial U}{\partial x}\Big)^2\,dx\,\frac{\partial^2 U}{\partial x^2}\,, \tag{3.1.8}$$

and multiply (3.1.8) scalarly by $\partial U/\partial t$. We deduce

$$\frac{1}{2}\,\frac{d}{dt}\int_0^1 \Big(\frac{\partial U}{\partial t}\Big)^2 dx + (1+S)^p \int_0^1 \Big(\frac{\partial^2 U}{\partial x \partial t}\Big)^2 dx + p(1+S)^{p-1}\int_0^1 \Big(\frac{\partial U}{\partial x}\Big)^2 dx \int_0^1 \frac{\partial U}{\partial x}\,\frac{\partial^2 U}{\partial x \partial t}\,dx = 0,$$

or

$$\frac{d}{dt}\int_0^1 \Big(\frac{\partial U}{\partial t}\Big)^2 dx + 2(1+S)^p \int_0^1 \Big(\frac{\partial^2 U}{\partial x \partial t}\Big)^2 dx = -2p(1+S)^{p-1}\int_0^1 \Big(\frac{\partial U}{\partial x}\Big)^2 dx \int_0^1 \frac{\partial U}{\partial x}\,\frac{\partial^2 U}{\partial x \partial t}\,dx.$$

Let us estimate the left-hand side of the latter equality

$$-2p(1+S)^{p-1}\int_0^1 \Big(\frac{\partial U}{\partial x}\Big)^2 dx \int_0^1 \frac{\partial U}{\partial x}\,\frac{\partial^2 U}{\partial x \partial t}\,dx$$

$$= -2\int_0^1 \bigg\{p(1+S)^{\frac{p}{2}-1}\Big[\int_0^1 \Big(\frac{\partial U}{\partial x}\Big)^2 dx\Big]\,\frac{\partial U}{\partial x}\bigg\}\bigg\{(1+S)^{\frac{p}{2}}\,\frac{\partial^2 U}{\partial x \partial t}\bigg\}\,dx,$$

from which, using the Schwarz inequality, we get

$$\frac{d}{dt}\int_0^1 \Big(\frac{\partial U}{\partial t}\Big)^2 dx + 2(1+S)^p \int_0^1 \Big(\frac{\partial^2 U}{\partial x \partial t}\Big)^2 dx$$
$$\le (1+S)^p \int_0^1 \Big(\frac{\partial^2 U}{\partial x \partial t}\Big)^2 dx + p^2(1+S)^{p-2}\Big[\int_0^1 \Big(\frac{\partial U}{\partial x}\Big)^2 dx\Big]^2 \int_0^1 \Big(\frac{\partial U}{\partial x}\Big)^2 dx,$$

or

$$\frac{d}{dt}\int_0^1 \Big(\frac{\partial U}{\partial t}\Big)^2 dx + (1+S)^p \int_0^1 \Big(\frac{\partial^2 U}{\partial x \partial t}\Big)^2 dx \le p^2(1+S)^{p-2}\Big[\int_0^1 \Big(\frac{\partial U}{\partial x}\Big)^2 dx\Big]^3. \tag{3.1.9}$$

Note that Theorem 3.1.2 enables us to estimate the function S,

$$S(t) = \int_0^t\int_0^1 \Big(\frac{\partial U}{\partial x}\Big)^2 dx\, d\tau = \int_0^t \Big\|\frac{\partial U}{\partial x}\Big\|^2 d\tau \le C\int_0^t \exp(-\tau)\, d\tau \le C.$$

Thus, we have

$$1 \le 1 + S(t) \le C. \tag{3.1.10}$$

Using the Poincaré inequality, Theorem 3.1.2 and relation (3.1.10), from (3.1.9) we get

$$\frac{d}{dt}\int_0^1 \Big(\frac{\partial U}{\partial t}\Big)^2 dx + \int_0^1 \Big(\frac{\partial U}{\partial t}\Big)^2 dx \le C\exp(-3\tau).$$

Multiplying the latter inequality by the function $\exp(t)$, we arrive at

$$\frac{d}{dt}\Big(\exp(t)\Big\|\frac{\partial U}{\partial t}\Big\|^2\Big) \le C\exp(-2t).$$

From this, we deduce

$$\exp(t)\Big\|\frac{\partial U}{\partial t}\Big\|^2 \le C\int_0^t \exp(-2\tau)\, d\tau.$$

Therefore, Theorem 3.1.3 is proved. □

Proof of Theorem 3.1.1. Let us now estimate $\partial^2 U/\partial x^2$ in the norm of the space $L_1(0,1)$. From (3.1.2) we have

$$\frac{\partial^2 U}{\partial x^2} = (1+S)^{-p}\,\frac{\partial U}{\partial t}\,. \tag{3.1.11}$$

So, applying again the Schwarz inequality, Theorem 3.1.3 and estimate (3.1.10), we derive

$$\int_0^1 \Big|\frac{\partial^2 U}{\partial x^2}\Big| dx \le \Big[\int_0^1 (1+S)^{-2p}\, dx\Big]^{\frac12}\Big[\int_0^1 \Big|\frac{\partial U}{\partial t}\Big|^2 dx\Big]^{\frac12} \le C\exp\Big(-\frac{t}{2}\Big),$$

whence, taking into account the relation

$$\frac{\partial U(x,t)}{\partial x} = \int_0^1 \frac{\partial U(y,t)}{\partial y}\, dy + \int_0^1\int_y^x \frac{\partial^2 U(\xi,t)}{\partial \xi^2}\, d\xi\, dy$$

and the boundary conditions (3.1.3), it follows that

$$\Big|\frac{\partial U(x,t)}{\partial x}\Big| = \Big|\int_0^1\int_y^x \frac{\partial^2 U(\xi,t)}{\partial \xi^2}\,d\xi\,dy\Big| \le \int_0^1 \Big|\frac{\partial^2 U(y,t)}{\partial y^2}\Big|\,dy \le C\exp\Big(-\frac{t}{2}\Big).$$

Let us now estimate $\partial U/\partial t$ in the norm of the space $C^1(0,1)$. We multiply equation (3.1.2) scalarly by $\partial^3 U/\partial x^2\partial t$. After using the formula of integrating by parts and the boundary conditions (3.1.3), we get

$$\frac{\partial U}{\partial t}\,\frac{\partial^2 U}{\partial x\partial t}\Big|_0^1 - \Big\|\frac{\partial^2 U}{\partial x\partial t}\Big\|^2 = (1+S)^p\int_0^1 \frac{\partial^2 U}{\partial x^2}\,\frac{\partial^3 U}{\partial x^2\partial t}\,dx, \tag{3.1.12}$$

$$\frac{1}{2}\,(1+S)^p\,\frac{d}{dt}\Big\|\frac{\partial^2 U}{\partial x^2}\Big\|^2 + \Big\|\frac{\partial^2 U}{\partial x\partial t}\Big\|^2 = 0,$$

or

$$\frac{d}{dt}\Big\|\frac{\partial^2 U}{\partial x^2}\Big\|^2 \le 0. \tag{3.1.13}$$

Note that from (3.1.12) we have

$$\begin{aligned}\Big\|\frac{\partial^2 U}{\partial x\partial t}\Big\|^2 &\le (1+S)^p\Big[\int_0^1\Big(\frac{\partial^2 U}{\partial x^2}\Big)^2 dx\Big]^{\frac12}\Big[\int_0^1\Big(\frac{\partial^3 U}{\partial x^2\partial t}\Big)^2 dx\Big]^{\frac12}\\ &\le \frac12\,(1+S)^p\Big\|\frac{\partial^2 U}{\partial x^2}\Big\|^2 + \frac12\,(1+S)^p\Big\|\frac{\partial^3 U}{\partial x^2\partial t}\Big\|^2.\end{aligned} \tag{3.1.14}$$

Multiplying (3.1.8) scalarly by $\partial^3 U/\partial x^2\partial t$,

$$\begin{aligned}&\frac{\partial^2 U}{\partial t^2}\,\frac{\partial^2 U}{\partial x\partial t}\Big|_0^1 - \int_0^1 \frac{\partial^3 U}{\partial x\partial t^2}\,\frac{\partial^2 U}{\partial x\partial t}\,dx\\ &\qquad = (1+S)^p\Big\|\frac{\partial^3 U}{\partial x^2\partial t}\Big\|^2 + p(1+S)^{p-1}\int_0^1\Big(\frac{\partial U}{\partial x}\Big)^2 dx\int_0^1 \frac{\partial^2 U}{\partial x^2}\,\frac{\partial^3 U}{\partial x^2\partial t}\,dx,\end{aligned}$$

and taking into account the boundary conditions (3.1.3), we obtain

$$\begin{aligned}&\frac{d}{dt}\Big\|\frac{\partial^2 U}{\partial x\partial t}\Big\|^2 + 2(1+S)^p\Big\|\frac{\partial^3 U}{\partial x^2\partial t}\Big\|^2\\ &\qquad = -2\int_0^1\Big\{p(1+S)^{\frac p2-1}\Big[\int_0^1\Big(\frac{\partial U}{\partial x}\Big)^2 dx\Big]\frac{\partial^2 U}{\partial x^2}\Big\}\Big\{(1+S)^{\frac p2}\,\frac{\partial^3 U}{\partial x^2\partial t}\Big\}\,dx.\end{aligned}$$

From this, using the Schwarz inequality once again, we obtain

$$\begin{aligned}&\frac{d}{dt}\Big\|\frac{\partial^2 U}{\partial x\partial t}\Big\|^2 + 2(1+S)^p\Big\|\frac{\partial^3 U}{\partial x^2\partial t}\Big\|^2\\ &\qquad \le (1+S)^p\Big\|\frac{\partial^3 U}{\partial x^2\partial t}\Big\|^2 + p^2(1+S)^{p-2}\Big[\int_0^1\Big(\frac{\partial U}{\partial x}\Big)^2 dx\Big]^2\int_0^1\Big(\frac{\partial^2 U}{\partial x^2}\Big)^2 dx,\end{aligned}$$

or

$$\frac{d}{dt}\Big\|\frac{\partial^2 U}{\partial x\partial t}\Big\|^2 + (1+S)^p\Big\|\frac{\partial^3 U}{\partial x^2\partial t}\Big\|^2 \le p^2(1+S)^{p-2}\Big[\int_0^1\Big(\frac{\partial U}{\partial x}\Big)^2 dx\Big]^2\int_0^1\Big(\frac{\partial^2 U}{\partial x^2}\Big)^2 dx.$$

Using Theorem 3.1.2, relations (3.1.10), (3.1.11) and Theorem 3.1.3, we arrive at

$$\frac{d}{dt}\Big\|\frac{\partial^2 U}{\partial x\partial t}\Big\|^2 + (1+S)^p\Big\|\frac{\partial^3 U}{\partial x^2\partial t}\Big\|^2 \le C\exp(-3t). \tag{3.1.15}$$

Combining (3.1.6), (3.1.7) and (3.1.13)–(3.1.15), we get

$$\begin{aligned}\|U\|^2 + \frac{d}{dt}\|U\|^2 + \Big\|\frac{\partial U}{\partial x}\Big\|^2 + \frac{d}{dt}\Big\|\frac{\partial U}{\partial x}\Big\|^2 + 2(1+S)^p\Big\|\frac{\partial^2 U}{\partial x^2}\Big\|^2 \\ + \frac{d}{dt}\Big\|\frac{\partial^2 U}{\partial x^2}\Big\|^2 + \Big\|\frac{\partial^2 U}{\partial x\partial t}\Big\|^2 + \frac{d}{dt}\Big\|\frac{\partial^2 U}{\partial x\partial t}\Big\|^2 + (1+S)^p\Big\|\frac{\partial^3 U}{\partial x^2\partial t}\Big\|^2 \\ \le \frac{1}{2}(1+S)^p\Big\|\frac{\partial^2 U}{\partial x^2}\Big\|^2 + \frac{1}{2}(1+S)^p\Big\|\frac{\partial^3 U}{\partial x^2\partial t}\Big\|^2 + C\exp(-3t). \end{aligned} \tag{3.1.16}$$

After a simple transformation, keeping in mind estimate (3.1.10), from (3.1.16) we deduce

$$\begin{aligned}\|U\|^2 + \frac{d}{dt}\|U\|^2 + \Big\|\frac{\partial U}{\partial x}\Big\|^2 + \frac{d}{dt}\Big\|\frac{\partial U}{\partial x}\Big\|^2 \\ + \Big\|\frac{\partial^2 U}{\partial x^2}\Big\|^2 + \frac{d}{dt}\Big\|\frac{\partial^2 U}{\partial x^2}\Big\|^2 + \Big\|\frac{\partial^2 U}{\partial x\partial t}\Big\|^2 + \frac{d}{dt}\Big\|\frac{\partial^2 U}{\partial x\partial t}\Big\|^2 \le C\exp(-3t).\end{aligned}$$

From this, after multiplication by the function $\exp(t)$, we get

$$\frac{d}{dt}\left[\exp(t)\left(\|U\|^2 + \Big\|\frac{\partial U}{\partial x}\Big\|^2 + \Big\|\frac{\partial^2 U}{\partial x^2}\Big\|^2 + \Big\|\frac{\partial^2 U}{\partial x\partial t}\Big\|^2\right)\right] \le C\exp(-2t),$$

or

$$\|U\|^2 + \Big\|\frac{\partial U}{\partial x}\Big\|^2 + \Big\|\frac{\partial^2 U}{\partial x^2}\Big\|^2 + \Big\|\frac{\partial^2 U}{\partial x\partial t}\Big\|^2 \le C\exp(-t).$$

At last, taking into account the relation

$$\frac{\partial U(x,t)}{\partial t} = \int_0^1 \frac{\partial U(y,t)}{\partial t}\,dy + \int_0^1\!\!\int_y^x \frac{\partial^2 U(\xi,t)}{\partial \xi\partial t}\,d\xi\,dy$$

and Theorem 3.1.3, we obtain

$$\Big|\frac{\partial U(x,t)}{\partial t}\Big| \le \left[\int_0^1 \Big(\frac{\partial U(x,t)}{\partial t}\Big)^2 dx\right]^{\frac12} + \left[\int_0^1 \Big(\frac{\partial^2 U(x,t)}{\partial x\partial t}\Big)^2 dx\right]^{\frac12} \le C\Big(-\frac{t}{2}\Big).$$

So, the main Theorem 3.1.1 of this subsection is proved. □

Remark 3.1.1. The large time behavior to the solutions of the initial-boundary value problems for (3.1.2) type models for the case $-1/2 < p < 0$ is studied in [32, 46].

Remark 3.1.2. The existence of globally defined solutions of problem (3.1.2)–(3.1.5) can be reobtained by a routine procedure. One can first establish the existence of local solutions at a maximal time interval and then find from the obtained estimates that this solution cannot escape at a finite time.

3.1.3 Asymptotic behavior of solutions with nonhomogeneous condition on a part of the boundary

Let us consider the following initial-boundary value problem:

$$\frac{\partial U}{\partial t} = a(S)\,\frac{\partial^2 U}{\partial x^2}\,, \quad (x,t) \in Q = (0,1)\times(0,\infty), \tag{3.1.17}$$

$$U(0,t) = 0, \quad U(1,t) = \psi, \quad t \ge 0, \tag{3.1.18}$$

$$U(x,0) = U_0(x), \quad x \in [0,1], \tag{3.1.19}$$

where

$$S(t)=\int_0^t\int_0^1\Big(\frac{\partial U}{\partial x}\Big)^2\,dx\,d\tau, \tag{3.1.20}$$

$a(S)=(1+S)^p$, $p>0$, $\psi=Const>0$; $U_0=U_0(x)$ is a given function.

Remark 3.1.3. It should be noted that the boundary conditions (3.1.18) are, just as in Chapter 2, used by taking into account the physical problems considered in [25].

The main purpose of this section is to prove the following statement.

Theorem 3.1.4. If $a(S)=(1+S)^p$, $p>0$, $\psi=Const>0$; $U_0\in W_2^2(0,1)$, $U_0(0)=0$, $U_0(1)=\psi$, then for the solution of problem (3.1.17)–(3.1.20) the following estimates are true:

$$\Big|\frac{\partial U(x,t)}{\partial x}-\psi\Big|\le Ct^{-1-p},\quad \Big|\frac{\partial U(x,t)}{\partial t}\Big|\le Ct^{-1},\ \ t\ge 1.$$

Before we proceed to proving Theorem 3.1.4, we establish some auxiliary lemmas.

Lemma 3.1.1. The following estimates are true:

$$\varphi^{\frac{1}{1+2p}}(t)\le 1+S(t)\le C\varphi^{\frac{1}{1+2p}}(t),\ \ t\ge 0,$$

where

$$\varphi(t)=1+\int_0^t\int_0^1(1+S)^{2p}\Big(\frac{\partial U}{\partial x}\Big)^2\,dx\,d\tau. \tag{3.1.21}$$

Proof. From (3.1.20) it follows that

$$\frac{dS}{dt}=\int_0^1\Big(\frac{\partial U}{\partial x}\Big)^2\,dx,\ \ S(0)=0. \tag{3.1.22}$$

Let us multiply the first identity of (3.1.22) by $(1+S)^{2p}$ and introduce the following notation

$$\sigma=(1+S)^p\,\frac{\partial U}{\partial x}\,.$$

We have

$$\frac{1}{1+2p}\,\frac{dS^{1+2p}}{dt}=\int_0^1\sigma^2\,dx.$$

Integrating this relation on $(0,t)$, we arrive at

$$\frac{1}{1+2p}\,(1+S)^{1+2p}=\int_0^t\int_0^1\sigma^2\,dx\,d\tau+\frac{1}{1+2p}\,.$$

Note that $0<\frac{1}{1+2p}<1$. So, we get

$$\varphi^{\frac{1}{1+2p}}(t)\le 1+S(t)\le\big[(1+2p)\varphi(t)\big]^{\frac{1}{1+2p}}.$$

Thus, Lemma 3.1.1 is proved. □

Lemma 3.1.2. The following estimates are true:

$$c\varphi^{\frac{2p}{1+2p}}(t)\le\int_0^1\sigma^2\,dx\le C\varphi^{\frac{2p}{1+2p}}(t),\ \ t\ge 0.$$

Proof. Taking into account Lemma 3.1.1, we get

$$\int_0^1 \sigma^2\,dx = \int_0^1 (1+S)^{2p}\Big(\frac{\partial U}{\partial x}\Big)^2\,dx \ge \varphi^{\frac{2p}{1+2p}}(t)\int_0^1 \Big(\frac{\partial U}{\partial x}\Big)^2\,dx$$

$$\ge \varphi^{\frac{2p}{1+2p}}(t)\bigg[\int_0^1 \frac{\partial U}{\partial x}\,dx\bigg]^2 = \psi^2\varphi^{\frac{2p}{1+2p}}(t),$$

or

$$\int_0^1 \sigma^2\,dx \ge c\varphi^{\frac{2p}{1+2p}}(t). \tag{3.1.23}$$

Let us multiply equation (3.1.17) scalarly by $(1+S)^{-p}\partial U/\partial t$. Using the formula of integration by parts and the boundary conditions (3.1.18), we have

$$\int_0^1 (1+S)^{-p}\Big(\frac{\partial U}{\partial t}\Big)^2\,dx + \frac{1}{2}\,\frac{d}{dt}\int_0^1 \Big(\frac{\partial U}{\partial x}\Big)^2\,dx = 0.$$

After integration from 0 to t, we arrive at

$$\int_0^t\!\!\int_0^1 (1+S)^{-p}\Big(\frac{\partial U}{\partial \tau}\Big)^2\,dx\,d\tau + \frac{1}{2}\int_0^1 \Big(\frac{\partial U}{\partial x}\Big)^2\,dx = C.$$

From this, we get

$$\int_0^1 \Big(\frac{\partial U}{\partial x}\Big)^2\,dx \le C. \tag{3.1.24}$$

Using (3.1.24) and Lemma 3.1.1, we conclude

$$\int_0^1 \sigma^2\,dx = (1+S)^{2p}\int_0^1 \Big(\frac{\partial U}{\partial x}\Big)^2\,dx \le C\varphi^{\frac{2p}{1+2p}}(t).$$

Now, taking into account (3.1.23), from the latter inequality the proof of Lemma 3.1.2 is complete. □

From Lemma 3.1.2 and relation (3.1.21) we have the following estimates:

$$c\varphi^{\frac{2p}{1+2p}}(t) \le \frac{d\varphi(t)}{dt} \le C\varphi^{\frac{2p}{1+2p}}(t),\quad t \ge 0. \tag{3.1.25}$$

Lemma 3.1.3. $\partial U/\partial t$ satisfy the inequality

$$\int_0^1 \Big(\frac{\partial U}{\partial t}\Big)^2\,dx \le C\varphi^{-\frac{2}{1+2p}}(t).$$

Proof. Using the Poincaré inequality, Lemma 3.1.1 and relation (3.1.24), we get

$$\frac{d}{dt}\int_0^1 \Big(\frac{\partial U}{\partial t}\Big)^2\,dx + \varphi^{\frac{p}{1+2p}}(t)\int_0^1 \Big(\frac{\partial U}{\partial t}\Big)^2\,dx \le C\varphi^{\frac{p-2}{1+2p}}(t).$$

Using Gronwall's inequality, we have

$$\int_0^1 \Big(\frac{\partial U}{\partial t}\Big)^2 dx \le \exp\Big(-\int_0^t \varphi^{\frac{p}{1+2p}}(\tau)\,d\tau\Big) \times \Big[\int_0^1 \Big(\frac{\partial U}{\partial t}\Big)^2 dx\Big|_{t=0} + C\int_0^t \exp\Big(\int_0^\tau \varphi^{\frac{p}{1+2p}}(\xi)\,d\xi\Big)\varphi^{\frac{p-2}{1+2p}}(\tau)\,d\tau\Big]. \tag{3.1.26}$$

Noting that $\varphi(t)\ge 1$, applying the L'Hopital rule and estimate (3.1.25), we obtain

$$\begin{aligned}\lim_{t\to\infty}&\frac{\int_0^t \exp\Big(\int_0^\tau \varphi^{\frac{p}{1+2p}}(\xi)\,d\xi\Big)\varphi^{\frac{p-2}{1+2p}}(\tau)\,d\tau}{\exp\Big(\int_0^t \varphi^{\frac{p}{1+2p}}(\tau)\,d\tau\Big)\varphi^{-\frac{2}{1+2p}}(t)}\\
&= \lim_{t\to\infty}\frac{\exp\Big(\int_0^t \varphi^{\frac{p}{1+2p}}(\tau)\,d\tau\Big)\varphi^{\frac{p-2}{1+2p}}(t)}{\exp\Big(\int_0^t \varphi^{\frac{p}{1+2p}}(\tau)\,d\tau\Big)\Big(\varphi^{\frac{p-2}{1+2p}}(t) - \frac{2}{1+2p}\varphi^{\frac{-3-2p}{1+2p}}(t)\,\frac{d\varphi}{dt}\Big)}\\
&\le \lim_{t\to\infty}\frac{1}{1-\frac{C}{1+2p}\varphi^{-\frac{p+1}{1+2p}}(t)} \le C.\end{aligned} \tag{3.1.27}$$

Therefore, Lemma 3.1.3 follows from (3.1.26) and (3.1.27). □

Proof of Theorem 3.1.4. Now, according to the method applied in Subsection 3.1.1, taking into account Lemmas 3.1.1 and 3.1.3, we derive

$$\begin{aligned}\Big|\frac{\partial U(x,t)}{\partial x} - \psi\Big| &= \Big|\int_0^1\int_y^x \frac{\partial^2 U(\xi,t)}{\partial \xi^2}\,d\xi\,dy\Big| \le \int_0^1 \Big|\frac{\partial^2 U(x,t)}{\partial x^2}\Big|\,dx \le \int_0^1 \Big|(1+S)^{-p}\frac{\partial U}{\partial t}\Big|\,dx\\
&\le \Big[\int_0^1 (1+S)^{-2p}\,dx\Big]^{\frac12}\Big[\int_0^1 \Big|\frac{\partial U}{\partial t}\Big|^2 dx\Big]^{\frac12} \le C\varphi^{-\frac{p}{1+2p}}(t)\varphi^{-\frac{1}{1+2p}}(t) = C\varphi^{-\frac{p+1}{1+2p}}(t).\end{aligned}$$

Hence, we have

$$\Big|\frac{\partial U(x,t)}{\partial x} - \psi\Big| \le C\varphi^{-\frac{p+1}{1+2p}}(t). \tag{3.1.28}$$

Let us now estimate $\partial U/\partial t$. To this end, let us multiply (3.1.8) by $\varphi^{\frac{2}{1+2p}}(t)$. Integrating on $(0,t)$, using the formula of integrating by parts, estimates (3.1.24), (3.1.25) and Lemmas 3.1.1, 3.1.3, we get

$$\int_0^t \varphi^{\frac{2}{1+2p}}(\tau)\frac{d}{d\tau}\int_0^1 \Big(\frac{\partial U}{\partial \tau}\Big)^2 dx\,d\tau + \int_0^t \varphi^{\frac{2}{1+2p}}(\tau)(1+S)^p\int_0^1 \Big(\frac{\partial^2 U}{\partial\tau\partial x}\Big)^2 dx\,d\tau \le C\int_0^t \varphi^{\frac{p}{1+2p}}(\tau)\,d\tau,$$

$$\begin{aligned}\int_0^t \varphi^{\frac{2}{1+2p}}(\tau)\varphi^{\frac{p}{1+2p}}(\tau)\int_0^1 \Big(\frac{\partial^2 U}{\partial\tau\partial x}\Big)^2 dx\,d\tau &\le -\varphi^{\frac{2}{1+2p}}(t)\int_0^1 \Big(\frac{\partial U}{\partial t}\Big)^2 dx + \int_0^1 \Big(\frac{\partial U}{\partial t}\Big)^2 dx\Big|_{t=0}\\
&+\frac{2}{1+2p}\int_0^t \varphi^{\frac{1-2p}{1+2p}}(\tau)\,\frac{d\varphi}{d\tau}\int_0^1 \Big(\frac{\partial U}{\partial \tau}\Big)^2 dx\,d\tau + C_1\int_0^t \varphi^{\frac{-p}{1+2p}}(\tau)\,\frac{d\varphi}{d\tau}\,d\tau\\
&\le C_2 + C_3\int_0^t \varphi^{-\frac{1}{1+2p}}(\tau)\,d\tau + C_4\big(\varphi^{\frac{p+1}{1+2p}}(t)-1\big) \le C\varphi^{\frac{p+1}{1+2p}}(t),\end{aligned}$$

or

$$\int_0^t \varphi^{\frac{p+2}{1+2p}}(\tau) \int_0^1 \left(\frac{\partial^2 U}{\partial \tau \partial x}\right)^2 dx\, d\tau \le C\varphi^{\frac{p+1}{1+2p}}(t). \tag{3.1.29}$$

Multiplying equation (3.1.8) scalarly by $\varphi^{\frac{3}{1+2p}}(t)\partial^2 U/\partial t^2$, applying the formula of integration by parts, the Schwarz inequality, Lemma 3.1.1 and a priori estimates (3.1.25), (3.1.28), (3.1.29), we get

$$\int_0^1 \varphi^{\frac{3}{1+2p}}(t)\left(\frac{\partial^2 U}{\partial t^2}\right)^2 dx + \int_0^1 \varphi^{\frac{3}{1+2p}}(t)(1+S)^p \frac{\partial^2 U}{\partial t \partial x}\,\frac{\partial^3 U}{\partial t^2 \partial x}\, dx$$

$$+p\int_0^1 \varphi^{\frac{3}{1+2p}}(t)(1+S)^{p-1}\int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 dx\, \frac{\partial U}{\partial x}\,\frac{\partial^3 U}{\partial t^2 \partial x}\, dx = 0,$$

$$\int_0^t\!\!\int_0^1 \varphi^{\frac{3}{1+2p}}(\tau)\left(\frac{\partial^2 U}{\partial \tau^2}\right)^2 dx\, d\tau + \frac{1}{2}\int_0^t\!\!\int_0^1 \varphi^{\frac{3}{1+2p}}(\tau)(1+S)^p \frac{\partial}{\partial \tau}\left(\frac{\partial^2 U}{\partial \tau \partial x}\right)^2 dx\, d\tau$$

$$+p\int_0^t \varphi^{\frac{3}{1+2p}}(\tau)(1+S)^{p-1}\int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 dx \int_0^1 \frac{\partial U}{\partial x}\,\frac{\partial}{\partial \tau}\left(\frac{\partial^2 U}{\partial \tau \partial x}\right) dx\, d\tau = 0,$$

$$\frac{1}{2}\varphi^{\frac{3}{1+2p}}(t)(1+S)^p\int_0^1 \left(\frac{\partial^2 U}{\partial t \partial x}\right)^2 dx - \frac{1}{2}\int_0^1 \left(\frac{\partial^2 U}{\partial t \partial x}\right)^2 dx\bigg|_{t=0}$$

$$\le \frac{3}{2+4p}\int_0^t\!\!\int_0^1 \varphi^{\frac{2-2p}{1+2p}}(\tau)\,\frac{d\varphi}{d\tau}(1+S)^p\left(\frac{\partial^2 U}{\partial \tau \partial x}\right)^2 dx\, d\tau$$

$$+\frac{p}{2}\int_0^t \varphi^{\frac{3}{1+2p}}(\tau)(1+S)^{p-1}\int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 dx \int_0^1 \left(\frac{\partial^2 U}{\partial \tau \partial x}\right)^2 dx\, d\tau$$

$$-p\varphi^{\frac{3}{1+2p}}(t)(1+S)^{p-1}\int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 dx \int_0^1 \frac{\partial U}{\partial x}\,\frac{\partial^2 U}{\partial t \partial x}\, dx + p\int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 dx \int_0^1 \frac{\partial U}{\partial x}\,\frac{\partial^2 U}{\partial t \partial x}\, dx\bigg|_{t=0}$$

$$+\frac{3p}{1+2p}\int_0^t \varphi^{\frac{2-2p}{1+2p}}(\tau)\,\frac{d\varphi}{d\tau}\,(1+S)^{p-1}\int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 dx \int_0^1 \frac{\partial U}{\partial x}\,\frac{\partial^2 U}{\partial \tau \partial x}\, dx\, d\tau$$

$$+p(p-1)\int_0^t \varphi^{\frac{3}{1+2p}}(\tau)(1+S)^{p-2}\left[\int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 dx\right]^2 \int_0^1 \frac{\partial U}{\partial x}\,\frac{\partial^2 U}{\partial \tau \partial x}\, dx\, d\tau$$

$$+p\int_0^t \varphi^{\frac{3}{1+2p}}(\tau)(1+S)^{p-1}\frac{d}{d\tau}\int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 dx \int_0^1 \frac{\partial U}{\partial x}\,\frac{\partial^2 U}{\partial \tau \partial x}\, dx\, d\tau$$

$$+p\int_0^t \varphi^{\frac{3}{1+2p}}(\tau)(1+S)^{p-1}\int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 dx \int_0^1 \left(\frac{\partial^2 U}{\partial \tau \partial x}\right)^2 dx\, d\tau$$

$$\le C_1\varphi^{\frac{p+1}{1+2p}}(t) + C_2\varphi^{\frac{p+1}{1+2p}}(t) + \frac{1}{4}\varphi^{\frac{3}{1+2p}}(t)(1+S)^p\int_0^1 \left(\frac{\partial^2 U}{\partial t \partial x}\right)^2 dx$$

$$+C_3\varphi^{\frac{3}{1+2p}}(t)(1+S)^{p-2} + p\left\|\frac{\partial U}{\partial x}\right\|^3\left\|\frac{\partial^2 U}{\partial x \partial t}\right\|\bigg|_{t=0}$$

$$+C_4\int_0^t \varphi^{\frac{2}{1+2p}}(\tau)(1+S)^p\int_0^1\Big(\frac{\partial^2 U}{\partial\tau\partial x}\Big)^2\,dx\,d\tau+C_4\int_0^t \varphi^{\frac{2}{1+2p}}(\tau)(1+S)^{p-2}\,d\tau$$

$$+C_5\int_0^t \varphi^{\frac{2}{1+2p}}(\tau)(1+S)^p\int_0^1\Big(\frac{\partial^2 U}{\partial\tau\partial x}\Big)^2\,dx\,d\tau+C_5\int_0^t \varphi^{\frac{4}{1+2p}}(\tau)(1+S)^{p-4}\,d\tau$$

$$+C_6\int_0^t \varphi^{\frac{3}{1+2p}}(\tau)(1+S)^{p-1}\Big[\int_0^1\Big(\frac{\partial U}{\partial x}\Big)^2\,dx\Big]^{\frac12}$$

$$\times\Big[\int_0^1\Big(\frac{\partial^2 U}{\partial x\partial\tau}\Big)^2\,dx\Big]^{\frac12}\Big[\int_0^1\Big(\frac{\partial U}{\partial x}\Big)^2\,dx\Big]^{\frac12}\Big[\int_0^1\Big(\frac{\partial^2 U}{\partial x\partial\tau}\Big)^2\,dx\Big]^{\frac12}\,d\tau$$

$$+C_7\int_0^t \varphi^{\frac{3}{1+2p}}(\tau)(1+S)^{p-1}\int_0^1\Big(\frac{\partial^2 U}{\partial\tau\partial x}\Big)^2\,dx\,d\tau.$$

From this, taking again into account Lemma 3.1.1 and estimates (3.1.23), (3.1.24), after simple transformations we get

$$\frac14\,\varphi^{\frac{p+3}{1+2p}}(t)\int_0^1\Big(\frac{\partial^2 U}{\partial t\partial x}\Big)^2\,dx\le C_8\varphi^{\frac{p+1}{1+2p}}(t)+C_9\int_0^t\varphi^{\frac{p}{1+2p}}(\tau)\,d\tau$$

$$+\,C_{10}\int_0^t \varphi^{\frac{3}{1+2p}}(\tau)(1+S)^{p-1}\int_0^1\Big(\frac{\partial^2 U}{\partial\tau\partial x}\Big)^2\,dx\,d\tau+C_{11}$$

$$\le C_{12}\varphi^{\frac{p+1}{1+2p}}(t),$$

or, finally,

$$\int_0^1\Big(\frac{\partial^2 U}{\partial t\partial x}\Big)^2\,dx\le C\varphi^{-\frac{2}{1+2p}}(t).$$

From this, according to the scheme of Chapter 2, we obtain

$$\Big|\frac{\partial U(x,t)}{\partial t}\Big|\le C\varphi^{-\frac{1}{1+2p}}(t). \tag{3.1.30}$$

After integration, from (3.1.24), it is easy to show that

$$ct^{1+2p}\le\varphi(t)\le Ct^{1+2p},\quad t\ge 1.$$

From this, taking into account estimates (3.1.28) and (3.1.30), we obtain the validity of Theorem 3.1.4. □

Remark 3.1.4. The existence of globally defined solutions of problems (3.1.2)–(3.1.5) and (3.1.17)–(3.1.20) can now be reobtained by a routine procedure. One first establishes the existence of local solutions at a maximal time interval and then, relying on the obtained estimates, concludes that this solution cannot vanish at a finite time.

3.2 Stability and convergence of semi-discrete and finite difference schemes

The purpose of this section is to study the semi-discrete and finite difference schemes for equation (3.1.1). Here we consider the case $a(S)=(1+S)^p$, $0<p\le 1$. Note that the difference schemes for these models were investigated in [38, 39]. The difference schemes for a certain nonlinear parabolic integro-differential model, similar to (3.1.1), were studied in [10, 19, 30, 44, 52, 65–67, 74].

3.2.1 Semi-discrete scheme

Consider the problem

$$\frac{\partial U}{\partial t} - \left[1 + \int_0^t\int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 dx\, d\tau\right]^p \frac{\partial^2 U}{\partial x^2} = f(x,t), \tag{3.2.1}$$

$$U(0,t) = U(1,t) = 0, \tag{3.2.2}$$

$$U(x,0) = U_0(x), \tag{3.2.3}$$

in the rectangle $Q_T = (0,1)\times(0,T)$, where T is a positive constant, $f = f(x,t)$ and $U_0 = U_0(x)$ are the given functions of their arguments.

As in Chapter 2, we introduce a net whose mesh points are denoted by $x_i = ih$, $i = 0,1,\dots,M$, with $h = 1/M$. The boundaries are specified by $i = 0$ and $i = M$. Let $u_i = u_i(t)$ be a semi-discrete approximation for (x_i,t). The exact solution to the problem for (x_i,t), denoted by $U_i = U_i(t)$, is assumed to exist and be smooth enough. From the boundary conditions (3.2.2) we have $u_0(t) = u_M(t) = 0$. For other points x_i, $i = 1,2,\dots,M-1$, the AIDE (3.2.1) will be replaced by approximation of the space derivatives by the forward and backward differences and spatial integral will be approximated by quadrature, say right rectangular formula. Once again, for the forward and backward differences we use the following notations:

$$u_{x,i}(t) = \frac{u_{i+1}(t) - u_i(t)}{h}, \quad u_{\overline{x},i}(t) = \frac{u_i(t) - u_{i-1}(t)}{h}.$$

Note that the values $u_i(0)$, $i = 1,2,\dots,M-1$, can be computed from the initial condition (3.2.3):

$$u_i(0) = U_{0,i}, \;\; i = 1,2,\dots,M-1.$$

Therefore, the semi-discrete problem corresponding to (3.2.1)–(3.2.3) is

$$\frac{du_i}{dt} - \left[1 + h\sum_{l=1}^{M}\int_0^t (u_{\overline{x},l})^2\, d\tau\right]^p u_{\overline{x}x,i} = f(x_i,t), \;\; i = 1,2,\dots,M-1, \tag{3.2.4}$$

$$u_0(t) = u_M(t) = 0, \tag{3.2.5}$$

$$u_i(0) = U_{0,i}, \;\; i = 0,1,\dots,M. \tag{3.2.6}$$

Thus, we have obtained the Cauchy problem (3.2.4)–(3.2.6) for a nonlinear system of OIDEs.

Multiplying (3.2.4) by $u(t) = (u_1(t), u_2(t),\dots,u_{M-1}(t))$, using the discrete analogue of the integration by parts and the Poincaré inequality, we get

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_h^2 + \|u_{\overline{x}}(t)]|_h^2 \le (f(t),u(t))_h \le \frac{1}{2}\|f(t)\|_h^2 + \frac{1}{2}\|u(t)\|_h^2 \le \frac{1}{2}\|f(t)\|_h^2 + \frac{1}{2}\|u_{\overline{x}}(t)]|_h^2,$$

where $f(t) = (f_1(t), f_2(t),\dots,f_{M-1}(t))$, $f_i(t) = f(x_i,t)$. So, we have

$$\|u(t)\|_h^2 + \int_0^t \|u_{\overline{x}}]|_h^2\, d\tau \le C. \tag{3.2.7}$$

Here and below, in the investigation of (3.2.4)–(3.2.6), C denotes a positive constant independent of h.

Remark 3.2.1. The a priori estimate (3.1.28) guarantees the global solvability of scheme (3.2.4)–(3.2.6). Note that applying the technique as in proving the convergence of theorem below, it is not difficult to prove the uniqueness and stability of the solution of scheme (3.2.4)–(3.2.6), as well.

The first result of this section is formulated in the form of the following statement.

Theorem 3.2.1. If $0 < p \le 1$ and problem (3.2.1)–(3.2.3) has a sufficiently smooth solution $U = U(x,t)$, then the solution $u = u(t) = (u_1(t), u_2(t), \dots, u_{M-1}(t))$ of problem (3.2.4)–(3.2.6) tends to the solution of the continuous problem (3.2.1)–(3.2.3) $U = U(t) = (U_1(t), U_2(t), \dots, U_{M-1}(t))$ as $h \to 0$, and the following estimate is true:

$$\|u(t) - U(t)\|_h \le Ch. \tag{3.2.8}$$

Proof. For the exact solution $U = U(x,t)$, we have

$$\frac{dU_i}{dt} - \left[1 + h\sum_{l=1}^{M}\int_0^t (U_{\overline{x},l})^2\,d\tau\right]^p U_{\overline{x}x,i} = f(x_i,t) - \psi_i(t), \quad i = 1,2,\dots,M-1, \tag{3.2.9}$$

$$U_0(t) = U_M(t) = 0, \tag{3.2.10}$$

$$U_i(0) = U_{0,i}, \quad i = 0,1,\dots,M, \tag{3.2.11}$$

where

$$\psi_i(t) = O(h).$$

Let $z_i(t) = u_i(t) - U_i(t)$ be the difference between approximate and exact solutions. From (3.2.4)–(3.2.6) and (3.2.9)–(3.2.11), we have

$$\frac{dz_i}{dt} - \left\{\left[1 + h\sum_{l=1}^{M}\int_0^t (u_{\overline{x},l})^2\,d\tau\right]^p u_{\overline{x},i} - \left[1 + h\sum_{l=1}^{M}\int_0^t (U_{\overline{x},l})^2\,d\tau\right]^p U_{\overline{x},i}\right\}_x = \psi_i(t), \tag{3.2.12}$$

$$z_0(t) = z_M(t) = 0, \tag{3.2.13}$$

$$z_i(0) = 0. \tag{3.2.14}$$

Multiplying (3.2.12) by $z(t) = (z_1(t), z_2(t), \dots, z_{M-1}(t))$, using (3.2.13) and the discrete analogue of the integration by parts, we get

$$\frac{1}{2}\frac{d}{dt}\|z\|_h^2 + h\sum_{i=1}^{M}\left\{\left[1 + h\sum_{l=1}^{M}\int_0^t (u_{\overline{x},l})^2\,d\tau\right]^p u_{\overline{x},i} - \left[1 + h\sum_{l=1}^{M}\int_0^t (U_{\overline{x},l})^2\,d\tau\right]^p U_{\overline{x},i}\right\}(u_{\overline{x},i} - U_{\overline{x},i}) = \sum_{i=1}^{M-1}\psi_i z_i h. \tag{3.2.15}$$

Note that

$$h\sum_{i=1}^{M}\left\{\left(1 + h\sum_{\ell=1}^{M}\int_0^t (u_{\overline{x},\ell})^2\,d\tau\right)^p u_{\overline{x},i} - \left(1 + h\sum_{\ell=1}^{M}\int_0^t (U_{\overline{x},\ell})^2\,d\tau\right)^p U_{\overline{x},i}\right\}(u_{\overline{x},i} - U_{\overline{x},i})$$

$$= h\sum_{i=1}^{M}\int_0^1 \left\{\frac{d}{d\mu}\left(1 + h\sum_{\ell=1}^{M}\int_0^t \left[U_{\overline{x},\ell} + \mu(u_{\overline{x},\ell} - U_{\overline{x},\ell})\right]^2 d\tau\right)^p \left[U_{\overline{x},i} + \mu(u_{\overline{x},i} - U_{\overline{x},i})\right]\right\} d\mu\,(u_{\overline{x},i} - U_{\overline{x},i})$$

$$= 2ph\sum_{i=1}^{M}\int_0^1 \left(1 + h\sum_{\ell=1}^{M}\int_0^t \left[U_{\overline{x},\ell} + \mu(u_{\overline{x},\ell} - U_{\overline{x},\ell})\right]^2 d\tau\right)^{p-1}$$

$$\times h\sum_{\ell=1}^{M}\int_0^t \left[U_{\overline{x},\ell} + \mu(u_{\overline{x},\ell} - U_{\overline{x},\ell})\right](u_{\overline{x},\ell} - U_{\overline{x},\ell})\,d\tau \left[U_{\overline{x},i} + \mu(u_{\overline{x},i} - U_{\overline{x},i})\right] d\mu\,(u_{\overline{x},i} - U_{\overline{x},i})$$

$$+ h\sum_{i=1}^{M}\int_0^1 \left(1 + h\sum_{\ell=1}^{M}\int_0^t \left[U_{\overline{x},\ell} + \mu(u_{\overline{x},\ell} - U_{\overline{x},\ell})\right]^2 d\tau\right)^p (u_{\overline{x},i} - U_{\overline{x},i})\,d\mu\,(u_{\overline{x},i} - U_{\overline{x},i})$$

$$= 2p \int_0^1 \left(1 + h \sum_{\ell=1}^{M} \int_0^t \left[U_{\overline{x},\ell} + \mu(u_{\overline{x},\ell} - U_{\overline{x},\ell})\right]^2 d\tau\right)^{p-1}$$

$$\times\, h \sum_{\ell=1}^{M} \int_0^t \left[U_{\overline{x},\ell} + \mu(u_{\overline{x},\ell} - U_{\overline{x},\ell})\right](u_{\overline{x},\ell} - U_{\overline{x},\ell})\, d\tau\, h \sum_{i=1}^{M} \left[U_{\overline{x},i} + \mu(u_{\overline{x},i} - U_{\overline{x},i})\right](u_{\overline{x},i} - U_{\overline{x},i})\, d\mu$$

$$+ \int_0^1 \left(1 + h \sum_{\ell=1}^{M} \int_0^t \left[U_{\overline{x},\ell} + \mu(u_{\overline{x},\ell} - U_{\overline{x},\ell})\right]^2 d\tau\right)^p h \sum_{i=1}^{M} (u_{\overline{x},i} - U_{\overline{x},i})^2\, d\mu$$

$$= 2p \int_0^1 \left(1 + h \sum_{\ell=1}^{M} \int_0^t \left[U_{\overline{x},\ell} + \mu(u_{\overline{x},\ell} - U_{\overline{x},\ell})\right]^2 d\tau\right)^{p-1} \xi(\mu)\, \frac{d\xi(\mu)}{dt}\, d\mu$$

$$+ \int_0^1 \left(1 + h \sum_{\ell=1}^{M} \int_0^t \left[U^k_{\overline{x},\ell} + \mu(u^k_{\overline{x},\ell} - U^k_{\overline{x},\ell})\right]^2 d\tau\right)^p h \sum_{i=1}^{M} (z_{\overline{x}})^2\, d\mu,$$

where

$$\xi(\mu) = h \sum_{\ell=1}^{M} \int_0^t \left[U_{\overline{x},\ell} + \mu(u_{\overline{x},\ell} - U_{\overline{x},\ell})\right](u_{\overline{x},\ell} - U_{\overline{x},\ell})\, d\tau,$$

$$\xi(\mu)\big|_{t=0} = 0,$$

and therefore,

$$\frac{d\xi(\mu)}{dt} = h \sum_{\ell=1}^{M} \left[U_{\overline{x},\ell} + \mu(u_{\overline{x},\ell} - U_{\overline{x},\ell})\right](u_{\overline{x},\ell} - U_{\overline{x},\ell}).$$

Introducing the notation

$$s(\mu) = h \sum_{\ell=1}^{M} \int_0^t \left[U_{\overline{x},\ell} + \mu(u_{\overline{x},\ell} - U_{\overline{x},\ell})\right]^2 d\tau,$$

from the previous equality, we have

$$h \sum_{i=1}^{M} \left\{ \left(1 + h \sum_{\ell=1}^{M} \int_0^t (u_{\overline{x},\ell})^2\, d\tau\right)^p u_{\overline{x},i} - \left(1 + h \sum_{\ell=1}^{M} \int_0^t (U_{\overline{x},\ell})^2\, d\tau\right)^p U_{\overline{x},i} \right\} (u_{\overline{x},i} - U_{\overline{x},i})$$

$$= 2p \int_0^1 (1 + s(\mu))^{p-1} \xi(\mu)\, \frac{d\xi(\mu)}{dt}\, d\mu + \int_0^1 (1 + s(\mu))^p \|z_{\overline{x}}]|^2\, d\mu$$

$$= p \int_0^1 (1 + s(\mu))^{p-1}\, \frac{d\xi^2(\mu)}{dt}\, d\mu + \int_0^1 (1 + s(\mu))^p \|z_{\overline{x}}]|^2\, d\mu.$$

After substituting this equality in (3.2.15), we get

$$\frac{1}{2}\, \frac{d}{dt}\, \|z\|_h^2 + p \int_0^1 (1 + s(\mu))^{p-1}\, \frac{d\xi^2(\mu)}{dt}\, d\mu + \int_0^1 (1 + s(\mu))^p \|z_{\overline{x}}]|^2\, d\mu = (\psi, z). \tag{3.2.16}$$

Integrating the obtained equality (3.2.16) on $(0, t)$, we have

$$\|z\|_h^2 + 2p \int_0^t \int_0^1 (1 + s(\mu))^{p-1}\, \frac{d\xi^2(\mu)}{dt}\, d\mu\, d\tau + \int_0^t \int_0^1 (1 + s(\mu))^p \|z_{\overline{x}}]|^2\, d\mu\, d\tau = \int_0^t (\psi, z)\, d\tau.$$

Using the formula of integration by parts, we get

$$\|z\|_h^2 + 2p\int_0^1 (1+s(\mu))^{p-1}\xi^2(\mu)\,d\mu - 2p(p-1)\int_0^t\int_0^1 (1+s(\mu))^{p-2}\,\frac{ds(\mu)}{dt}\,\xi^2(\mu\,)d\mu\,d\tau$$

$$+\int_0^t\int_0^1 (1+s(\mu))^p\|z_{\overline{x}}]|^2\,d\mu\,d\tau = \int_0^t (\psi,z)\,d\tau,$$

or from the definition of $s\,(\mu)$, we have

$$\|z\|_h^2 + 2p\int_0^1 (1+s(\mu))^{p-1}\xi^2(\mu)\,d\mu$$

$$-\,2p(p-1)\int_0^t\int_0^1 (1+s(\mu))^{p-2}h\sum_{\ell=1}^{M}\big[U_{\overline{x},\ell}+\mu(u_{\overline{x},\ell}-U_{\overline{x},\ell})\big]^2\xi^2(\mu)\,d\mu\,d\tau$$

$$+\int_0^t\int_0^1 (1+s(\mu))^p\|z_{\overline{x}}]|^2\,d\mu\,d\tau = \int_0^t (\psi,z)\,d\tau.$$

Taking into account the restriction $0<p\le 1$, from the last equality, we have

$$\|z(t)\|^2 \le \int_0^t \|z(\tau)\|^2\,d\tau + \int_0^t \|\psi_i\|^2\,d\tau. \tag{3.2.17}$$

From (3.2.17) we get (3.2.8), and hence Theorem 3.2.1 is proved. □

3.2.2 Finite difference scheme

In the rectangle $[0,1]\times[0,T]$, where T is a positive constant, let us study a finite difference scheme for the following initial-boundary value problem:

$$\frac{\partial U}{\partial t} - \left(1+\int_0^t\int_0^1\left(\frac{\partial U}{\partial x}\right)^2 dx\,d\tau\right)^p\frac{\partial^2 U}{\partial x^2} = f(x,t), \tag{3.2.18}$$

$$U(0,t)=U(1,t)=0, \tag{3.2.19}$$

$$U(x,0)=U_0(x), \tag{3.2.20}$$

where $0<p\le 1$ and U_0 is a given function.

As earlier, let us introduce a net on $[0,1]\times[0,T]$ with mesh points denoted by $(x_i,t_j)=(ih,j\tau)$, where, $i=0,1,\dots,M$; $j=0,1,\dots,N$ with $h=1/M$, $\tau=T/N$. The initial line is denoted by $j=0$. The discrete approximation for (x_i,t_j) is designated by u_i^j and the exact solution to problem (3.2.18)–(3.2.20) by U_i^j. We again follow the known notations of the backward and forward derivatives and the inner products.

For problem (3.2.18)–(3.2.20), let us consider the following finite difference scheme:

$$\frac{u_i^{j+1}-u_i^j}{\tau} - \Big(1+\tau h\sum_{k=1}^{j+1}\sum_{\ell=1}^{M}(u_{\overline{x},\ell}^k)^2\Big)^p u_{\overline{x}x,i}^{j+1} = f_i^j,\ \ i=1,2,\dots,M-1;\ \ j=0,1,\dots,N-1, \tag{3.2.21}$$

$$u_0^j = u_M^j = 0,\ \ j=0,1,\dots,N, \tag{3.2.22}$$

$$u_i^0 = U_{0,i},\ \ i=0,1,\dots,M. \tag{3.2.23}$$

Multiplying (3.2.21) scalarly by u_i^{j+1}, it is not difficult to get the inequality

$$\|u^n\|^2 + \tau \sum_{j=1}^{n} \|u_{\overline{x}}^j]|^2 < C, \quad n = 1, 2, \dots, N; \tag{3.2.24}$$

here and below in this subsection C is a positive constant independent of τ and h.

Remark 3.2.2. The a priori estimate (3.2.24) guarantees the solvability of scheme (3.2.21)–(3.2.23) by using the Brouwer's fixed-point lemma (see, e.g., [54] or Section 4.1). Note that applying the same technique as in proving the convergence theorem below, it is not difficult to prove the uniqueness and stability of the solution of the scheme (3.2.21)–(3.2.23), as well.

The main statement of the present subsection can be stated as follows.

Theorem 3.2.2. If $0 < p \le 1$ and problem (3.2.18)–(3.2.20) has a sufficiently smooth solution $U(x,t)$, then the solution $u^j = (u_1^j, u_2^j, \dots, u_{M-1}^j)$, $j = 1, 2, \dots, N$, of the difference scheme (3.2.21)–(3.2.23) tends to the solution of the continuous problem $U^j = (U_1^j, U_2^j, \dots, U_{M-1}^j)$, $j = 1, 2, \dots, N$, as $\tau \to 0$, $h \to 0$ and the following estimate is true

$$\|u^j - U^j\| \le C(\tau + h). \tag{3.2.25}$$

Proof. Let us introduce the difference $z_i^j = u_i^j - U_i^j$ to get the relations

$$z_{t,i}^{j+1} - \left\{ \Big(1 + \tau h \sum_{k=1}^{j+1} \sum_{\ell=1}^{M} (u_{\overline{x},\ell}^k)^2 \Big)^p u_{\overline{x},i}^{j+1} - \Big(1 + \tau h \sum_{k=1}^{j+1} \sum_{\ell=1}^{M} (U_{\overline{x},\ell}^k)^2 \Big)^p U_{\overline{x},i}^{j+1} \right\}_x = -\psi_i^j, \tag{3.2.26}$$

$$z_0^j = z_M^j = 0,$$

$$z_i^0 = 0,$$

where

$$\psi_i^j = O(\tau + h).$$

Multiplying (3.2.26) scalarly by $\tau z^{j+1} = \tau(z_1^{j+1}, z_2^{j+1}, \dots, z_{M-1}^{j+1})$ and using the discrete analogue of the formula of integration by parts, we get

$$\begin{aligned} \|z^{j+1}\|^2 - (z^{j+1}, z^j) + \tau h \sum_{i=1}^{M} \bigg\{ \Big(1 + \tau h \sum_{k=1}^{j+1} \sum_{\ell=1}^{M} (u_{\overline{x},\ell}^k)^2 \Big)^p u_{\overline{x},i}^{j+1} \\ - \Big(1 + \tau h \sum_{k=1}^{j+1} \sum_{\ell=1}^{M} (U_{\overline{x},\ell}^k)^2 \Big)^p U_{\overline{x},i}^{j+1} \bigg\} z_{\overline{x},i}^{j+1} = -\tau(\psi^j, z^{j+1}). \end{aligned} \tag{3.2.27}$$

Note that

$$\begin{aligned} & h \sum_{i=1}^{M} \bigg\{ \Big(1 + \tau h \sum_{k=1}^{j+1} \sum_{\ell=1}^{M} (u_{\overline{x},\ell}^k)^2 \Big)^p u_{\overline{x},i}^{j+1} - \Big(1 + \tau h \sum_{k=1}^{j+1} \sum_{\ell=1}^{M} (U_{\overline{x},\ell}^k)^2 \Big)^p U_{\overline{x},i}^{j+1} \bigg\} (u_{\overline{x},i}^{j+1} - U_{\overline{x},i}^{j+1}) \\ & = h \sum_{i=1}^{M} \int_0^1 \bigg\{ \frac{d}{d\mu} \Big(1 + \tau h \sum_{k=1}^{j+1} \sum_{\ell=1}^{M} \big[U_{\overline{x},\ell}^k + \mu(u_{\overline{x},\ell}^k - U_{\overline{x},\ell}^k)\big]^2 \Big)^p \big[U_{\overline{x},i}^{j+1} + \mu(u_{\overline{x},i}^{j+1} - U_{\overline{x},i}^{j+1})\big] \bigg\} \, d\mu \, (u_{\overline{x},i}^{j+1} - U_{\overline{x},i}^{j+1}) \\ & = 2ph \sum_{i=1}^{M} \int_0^1 \Big(1 + \tau h \sum_{k=1}^{j+1} \sum_{\ell=1}^{M} \big[U_{\overline{x},\ell}^k + \mu(u_{\overline{x},\ell}^k - U_{\overline{x},\ell}^k)\big]^2 \Big)^{p-1} \\ & \times \tau h \sum_{k=1}^{j+1} \sum_{\ell=1}^{M} \big[U_{\overline{x},\ell}^k + \mu(u_{\overline{x},\ell}^k - U_{\overline{x},\ell}^k)\big] (u_{\overline{x},\ell}^k - U_{\overline{x},\ell}^k) \big[U_{\overline{x},i}^{j+1} + \mu(u_{\overline{x},i}^{j+1} - U_{\overline{x},i}^{j+1})\big] \, d\mu \, (u_{\overline{x},i}^{j+1} - U_{\overline{x},i}^{j+1}) \end{aligned}$$

$$+\,h\sum_{i=1}^{M}\int\limits_0^1\bigg(1+\tau h\sum_{k=1}^{j+1}\sum_{\ell=1}^{M}\big[U_{\overline{x},\ell}^k+\mu(u_{\overline{x},\ell}^k-U_{\overline{x},\ell}^k)\big]^2\bigg)^p(u_{\overline{x},i}^{j+1}-U_{\overline{x},i}^{j+1})\,d\mu\,(u_{\overline{x},i}^{j+1}-U_{\overline{x},i}^{j+1})$$

$$=2p\int\limits_0^1\bigg(1+\tau h\sum_{k=1}^{j+1}\sum_{\ell=1}^{M}\big[U_{\overline{x},\ell}^k+\mu(u_{\overline{x},\ell}^k-U_{\overline{x},\ell}^k)\big]^2\bigg)^{p-1}$$

$$\times\,\tau h\sum_{k=1}^{j+1}\sum_{\ell=1}^{M}\big[U_{\overline{x},\ell}^k+\mu(u_{\overline{x},\ell}^k-U_{\overline{x},\ell}^k)\big](u_{\overline{x},\ell}^k-U_{\overline{x},\ell}^k)h\sum_{i=1}^{M}\big[U_{\overline{x},i}^{j+1}+\mu(u_{\overline{x},i}^{j+1}-U_{\overline{x},i}^{j+1})\big](u_{\overline{x},i}^{j+1}-U_{\overline{x},i}^{j+1})\,d\mu$$

$$+\int\limits_0^1\bigg(1+\tau h\sum_{k=1}^{j+1}\sum_{\ell=1}^{M}\big[U_{\overline{x},\ell}^k+\mu(u_{\overline{x},\ell}^k-U_{\overline{x},\ell}^k)\big]^2\bigg)^p h\sum_{i=1}^{M}(u_{\overline{x},i}^{j+1}-U_{\overline{x},i}^{j+1})^2\,d\mu$$

$$=2p\int\limits_0^1\bigg(1+\tau h\sum_{k=1}^{j+1}\sum_{\ell=1}^{M}\big[U_{\overline{x},\ell}^k+\mu(u_{\overline{x},\ell}^k-U_{\overline{x},\ell}^k)\big]^2\bigg)^{p-1}\xi^{j+1}(\mu)\xi_t^j(\mu)\,d\mu$$

$$+\int\limits_0^1\bigg(1+\tau h\sum_{k=1}^{j+1}\sum_{\ell=1}^{M}\big[U_{\overline{x},\ell}^k+\mu(u_{\overline{x},\ell}^k-U_{\overline{x},\ell}^k)\big]^2\bigg)^p h\sum_{i=1}^{M}(z_{\overline{x}}^{j+1})^2\,d\mu,$$

where

$$\xi^{j+1}(\mu)=\tau h\sum_{k=1}^{j+1}\sum_{\ell=1}^{M}\big[U_{\overline{x},\ell}^k+\mu(u_{\overline{x},\ell}^k-U_{\overline{x},\ell}^k)\big](u_{\overline{x},\ell}^k-U_{\overline{x},\ell}^k),$$

$$\xi^0(\mu)=0,$$

and therefore,

$$\xi_t^j(\mu)=h\sum_{\ell=1}^{M}\big[U_{\overline{x},\ell}^{j+1}+\mu(u_{\overline{x},\ell}^{j+1}-U_{\overline{x},\ell}^{j+1})\big](u_{\overline{x},\ell}^{j+1}-U_{\overline{x},\ell}^{j+1}).$$

Introducing the notation

$$s^{j+1}(\mu)=\tau h\sum_{k=1}^{j+1}\sum_{\ell=1}^{M}\big[U_{\overline{x},\ell}^k+\mu(u_{\overline{x},\ell}^k-U_{\overline{x},\ell}^k)\big]^2,$$

from the previous equality, we have

$$h\sum_{i=1}^{M}\bigg\{\bigg(1+\tau h\sum_{k=1}^{j+1}\sum_{\ell=1}^{M}(u_{\overline{x},\ell}^k)^2\bigg)^p u_{\overline{x},i}^{j+1}-\bigg(1+\tau h\sum_{k=1}^{j+1}\sum_{\ell=1}^{M}(U_{\overline{x},\ell}^k)^2\bigg)^p U_{\overline{x},i}^{j+1}\bigg\}(u_{\overline{x},i}^{j+1}-U_{\overline{x},i}^{j+1})$$

$$=2p\int\limits_0^1(1+s^{j+1}(\mu))^{p-1}\xi^{j+1}\xi_t^j\,d\mu+\int\limits_0^1(1+s^{j+1}(\mu))^p\|z_{\overline{x}}^{j+1}]|^2\,d\mu.$$

After substituting this equality into (3.2.27), we get

$$\|z^{j+1}\|^2-(z^{j+1},z^j)+2\tau p\int\limits_0^1(1+s^{j+1}(\mu))^{p-1}\xi^{j+1}(\mu)\xi_t^j(\mu)\,d\mu$$

$$+\,\tau\int\limits_0^1(1+s^{j+1}(\mu))^p\|z_{\overline{x}}^{j+1}]|^2\,d\mu=-\tau(\psi^j,z^{j+1}). \qquad (3.2.28)$$

Taking into account the restriction $p > 0$ and the relations $s^{j+1}(\mu) \geq 0$,

$$(r^{j+1}, r^j) = \frac{1}{2}\,\|r^{j+1}\|^2 + \frac{1}{2}\,\|r^j\|^2 - \frac{1}{2}\,\|r^{j+1} - r^j\|^2,$$
$$\tau\xi^{j+1}\xi_t^j = \frac{1}{2}\,(\xi^{j+1})^2 - \frac{1}{2}\,(\xi^j)^2 + \frac{\tau^2}{2}\,(\xi_t^j)^2,$$

from (3.2.28), we have

$$\begin{aligned}&\|z^{j+1}\|^2 - \frac{1}{2}\,\|z^{j+1}\|^2 - \frac{1}{2}\,\|z^j\|^2 \\ &\quad + \frac{1}{2}\,\|z^{j+1} - z^j\|^2 + p\int_0^1 (1 + s^{j+1}(\mu))^{p-1}\left[(\xi^{j+1}(\mu))^2 - (\xi^j(\mu))^2\right] d\mu \\ &\quad + \tau^2 p\int_0^1 (1 + s^{j+1}(\mu))^{p-1}(\xi_t^j(\mu))^2\, d\mu + \tau\|z_{\overline{x}}^{j+1}]|^2 \leq -\tau(\psi^j, z^{j+1}).\end{aligned} \tag{3.2.29}$$

From (3.2.29), we arrive at

$$\begin{aligned}&\frac{1}{2}\,\|z^{j+1}\|^2 - \frac{1}{2}\,\|z^j\|^2 + \frac{\tau^2}{2}\,\|z_t^j\|^2 \\ &\quad + p\int_0^1 (1 + s^{j+1}(\mu))^{p-1}\left[(\xi^{j+1}(\mu))^2 - (\xi^j(\mu))^2\right] d\mu + \tau\|z_{\overline{x}}^{j+1}]|^2 \leq \frac{\tau}{2}\,\|\psi^j\|^2 + \frac{\tau}{2}\,\|z^{j+1}\|^2.\end{aligned} \tag{3.2.30}$$

Using a discrete analogue of the Poincaré inequality $\|z^{j+1}\|^2 \leq \|z_{\overline{x}}^{j+1}]|^2$, from (3.2.30), we get

$$\begin{aligned}&\|z^{j+1}\|^2 - \|z^j\|^2 + \tau^2\|z_t^j\|^2 \\ &\quad + 2p\int_0^1 (1 + s^{j+1}(\mu))^{p-1}\left[(\xi^{j+1}(\mu))^2 - (\xi^j(\mu))^2\right] d\mu + \tau\|z_{\overline{x}}^{j+1}]|^2 \leq \tau\|\psi^j\|^2.\end{aligned} \tag{3.2.31}$$

Summing-up (3.2.31) from $j = 0$ to $j = n - 1$, we arrive at

$$\begin{aligned}&\|z^n\|^2 + \tau^2\sum_{j=0}^{n-1}\|z_t^j\|^2 \\ &\quad + 2p\sum_{j=0}^{n-1}\int_0^1 (1 + s^{j+1}(\mu))^{p-1}\left[(\xi^{j+1}(\mu))^2 - (\xi^j(\mu))^2\right] d\mu + \tau\sum_{j=0}^{n-1}\|z_{\overline{x}}^{j+1}]|^2 \leq \tau\sum_{j=0}^{n-1}\|\psi^j\|^2.\end{aligned} \tag{3.2.32}$$

Note that since $s^{j+1}(\mu) \geq s^j(\mu)$ and $p \leq 1$, for the second line of the latter formula we have

$$\begin{aligned}&\sum_{j=0}^{n-1}(1 + s^{j+1}(\mu))^{p-1}\left[(\xi^{j+1}(\mu))^2 - (\xi^j(\mu))^2\right] \\ &\quad = (1 + s^1(\mu))^{p-1}(\xi^1(\mu))^2 - (1 + s^1(\mu))^{p-1}(\xi^0(\mu))^2 + (1 + s^2(\mu))^{p-1}(\xi^2(\mu))^2 \\ &\quad - (1 + s^2(\mu))^{p-1}(\xi^1(\mu))^2 + \cdots + (1 + s^n(\mu))^{p-1}(\xi^n(\mu))^2 - (1 + s^n(\mu))^{p-1}(\xi^{n-1}(\mu))^2 \\ &\quad = (1 + s^n(\mu))^{p-1}(\xi^n(\mu))^2 + \sum_{j=1}^{n-1}\left[(1 + s^j(\mu))^{p-1} - (1 + s^{j+1}(\mu))^{p-1}\right](\xi^j(\mu))^2 \geq 0.\end{aligned}$$

Taking into account the last relation and (3.2.32), one can deduce

$$\|z^n\|^2 + \tau^2\sum_{j=0}^{n-1}\|z_t^j\|^2 + \tau\sum_{j=0}^{n-1}\|z_{\overline{x}}^{j+1}]|^2 \leq \tau\sum_{j=0}^{n-1}\|\psi^j\|^2. \tag{3.2.33}$$

From (3.2.33), we get (3.2.25), and thus, Theorem 3.2.2 is proved. □

Note that using the same approach of investigation, the second order difference scheme for problem (3.2.18)–(3.2.20) can be studied.

3.2.3 Numerical implementation remarks

Let us now speak about the numerical implementation of the discrete problem (3.2.21)–(3.2.23). Note that (3.2.21) can be rewritten as

$$\frac{1}{\tau}\,u_i^{j+1} - A(u^{j+1})\,\frac{u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}}{h^2} - f_i^j - \frac{1}{\tau}\,u_i^j = 0, \quad i = 1, \dots, M-1,$$

where

$$A(u^{j+1}) = \Big[1 + \tau h \sum_{\ell=1}^{M} \sum_{k=1}^{j+1} \Big(\frac{u_\ell^k - u_{\ell-1}^k}{h}\Big)^2\Big]^p.$$

This system can be written in a matrix form

$$H(u^{j+1}) \equiv G(u^{j+1}) - \frac{1}{\tau}\,u^j - f^j = 0.$$

The vector u contains all the unknowns $u_1, \dots, u_{M-1}$ at the level indicated by superscript. The vector G is given by

$$G(u^{j+1}) = T(u^{j+1})u^{j+1},$$

where the matrix T is symmetric and tridiagonal with the elements

$$T_{ir} = \begin{cases} \dfrac{1}{\tau} + 2\dfrac{A}{h^2}\,, & r = i, \\ -\dfrac{A}{h^2}\,, & r = i \pm 1. \end{cases}$$

The Newton method for the system is given by

$$\nabla H(u^{j+1})|^{(n)}(u^{j+1}|^{(n+1)} - u^{j+1}|^{(n)}) = -H(u^{j+1})|^{(n)}.$$

The elements of the matrix $\nabla H\left(u^{j+1}\right)$ require the derivative of A. The elements are

$$\nabla H(u^{j+1})|_{ir} = \begin{cases} \dfrac{1}{\tau} + \dfrac{2}{h^2}\,A(u^{j+1}) - \dfrac{\partial A(u^{j+1})}{\partial u_i^{j+1}}\,\delta_i^{j+1}, & r = i, \\ -\delta_i^{j+1}\,\dfrac{\partial A(u^{j+1})}{\partial u_r^{j+1}} - \dfrac{1}{h^2}\,(u^{j+1}), & r = i \pm 1, \\ -\delta_i^{j+1}\,\dfrac{\partial A(u^{j+1})}{\partial u_r^{j+1}}\,, & \text{otherwise}, \end{cases}$$

where

$$\delta_i^{j+1} = u_{\overline{x}x,i}^{j+1} = \frac{u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}}{h^2}\,.$$

To evaluate the partial derivatives, we use

$$\frac{\partial A}{\partial u_r^{j+1}} = \frac{\partial}{\partial u_r^{j+1}} \left[1 + \tau h \sum_{\ell=1}^{M} \sum_{k=1}^{j+1} \Big(\frac{u_\ell^k - u_{\ell-1}^k}{h}\Big)^2\right]^p$$

$$= p\left[1 + \tau h \sum_{\ell=1}^{M} \sum_{k=1}^{j+1} \Big(\frac{u_\ell^k - u_{\ell-1}^k}{h}\Big)^2\right]^{p-1} \frac{\partial}{\partial u_r^{j+1}} \left[C + \tau h\Big(\frac{u_r^{j+1} - u_{r-1}^{j+1}}{h}\Big)^2 + \tau h\Big(\frac{u_{r+1}^{j+1} - u_r^{j+1}}{h}\Big)^2\right]$$

$$= 2p\tau h\left[1 + \tau h \sum_{\ell=1}^{M} \sum_{k=1}^{j+1} \Big(\frac{u_\ell^k - u_{\ell-1}^k}{h}\Big)^2\right]^{p-1} \left[\frac{u_r^{j+1} - u_{r-1}^{j+1}}{h} \cdot \frac{1}{h} + \frac{u_{r+1}^{j+1} - u_r^{j+1}}{h} \cdot \Big(-\frac{1}{h}\Big)\right]$$

$$= -2p\tau h\left[1 + \tau h \sum_{\ell=1}^{M} \sum_{k=1}^{j+1} \Big(\frac{u_\ell^k - u_{\ell-1}^k}{h}\Big)^2\right]^{p-1} \frac{u_{r+1}^{j+1} - 2u_r^{j+1} + u_{r-1}^{j+1}}{h^2}.$$

Note that we have incorporated into the constant C all the terms that are independent of u_r^{j+1}.

Let us apply the Newton theorem for the convergence of the iterative process [62].

Theorem 3.2.3. Given the nonlinear system of equations

$$g_i(x_1, \dots, x_{M-1}) = 0, \ \ i = 1, 2, \dots, M-1.$$

If g_i are three times continuously differentiable in a region containing the solution $\xi_1, \dots, \xi_{M-1}$ and the Jacobian does not vanish in that region, then the Newton method converges at least quadratically.

In our case, we can write

$$g_i = u_i^{j+1} - \tau A(u^{j+1}) \frac{u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}}{h^2} - \tau f_i^j - u_i^j = 0, \ \ i = 1, \dots, M-1.$$

The Jacobian is the matrix ∇H computed above. The term $1/\tau$ on diagonal ensures that the Jacobian does not vanish. The differentiability is guaranteed, since ∇H is quadratic. The Newton method is valuable, because the matrix varies at every step of the iteration. One can use the Newton modified method (keeping the same matrix for several iterations), but the rate of convergence will be slower.

In our first numerical experiment we have chosen the right-hand side, so the exact solution is given by

$$U(x,t) = x(1-x)e^{-x-t}.$$

In this case, the right-hand side is

$$f(x,t) = -x(1-x)e^{-x-t}\Big(1 + \frac{1}{4}\, e^{-2t}t - \frac{3}{4}\, e^{-2-2t}t\Big)^p$$
$$\times \big(2e^{-x-t} - 2(1-x)e^{-x-t} + 2xe^{-x-t} + x(1-)xe^{-x-t}\big).$$

The parameters used are $M = 100$ which determines $h = 0.01$. Since the method is implicit, we can use $\tau = h$ and take 100 time steps. We plotted the numerical solution (marked with +) and the exact solution for $t = 0.5$ and $t = 1.0$ for different values of p ($p = 0.25, 0.5, 0.75, 1$; Figures 3.1–3.8), and it is clear that in all cases this two solutions are almost identical.

In our next experiment we have taken the zero right-hand side and the initial solution given by

$$U(x,0) = x(1-x) + x\big(e^{-x} - e^{-1}cos(30\pi x)\big).$$

In this case, we know that the solution will decay in time. The parameters M, h, τ are as before. In Figures 3.9 and 3.10 we plotted the initial solution and the numerical solution at five different times for the case $p = 1$. The same results are obtained for the cases $p = 0.25; 0.5; 0.75$. It is clear that the numerical solution approaches zero for all x. Therefore, the numerical solution of our experiment fully agree with the theoretical results in Theorem 3.1.2.

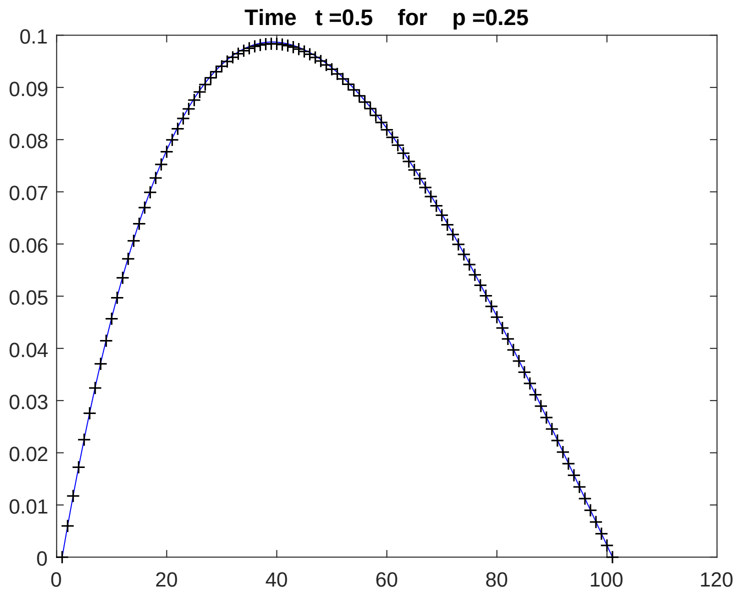

Figure 3.1. The solution at $t = 0.5$ for $p = 0.25$. The exact solution is a solid line and the numerical solution is marked by $+$.

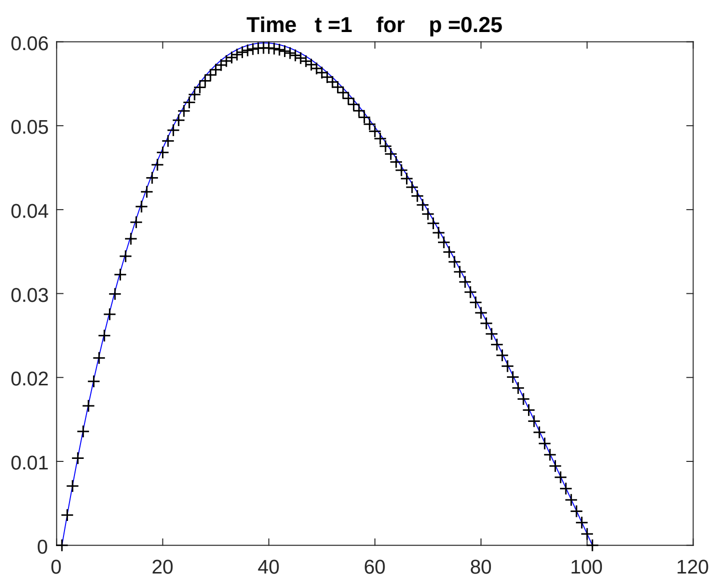

Figure 3.2. The solution at $t = 1$ for $p = 0.25$. The exact solution is a solid line and the numerical solution is marked by $+$.

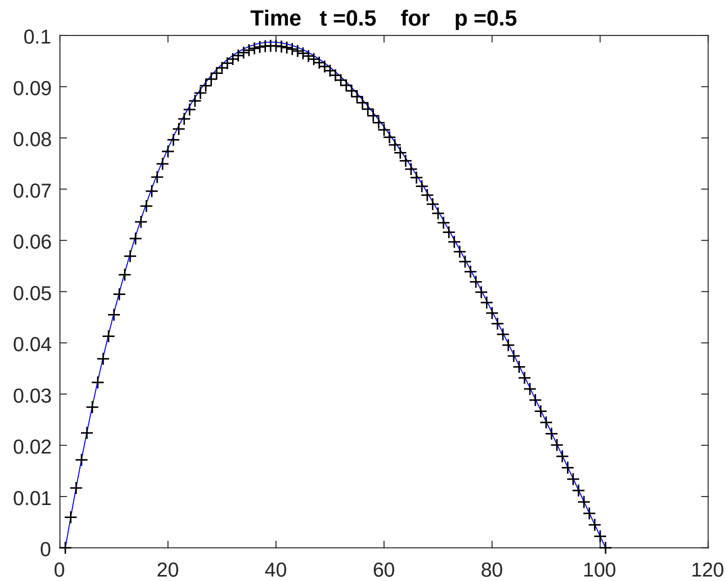

Figure 3.3. The solution at $t = 0.5$ for $p = 0.5$. The exact solution is a solid line and the numerical solution is marked by $+$.

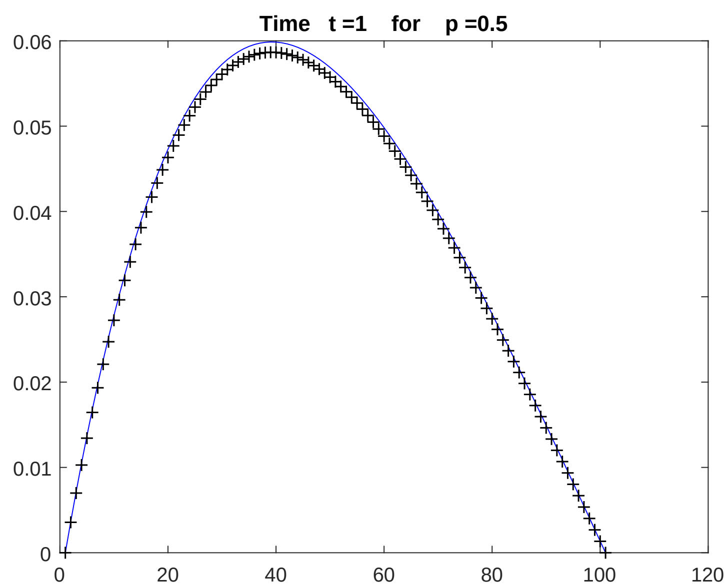

Figure 3.4. The solution at $t = 1$ for $p = 0.5$. The exact solution is a solid line and the numerical solution is marked by $+$.

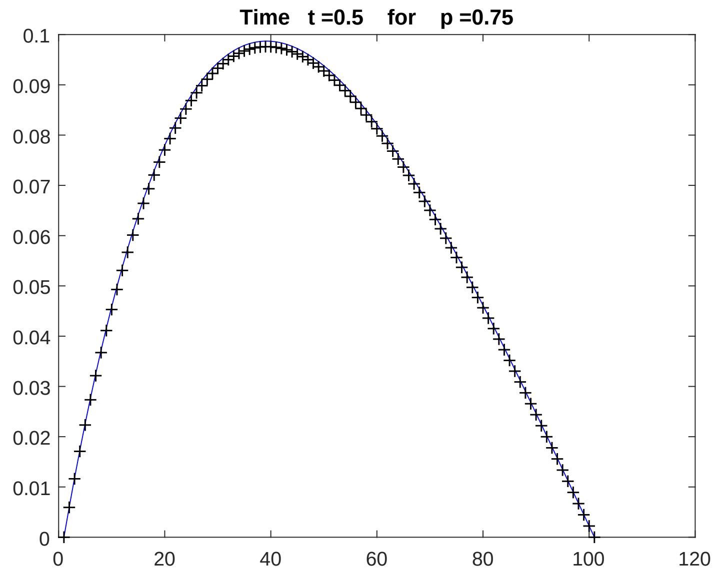

Figure 3.5. The solution at $t = 0.5$ for $p = 0.75$. The exact solution is a solid line and the numerical solution is marked by $+$.

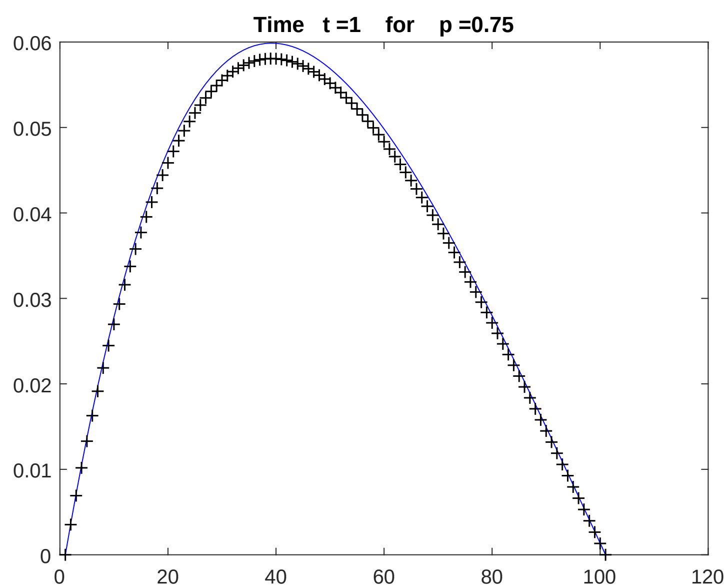

Figure 3.6. The solution at $t = 1$ for $p = 0.75$. The exact solution is a solid line and the numerical solution is marked by $+$.

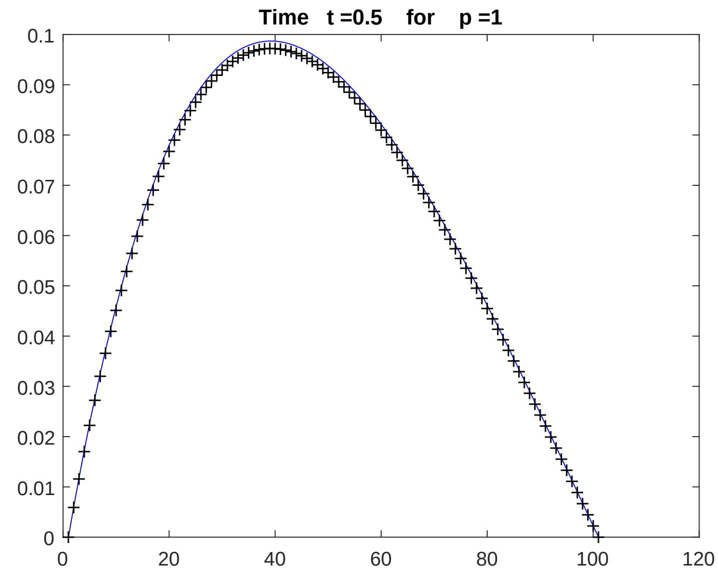

Figure 3.7. The solution at $t = 0.5$ for $p = 1$. The exact solution is a solid line and the numerical solution is marked by $+$.

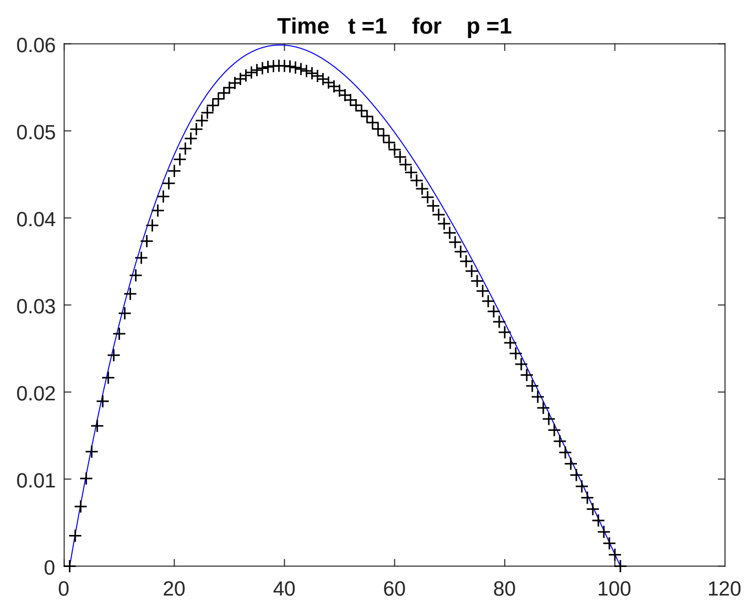

Figure 3.8. The solution at $t = 1$ for $p = 1$. The exact solution is a solid line and the numerical solution is marked by $+$.

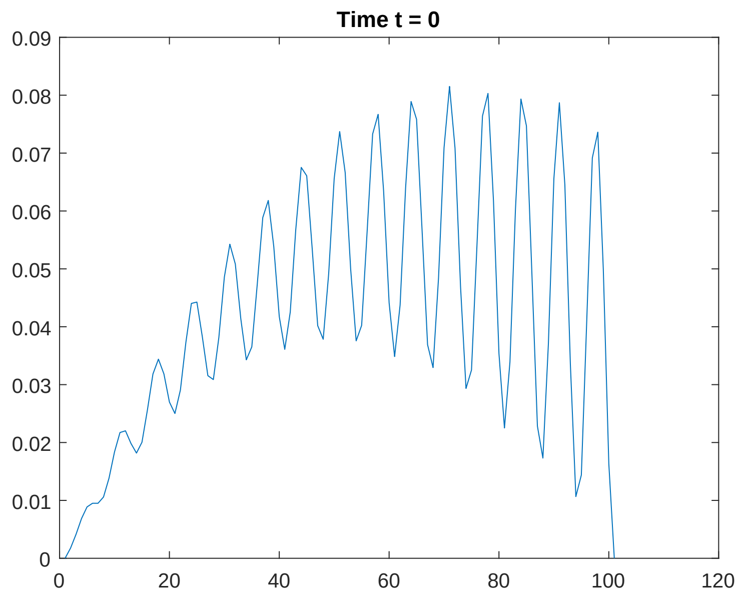

Figure 3.9. Initial solution.

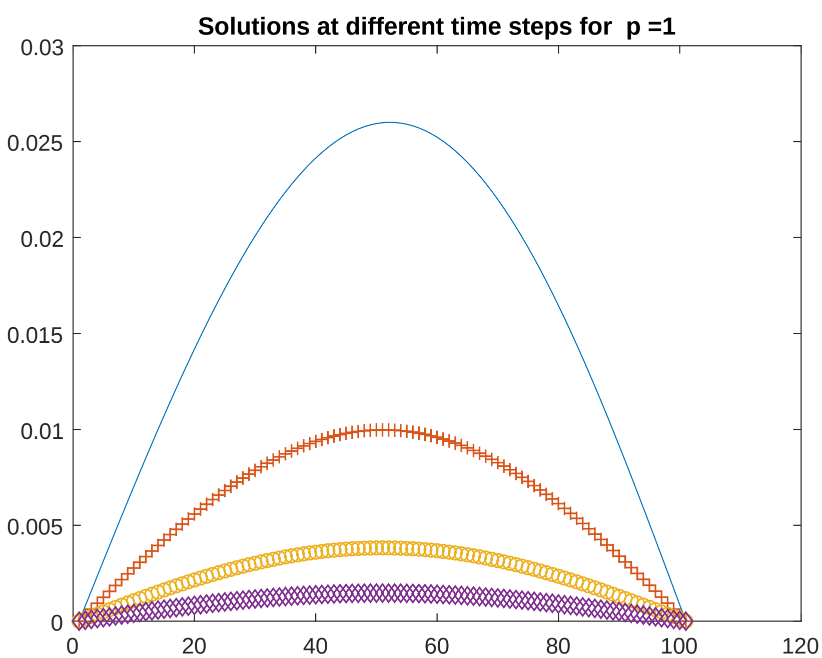

Figure 3.10. The numerical solutions at $t = 0.1, 0.2, 0.3, 0.4$ which are marked as '$-$', '$+$', '$\circ$', '$\diamond$', respectively, for the case $p = 1$.

Table 3.1 shows maximum of the absolute values of errors between the exact and numerical solutions for different time levels and different values of p.

As we can see, the maximum of the absolute values of differences between the exact and numerical solutions does not exceed the approximation error. Note also that the energy norm of the error decreases as it was expected by the theoretical researches. Several other numerical experiments were carried out which show the stability and convergence of scheme (3.2.21)–(3.2.23).

At the end of Chapter 3, let us make some conclusions. One nonlinear parabolic AIDE based on the system of Maxwell equations is considered. Large time behavior of solutions of the initial-boundary value problems for that model are studied. The corresponding semi-discrete and finite difference schemes are constructed and investigated. The stability and convergence of those schemes are proved. Results of numerical experiments with appropriate table and graphical illustrations are given. Results of numerical experiments fully agree with theoretical researches both, in the convergence of the finite discrete scheme, as well as in asymptotic behavior of a solution.

Table 3.1. Maximum of the absolute values of errors between the exact and numerical solutions for different time levels and different values of p.

t	Error for $p = 0.25$	Error for $p = 0.5$	Error for $p = 0.75$	Error for $p = 1$
0.2	0.00005013277686	0.00000033742015	0.00000142297535	0.00000286166125
0.4	0.00008803560728	0.00000267708729	0.00000387833216	0.00000997725914
0.5	0.00010724661584	0.00000488801011	0.00000543335746	0.00001519617372
0.6	0.00012672452711	0.00000790771200	0.00000720734945	0.00002157533554
0.8	0.00016681835775	0.00001649515863	0.00001138842646	0.00003789336764
1.0	0.00020916077114	0.00002898727812	0.00001632434202	0.00005892627813

Chapter 4

Unique solvability and asymptotic behavior of solutions for nonlinear multi-dimensional parabolic integro-differential problems

In Chapter 4, we investigate two classes of nonlinear parabolic type IDEs:

$$\frac{\partial U}{\partial t} - \sum_{i=1}^{n} D_i \left[a\left(\int_0^t |\nabla U|^q \, d\tau \right) |\nabla U|^{q-2} D_i U \right] = f(x,t),$$

and

$$\frac{\partial U}{\partial t} - a\left(\int_0^t \int_\Omega |\nabla U|^q \, dx \, d\tau \right) \sum_{i=1}^{n} D_i \left[|\nabla U|^{q-2} D_i U \right] = f(x,t),$$

where

$$D_i = \frac{\partial}{\partial x_i}\,, \quad \nabla = \Big(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \Big),$$

$q \geq 2$, $\Omega \subset R^n$ is a bounded domain, f and a are the given functions of their arguments.

These models are some generalizations of the equations considered in Chapters 1–3. They are based on the Maxwell system and arise at mathematical modeling of the process of penetration of an electromagnetic field in a medium, coefficient electroconductivity of which depends on temperature. Some peculiarities of these mathematical models are studied in Chapters 1–3. The above-pointed out IDEs are complex and has been managed to be investigated only for the particular cases so far (see, e.g., [3, 5, 6, 16, 17, 19, 20, 23, 24, 28, 31, 32, 34, 35, 37–39, 46, 49, 50, 53, 55] and the references therein). As to the similar type equations as given here, they were first suggested in [16, 20, 28] and then generalized in numerous other works.

In the present chapter we study the first type initial-boundary value problem. Investigations are carried out with the help of Galerkin's method and the method of compactness [54, 71]. Attention is also paid to the asymptotic behavior of solutions as $t \to \infty$. The chapter consists of five sections. Some designations, preliminary remarks and auxiliary statements are given in Section 4.1. Some features of the Volterra-type integro-differential problems are presented in Section 4.2. Section 4.3 is devoted to the unique solvability of the Volterra-type integro-differential problems. Asymptotic behavior of solutions for Volterra-type equations is given in Section 4.4. Some comments on the unique solvability and asymptotic behavior of the solutions for AIDEs are shortly given in Section 4.5.

4.1 Designations, preliminary remarks and auxiliary statements

Let Ω be a bounded area in the n-dimensional Euclidian space R^n, with a smooth enough boundary $\partial\Omega$, $x = (x_1, \dots, x_n) \in \Omega$; $\alpha = (\alpha_1, \dots, \alpha_n)$ is multi-index with the whole nonnegative components α_i and

$$D^\alpha U = D_1^{\alpha_1} \cdots D_n^{\alpha_n} U = \frac{\partial^{|\alpha|} U}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}, \quad |\alpha| = \alpha_1 + \cdots + \alpha_n.$$

Through $L_p(\Omega)$ is designated the Banach space of functions, integrable on Ω of degree p, $1 \le p < \infty$, and through $W_p^k(\Omega)$ are designated Sobolev spaces [54] consisting of the functions belonging to $L_p(\Omega)$ and having all generalized derivatives up to order k, inclusive, integrable on Ω with a degree p. The norm in $W_p^k(\Omega)$ is determined by the equality

$$\|U\|_{W_p^k(\Omega)} = \left[\int\limits_\Omega \sum_{|\alpha| \le k} |D^\alpha U|^p \, dx \right]^{\frac{1}{p}}.$$

Let $D(\Omega)$ be a space of indefinite differentiable functions with compact support in Ω. Through $\overset{\circ}{W}{}_p^k(\Omega)$ is designated a subspace of the space $W_p^k(\Omega)$, obtained by closure on the norm of the set $D(\Omega)$, and $W_q^{-k}(\Omega)$, $1/p + 1/q = 1$ denotes the space, conjugate to $\overset{\circ}{W}{}_p^k(\Omega)$.

Further, if X is the Banach space and $U(t)$, $0 \le t \le T$, is a measurable function with values in X, then $L_p(0, T; X)$ designates the Banach space with the norm

$$\|U\|_{L_p(0,T;X)} = \left[\int\limits_0^T \|U(t)\|_X^p \, dt \right]^{\frac{1}{p}}, \quad 1 \le p < \infty,$$

where $\|U(t)\|_X$ is the norm of functions U in the space X for fixed $t \in [0, T]$. In particular, we will use the spaces $L_p(0, T; \overset{\circ}{W}{}_p^k(\Omega))$ and $L_p(0, T; W_q^{-k}(\Omega))$. In order to prove the existence of solutions of a system of nonlinear algebraic equations obtained at application of Galerkin's method, we will use one of the variants of the Brouwer fixed point theorem. Let us formulate this statement [54].

For $\xi = (\xi_1, \dots, \xi_m)$ and $\eta = (\eta_1, \dots, \eta_m)$, we introduce the following Euclidean scalar product and the norm:

$$(\xi, \eta) = \sum_{i=1}^m \xi_i \eta_i, \quad |\xi| = (\xi, \xi)^{\frac{1}{2}}.$$

Lemma 4.1.1. Let $\xi \to P(\xi)$ be the function, continuous from R^m into itself such that for a suitable $\rho > 0$,

$$(P(\xi), \xi) \ge 0$$

for any ξ of the sphere $S_\rho = \{\xi : \ |\xi| = \rho\}$. Then there exists ξ, $|\xi| \le \rho$, such that

$$P(\xi) = 0.$$

The next lemma gives us an opportunity for further application of the obtained a priori estimations through the scalar multiplication of the initial operator by the elements of some complete system [54, 71].

Lemma 4.1.2. There exists the "basis" $\{w_j(x,t)\}$ of smooth enough functions in $\overline{Q} = \overline{\Omega} \times [0, T]$ such that the functions $\{Bw_j(x,t)\}$ form the "basis" in the space $L_p(0, T; \overset{\circ}{W}{}_p^k(\Omega))$, where

$$Bw = -\frac{\partial}{\partial t}\left[(T - t)\frac{\partial w}{\partial t}\right] - \psi \Delta w + \lambda w, \quad \Delta w = \sum_{i=1}^n \frac{\partial^2 w}{\partial x_i^2}, \quad \lambda > 0,$$

and $\psi \in C^\infty(\overline{\Omega})$; $\psi(x) > 0$ for $x \in \Omega$ and $\psi = \partial\psi/\partial\nu = 0$ on $\partial\Omega$, where ν is an outer normal to $\partial\Omega$.

To prove Lemma 4.1.2, we consider the following two auxiliary problems.

Problem 4.1.1. The sequence of functions g_k, defined on $[0,\,T]$, are constructed so that

$$-\frac{d}{dt}\Big[(T-t)\,\frac{dg_k}{dt}\Big]=\mu_k g_k,\quad g_k(0)=0,$$

$g_k(t)$ is bounded as $t\to T$, and the eigenvalues μ_k are positive, and the eigenfunctions are normalized by the condition

$$\int\limits_0^T g_k^2(t)\,dt=1.$$

Remark 4.1.1. The solutions of Problem 4.1.1 can be written in an analytic form as follows

$$g_k(t)=c_kJ_0\Big[\nu_k\Big(1-\frac{t}{T}\Big)^{\frac{1}{2}}\Big].$$

Here, J_0 is the first kind Bessel function, k represents the k-th positive root of function J_0, $\mu_k=\nu_k/4T$ and c_k is any constant, chosen from the normalization condition of the function g_k.

Problem 4.1.2. Let ξ_m, $m=1,2,\dots$, $\xi_m\in D(\Omega)$ be some complete system in $\overset{\circ}{W}{}_p^k(\Omega)$. We define V_{km} as the solutions of the following problem:

$$-\psi\Delta V_{km}+(\lambda+\mu_k)V_{km}=\xi_m,$$
$$V_{km}=0,\quad x\in\partial\Omega.$$

For large enough $\lambda>0$, Problem 4.1.2 has a unique solution, smooth in $\overline{\Omega}$ [54, 71]. Applying now the diagonal method if enumerate system of functions $\{V_{km}\}$ so that V_{km} will depend on one index, it is easy to establish that by linear combinations of the function of kind $B(V_{km}(x)g_k(t))$ it is possible to approximate any function $V\in L_p(0,T;\overset{\circ}{W}{}_p^k(\Omega))$ and thus they form the "basis" in the space $L_p(0,T;\overset{\circ}{W}{}_p^k(\Omega))$. Note that in many essentially nonlinear problems the application of usual Galerkin's method does not reach the goal, because the constructed in such a way sequence of Galerkin's approximation fails to take limit under the nonlinear functions. That is why the application of the modified Galerkin's method is needed with the basis defined by the operator B. For the limiting transition under the nonlinear members we use the following statements.

Lemma 4.1.3. Let Q be a bounded area in $R^n\times R$, h_m and h be the functions from $L_q(Q)$, $1<q<\infty$, such that

$$\|h_m\|_{L_q(Q)}\le C,\quad C=const,$$

$h_m\to h$ almost everywhere in Q. Then $h_m\to h$ weakly in $L_q(Q)$, i.e.,

$$\iint\limits_Q h_m(x,t)V(x,t)\,dx\,dt\longrightarrow\iint\limits_Q h(x,t)V(x,t)\,dx\,dt$$

for any $V\in L_q(Q)$, where $1/p+1/q=1$.

Theorem 4.1.1 (Valle–Pussen)**.** Let on the measurable set Q the family of measurable functions $M=\{h(x,t)\}$ be given. If there is a positive growing function $\phi(v)$ given for $v\ge 0$ and tending to $+\infty$ together with v, for which

$$\iint\limits_Q |h(x,t)|\phi(|h(x,t)|)\,dx\,dt<C,$$

where h is any function from M and C is a finite constant, independent of h, then the functions h are summable on Q, and their integrals are uniformly absolutely continuous.

Theorem 4.1.2 (Vitali)**.** Let on the measurable set Q the sequence of functions $h_1, h_2, \dots$ be convergent in a measure sense to the function h. If the functions of the sequence $\{h_m\}$ have uniformly absolutely continuous integrals, then h is summable and

$$\lim_{m\to\infty} \iint\limits_Q h_m(x,t)\,dx\,dt = \iint\limits_Q h(x,t)\,dx\,dt.$$

For the investigation of asymptotic behavior of solutions of the above-studied initial-boundary value problems we use the following statement (see, e.g., [43]).

Lemma 4.1.4. Let $h(\tau) \geq 0$ and for any $t \in [0,\infty)$ the inequalities

$$\int\limits_0^t h(\tau)\,d\tau < C, \quad \int\limits_0^t \Big|\frac{dh(\tau)}{d\tau}\Big|\,d\tau < C$$

be valid. Then

$$h(t) \to 0 \ \text{ as } \ t \to \infty.$$

4.2 Some features of the Volterra-type integro-differential problems

In the domain $Q = \Omega \times (0,T)$, $T = const > 0$, of the variables $x_1, x_2, \dots, x_n$, t, where $\Omega \subset R^n$ is the bounded domain with a sufficiently smooth boundary $\partial\Omega$, let us consider the following initial-boundary value problem:

$$\frac{\partial U}{\partial t} - \sum_{i=1}^n D_i\Big[a\Big(\int\limits_0^t |\nabla U|^q\,d\tau\Big)|\nabla U|^{q-2} D_i U\Big] = f(x,t), \tag{4.2.1}$$

$$U(x,t) = 0, \ \ (x,t) \in \partial\Omega \times (0,T), \tag{4.2.2}$$

$$U(x,0) = 0, \ \ x \in \Omega. \tag{4.2.3}$$

Here, $a = a(S)$ is a given function of its argument, $q \geq 2$,

$$|\nabla U| = \Big(\sum_{k=1}^n |D_k U|^2\Big)^{\frac{1}{2}}.$$

We seek for a solution of the problem in the space $L_{pq+q}(0,T;\overset{\circ}{W}{}^1_{pq+q}(\Omega))$ and $\partial U/\partial t \in L_2(Q)$. This solution satisfies the following integral identity

$$\iint\limits_Q \Big(\frac{\partial U}{\partial t} + AU\Big)V\,dx\,dt = \iint\limits_Q fV\,dx\,dt, \tag{4.2.4}$$

where V is an arbitrary function from the space $L_{pq+q}(0,T;\overset{\circ}{W}{}^1_{pq+q}(\Omega))$,

$$AU = -\sum_{i=1}^n D_i\Big[a\Big(\int\limits_0^t |\nabla U|^q\,d\tau\Big)|\nabla U|^{q-2} D_i U\Big].$$

The principal characteristic peculiarity of the equation of type (4.2.1) is connected with the appearance of the nonlinear terms depending on the time integral in the coefficients with high order derivatives. These circumstances require more different discussions than those usually necessary for the solution of local differential problems. Coefficient $a = a(S)$ in equation (4.2.1) consists of the integral defining the nonlocal operator A. The fact that the operator of type $\int\limits_0^t U\,d\tau$ is improvable, is often applied in the theory of differential equations. However, as is noted in [19, 49, 50], in the combination with spatial derivatives, the above-mentioned improvable characteristics may be lost. For example, the equation

$$\frac{\partial U}{\partial t}=\frac{\partial}{\partial x}\left[a\left(\int\limits_0^t\frac{\partial U}{\partial x}\,d\tau\right)\right],$$

in the nature of parabolic type, by the substitution of $\int\limits_0^t U\,d\tau=V$, is transformed into the following nonlinear hyperbolic equation

$$\frac{\partial^2 V}{\partial t^2}=\frac{\partial}{\partial x}\left[a\left(\frac{\partial V}{\partial x}\right)\right],$$

which, in general, has no global solution. This peculiarity gets a particular form in perspectives of solvability of such equations by requiring the restrictions on the coefficient a.

Note also that equation (4.2.1) admits the degeneration.

4.3 The unique solvability of Volterra-type integro-differential problems by modified Galerkin method

The principal aim of the present section is to prove the following statement.

Theorem 4.3.1. If

$$a(S)=(1+S)^p,\ \ S(x,t)=\int\limits_0^t|\nabla U|^q\,d\tau,\ \ 0<p\le 1,\ \ q\ge 2,$$
$$f,\ \frac{\partial f}{\partial t},\ \sqrt{\psi}\,\frac{\partial f}{\partial x_i}\in L_2(Q),\ \ f(x,0)=0,$$

then there exists the unique solution U of problem (4.2.1)–(4.2.3) satisfying identity (4.2.4) and having the following properties:

$$U\in L_{pq+q}(0,T;\overset{\circ}{W}{}^1_{pq+q}(\Omega)),\ \ \frac{\partial U}{\partial t}\in L_2(Q),$$
$$\sqrt{\psi}\,\frac{\partial}{\partial x_j}\left(\left|\frac{\partial U}{\partial x_i}\right|^{\frac{q-2}{2}}\frac{\partial U}{\partial x_i}\right)\in L_2(Q),\quad \sqrt{T-t}\,\frac{\partial}{\partial t}\left(\left|\frac{\partial U}{\partial x_i}\right|^{\frac{q-2}{2}}\frac{\partial U}{\partial x_i}\right)\in L_2(Q),\ \ i,j=1,\dots,n,$$

where $\psi\in C^\infty(\Omega)$, $\psi(x)>0$, $x\in\Omega$; $\psi=\partial\psi/\partial\nu=0$, $x\in\partial\Omega$, ν is the outer normal of $\partial\Omega$.

Proof of Theorem 4.3.1 is divided into several steps applying Galerkin's method and the method of compactness [54, 71]. One of the basic steps is getting the necessary a priori estimations.

Remark 4.3.1. Since in equation (4.2.1) we have nonlinear terms of the form

$$D_i\left[a\left(\int\limits_0^t|\nabla U|^q\,d\tau\right)|\nabla U|^{q-2}D_iU\right],\ \ i=1,\dots,n,$$

for the application of compactness method it is necessary to obtain a priori estimations for the second derivatives.

Approximate solution of problem (4.2.1)–(4.2.3) should be sought in the form

$$U_m(x,t)=\sum_{k=1}^m c_{mk}w_k(x,t),$$

where unknown coefficients c_{mk} are defined from the system of nonlinear algebraic equations of the form

$$\iint\limits_Q \Big(\frac{\partial U_m}{\partial t} + AU_m\Big) Bw_j \, dx\, dt = \iint\limits_Q fBw_j \, dx\, dt, \quad j = 1, \dots, m. \tag{4.3.1}$$

Here, $\{Bw_j\}$ form the complete system in the space $L_{pq+q}(0, T; \overset{\circ}{W}{}^1_{pq+q}(\Omega))$ [54, 71], and

$$Bw = -\frac{\partial}{\partial t}\Big[(T-t)\,\frac{\partial w}{\partial t}\Big] - \psi \Delta w + \lambda w, \quad \lambda = const > 0.$$

It is necessary to establish the solvability of system (4.3.1), i.e., the existence of Galerkin's approximations U_m. To this end, we prove the following statement beforehand.

Lemma 4.3.1. Galerkin's approximations U_m of problem (4.2.1)–(4.2.3) satisfy the inequality

$$\begin{aligned}
\iint\limits_Q \Big(\frac{\partial U_m}{\partial t} + AU_m\Big) BU_m \, dx\, dt &\geq C \iint\limits_Q \Big|\frac{\partial U_m}{\partial t}\Big|^2 \, dx\, dt + C \iint\limits_Q a(S_m)|\nabla U_m|^q \, dx\, dt \\
&+ C \iint\limits_Q a'(S_m)|\nabla U_m|^{2q} \, dx\, dt - C \iint\limits_Q (T-t) a''(S_m)|\nabla U_m|^{3q} \, dx\, dt \\
&+ C \sum_{i=1}^n \iint\limits_Q (T-t) a(S_m) \Big[\frac{\partial}{\partial t}\Big(\Big|\frac{\partial U_m}{\partial x_i}\Big|^{\frac{q-2}{2}} \frac{\partial U_m}{\partial x_i}\Big)\Big]^2 \, dx\, dt \\
&+ C \sum_{i,j=1}^n \iint\limits_Q \psi a(S_m \Big[\frac{\partial}{\partial x_j}\Big(\Big|\frac{\partial U_m}{\partial x_i}\Big|^{\frac{q-2}{2}} \frac{\partial U_m}{\partial x_i}\Big)\Big]^2 \, dx\, dt,
\end{aligned} \tag{4.3.2}$$

where

$$S_m(x,t) = \int\limits_0^t |\nabla U_m|^q \, d\tau.$$

Proof. To obtain relation (4.3.2), we have to estimate the following expressions:

$$J_1 = \iint\limits_Q \frac{\partial U_m}{\partial t} BU_m \, dx\, dt,$$

$$\begin{aligned}
J_2 = \iint\limits_Q AU_m BU_m \, dx\, dt &= \lambda \iint\limits_Q U_m AU_m \, dx\, dt \\
&- \iint\limits_Q \frac{\partial}{\partial t}\Big[(T-t)\,\frac{\partial U_m}{\partial t}\Big] AU_m \, dx\, dt - \iint\limits_Q \psi \Delta U_m AU_m \, dx\, dt = J_2^{(1)} + J_2^{(2)} + J_2^{(3)}.
\end{aligned}$$

Estimating these quantities, we have

$$\begin{aligned}
J_1 &= \iint\limits_Q \Big|\frac{\partial U_m}{\partial t}\Big|^2 \, dx\, dt - \int\limits_0^T \frac{T-t}{2}\,\frac{d}{dt} \int\limits_\Omega \Big|\frac{\partial U_m}{\partial t}\Big|^2 \, dx\, dt + \sum_{i=1}^n \iint\limits_Q \psi\, \frac{\partial}{\partial t}\,(D_i U_m) D_i U_m \, dx\, dt \\
&+ \sum_{i=1}^n \iint\limits_Q D_i \psi\, \frac{\partial U_m}{\partial t}\, D_i U_m \, dx\, dt + \frac{\lambda}{2} \iint\limits_Q \frac{\partial U_m^2}{\partial t} \, dx\, dt \\
&\geq \frac{1}{4} \iint\limits_Q \Big|\frac{\partial U_m}{\partial t}\Big|^2 \, dx\, dt - C \sum_{i=1}^n \iint\limits_Q |D_i U_m|^2 \, dx\, dt,
\end{aligned} \tag{4.3.3}$$

$$J_2^{(1)} = \lambda \sum_{i=1}^n \iint\limits_Q a(S_m)|\nabla U_m|^{q-2}|D_i U_m|^2 \, dx\, dt = \lambda \iint\limits_Q a(S_m)|\nabla U_m|^q \, dx\, dt. \tag{4.3.4}$$

Note that C in (4.3.3) denotes, generally speaking, various positive constants independent of λ, m and f. Estimating the terms $J_2^{(2)}$ and $J_2^{(3)}$, we have

$$\begin{aligned} J_2^{(2)} &= \sum_{i=1}^{n} \iint_Q D_i\big[a(S_m)|\nabla U_m|^{q-2} D_i U_m\big] \frac{\partial}{\partial t}\Big[(T-t)\frac{\partial U_m}{\partial t}\Big]\,dx\,dt \\ &= \sum_{i=1}^{n} \iint_Q (T-t)\,\frac{\partial}{\partial t}\big[a(S_m)|\nabla U_m t|^{q-2} D_i U_m\big]\,\frac{\partial}{\partial t}\,(D_i U_m)\,dx\,dt \\ &= \sum_{i=1}^{n} \iint_Q (T-t) a(S_m)|\nabla U_m|^{q-2}\Big|\frac{\partial}{\partial t}\,(D_i U_m)\Big|^2\,dx\,dt \\ &\quad + \sum_{i=1}^{n} \iint_Q (T-t) a'(S_m)|\nabla U_m|^{q}|\nabla U_m|^{q-2} D_i U_m\,\frac{\partial}{\partial t}(D_i U_m)\,dx\,dt \\ &\quad + \sum_{i=1}^{n} \iint_Q (T-t) a(S_m)\,\frac{\partial}{\partial t}\big(|\nabla U_m|^{q-2}\big) D_i U_m\,\frac{\partial}{\partial t}\,(D_i U_m)\,dx\,dt. \end{aligned}$$

Taking into account the identity

$$\begin{aligned} &\sum_{i=1}^{n} \iint_Q (T-t) a'(S_m)|\nabla U_m|^{2q-2} D_i U_m\,\frac{\partial}{\partial t}\,(D_i U_m)\,dx\,dt \\ &\qquad = \frac{1}{2q} \iint_Q (T-t) a'(S_m)|\nabla U_m|^{2q-2}\,\frac{\partial}{\partial t}\,(D_i U_m)^2\,dx\,dt \\ &\qquad = \frac{1}{2q} \iint_Q (T-t) a'(S_m)\,\frac{\partial}{\partial t}\,|\nabla U_m|^{2q}\,dx\,dt \\ &\qquad = \frac{1}{2q} \iint_Q a'(S_m)|\nabla U_m|^{2q}\,dx\,dt - \frac{1}{2q} \iint_Q (T-t) a''(S_m)|\nabla U_m|^{q}|\nabla U_m|^{2q}\,dx\,dt \\ &\qquad = \frac{1}{2q} \iint_Q a'(S_m)|\nabla U_m|^{2q}\,dx\,dt - \frac{1}{2q} \iint_Q (T-t) a''(S_m)|\nabla U_m|^{3q}\,dx\,dt, \end{aligned}$$

we get

$$\begin{aligned} &\sum_{i=1}^{n} \iint_Q (T-t) a'(S_m)|\nabla U_m|^{2q-2}\,\frac{\partial}{\partial t}\,|D_i U_m|^2\,dx\,dt \\ &\qquad = \frac{1}{q} \iint_Q a'(S_m)|\nabla U_m|^{2q}\,dx\,dt - \frac{1}{q} \iint_Q (T-t) a''(S_m)|\nabla U_m|^{3q}\,dx\,dt. \end{aligned} \tag{4.3.5}$$

Let us estimate now the last third term of $J_2^{(2)}$,

$$\begin{aligned} &\sum_{i=1}^{n} \iint_Q (T-t) a(S_m)\,\frac{\partial}{\partial t}\,(|\nabla U_m|^{2q-2}) D_i U_m\,\frac{\partial}{\partial t}(D_i U_m)\,dx\,dt \\ &\qquad = (q-2) \sum_{i,k=1}^{n} \iint_Q (T-t) a(S_m)|\nabla U_m|^{q-4} D_k U_m\,\frac{\partial}{\partial t}\,(D_k U_m) D_i U_m\,\frac{\partial}{\partial t}(D_i U_m)\,dx\,dt \\ &\qquad = (q-2) \sum_{i=1}^{n} \iint_Q (T-t) a(S_m)|\nabla U_m|^{q-4}\Big[\sum_{i=1}^{n} D_i U_m\,\frac{\partial}{\partial t}\,(D_i U_m)\Big]^2\,dx\,dt. \end{aligned}$$

Here we have applied easily verified identity

$$\frac{\partial}{\partial t}\left(|\nabla U_m|^{q-2}\right) = (q-2)|\nabla U_m|^{q-4} \sum_{k=1}^{n} D_k U_m \, \frac{\partial}{\partial t}\, (D_k U_m).$$

Using (4.3.5), for $J_2^{(2)}$, we have

$$\begin{aligned} J_2^{(2)} &= \sum_{i=1}^{n} \iint\limits_Q (T-t)a(S_m)|\nabla U_m|^{q-2} \left|\frac{\partial}{\partial t}\,(D_i U_m)\right|^2 dx\,dt \\ &\quad + \frac{1}{2q} \iint\limits_Q a'(S_m)|\nabla U_m|^{2q}\, dx\,dt - \frac{1}{2q}\iint\limits_Q (T-t)a''(S_m)|\nabla U_m|^{3q}\, dx\,dt \\ &\quad + (q-2)\iint\limits_Q (T-t)a(S_m)|\nabla U_m|^{q-4}\Big[\sum_{i=1}^{n} D_i U_m \,\frac{\partial}{\partial t}\,(D_i U_m)\Big]^2 dx\,dt. \end{aligned}$$

Taking into account the identity

$$|D_i U_m|^{\frac{q-2}{2}}\, \frac{\partial}{\partial t}\,(D_i U_m) = \frac{2}{q}\,\frac{\partial}{\partial t}\left(|D_i U_m|^{\frac{q-2}{2}} D_i U_m\right),$$

the first term of $J_2^{(2)}$ can be estimated as follows

$$\begin{aligned} &\sum_{i=1}^{n}\iint\limits_Q (T-t)a(S_m)|\nabla U_m|^{q-2}\Big[\frac{\partial}{\partial t}(D_i U_m)\Big]^2 dx\,dt \\ &\qquad = \sum_{i=1}^{n}\iint\limits_Q (T-t)a(S_m)\Big[\sum_{k=1}^{n}|D_k U_m|^2\Big]^{\frac{q-2}{2}}\left|\frac{\partial}{\partial t}(D_i U_m)\right|^2 dx\,dt \\ &\qquad \ge \sum_{i=1}^{n}\iint\limits_Q (T-t)a(S_m)|D_i U_m|^{q-2}\left|\frac{\partial}{\partial t}(D_i U_m)\right|^2 dx\,dt \\ &\qquad = \frac{4}{q^2}\sum_{i=1}^{n}\iint\limits_Q (T-t)a(S_m)\,\frac{\partial}{\partial t}\,\big[|D_i U_m|^{\frac{q-2}{2}} D_i U_m\big]^2 dx\,dt. \end{aligned}$$

Thus, for $J_2^{(2)}$, we finally get the following estimation

$$\begin{aligned} J_2^{(2)} &\ge \frac{4}{q^2}\sum_{i=1}^{n}\iint\limits_Q (T-t)a(S_m)\Big[\frac{\partial}{\partial t}\,\big(|D_i U_m|^{\frac{q-2}{2}} D_i U_m\big)\Big]^2 dx\,dt \\ &\quad + \frac{1}{2q} \iint\limits_Q a'(S_m)|\nabla U_m|^{2q}\, dx\,dt - \frac{1}{2q}\iint\limits_Q (T-t)a''(S_m)|\nabla U_m|^{3q}\, dx\,dt \\ &\quad + (q-2)\iint\limits_Q (T-t)a(S_m)|\nabla U_m|^{q-4}\Big[\sum_{i=1}^{n} D_i U_m \,\frac{\partial}{\partial t}\,(D_i U_m)\Big]^2 dx\,dt. \end{aligned} \tag{4.3.6}$$

Let us now estimate the term $J_2^{(3)}$. Towards this end, we rewrite it in the following form

$$\begin{aligned} J_2^{(3)} &= -\iint\limits_Q \psi A(U_m)\Delta U_m\, dx\,dt = -\sum_{j=1}^{n}\iint\limits_Q \psi A(U_m) D_j(D_j U_m)\, dx\,dt \\ &= -\sum_{j=1}^{n}\iint\limits_Q A(U_m)(D_j(\psi D_j U_m) - D_j\psi D_j U_m)\, dx\,dt \end{aligned}$$

$$= \sum_{i,j=1}^{n} \iint_{Q} D_i\big[a(S_m)|\nabla U_m|^{q-2} D_i U_m\big] D_i(\psi D_j U_m)\,dx\,dt + \sum_{j=1}^{n} \iint_{Q} A(U_m) D_j\psi D_j U_m\,dx\,dt$$

$$= \sum_{i,j=1}^{n} \iint_{Q} \psi D_j\big[a(S_m)|\nabla U_m|^{q-2} D_i U_m\big] D_i(D_j U_m)\,dx\,dt$$

$$+ \sum_{i,j=1}^{n} \iint_{Q} D_i\psi D_j\big[a(S_m)|\nabla U_m|^{q-2} D_i U_m\big] D_j U_m\,dx\,dt$$

$$- \sum_{i,j=1}^{n} \iint_{Q} D_j\psi D_i\big[a(S_m)|\nabla U_m|^{q-2} D_i U_m\big] D_j U_m\,dx\,dt.$$

Taking into account the easily verified identity

$$D_j\big(|\nabla U_m|^{q-2} D_i U_m\big) = (q-2)|\nabla U_m|^{q-4} \sum_{k=1}^{n} D_k U_m D_j(D_k U_m) D_i U_m + |\nabla U_m|^{q-2} D_j(D_i U_m),$$

we get

$$J_2^{(3)} = \sum_{i,j=1}^{n} \iint_{Q} \psi a(S_m)|\nabla U_m|^{q-2} |D_i(D_j U_m)|^2\,dx\,dt$$

$$+ q \sum_{i,j,k=1}^{n} \iint_{Q} \psi a'(S_m) \int_0^t |\nabla U_m|^{q-2} D_k U_m D_j(D_k U_m)\,d\tau\, |\nabla U_m|^{q-2} D_i U_m D_i(D_j U_m)\,dx\,dt$$

$$+ (q-2) \sum_{i,j,k=1}^{n} \iint_{Q} \psi a(S_m)|\nabla U_m|^{q-4} D_k U_m D_j(D_k U_m) D_i U_m D_i(D_j U_m)\,dx\,dt$$

$$+ \sum_{i,j=1}^{n} \iint_{Q} \psi a(S_m)|\nabla U_m|^{q-2} |D_i(D_j U_m)|^2\,dx\,dt$$

$$+ \sum_{i,j=1}^{n} \iint_{Q} D_i\psi a(S_m)|\nabla U_m|^{q-2} D_j(D_i U_m) D_j U_m\,dx\,dt$$

$$+ q \sum_{i,j,k=1}^{n} \iint_{Q} D_i\psi a'(S_m) \int_0^t |\nabla U_m|^{q-2} D_k U_m D_k(D_j U_m)\,d\tau\, |\nabla U_m|^{q-2} D_i U_m D_j U_m\,dx\,dt$$

$$+ (q-2) \sum_{i,j,k=1}^{n} \iint_{Q} D_i\psi a(S_m)|\nabla U_m|^{q-4} D_k U_m D_j(D_k U_m) D_i U_m D_j U_m\,dx\,dt$$

$$+ \sum_{i,j=1}^{n} \iint_{Q} D_i\psi a(S_m)|\nabla U_m|^{q-2} D_j(D_i U_m) D_j U_m\,dx\,dt$$

$$- \sum_{i,j=1}^{n} \iint_{Q} D_j\psi a(S_m)|\nabla U_m|^{q-2} D_i(D_i U_m) D_j U_m\,dx\,dt$$

$$- q \sum_{i,j,k=1}^{n} \iint_{Q} D_i\psi a'(S_m) \int_0^t |\nabla U_m|^{q-2} D_k U_m D_i(D_k U_m)\,d\tau\, |\nabla U_m|^{q-2} D_i U_m D_j U_m\,dx\,dt$$

$$- (q-2) \sum_{i,j,k=1}^{n} \iint_{Q} D_j\psi a(S_m)|\nabla U_m|^{q-4} D_k U_m D_i(D_k U_m) D_i U_m D_j U_m\,dx\,dt$$

$$-\sum_{i,j=1}^{n}\iint\limits_{Q} D_j\psi a(S_m)|\nabla U_m|^{q-2}D_i(D_iU_m)D_jU_m\,dx\,dt=\sum_{l=1}^{12}I_l.$$

Let us now estimate I_l, $l=1,2,\dots,12$. We have

$$I_1+I_4=2\sum_{i,j=1}^{n}\iint\limits_{Q}\psi a(S_m)|\nabla U_m|^{q-2}|D_i(D_jU_m)|^2\,dx\,dt. \tag{4.3.7}$$

Introducing the notation

$$\varphi_{kj}(x,t)=\int_0^t|\nabla U_m|^{q-2}D_kU_mD_j(D_kU_m)\,d\tau,$$

for I_2, we obtain

$$\begin{aligned}
I_2&=q\sum_{i,j,k=1}^{n}\iint\limits_{Q}\psi a'(S_m)\int_0^t|\nabla U_m|^{q-2}D_kU_mD_j(D_kU_m)\,d\tau\,|\nabla U_m|^{q-2}D_iU_mD_i(D_jU_m)\,dx\,dt\\
&=q\sum_{i,j,k=1}^{n}\iint\limits_{Q}\psi a'(S_m)\varphi_{kj}\frac{\partial\varphi_{ij}}{\partial t}\,dx\,dt=\frac{q}{2}\sum_{i,j,k=1}^{n}\iint\limits_{Q}\psi a'(S_m)\frac{\partial}{\partial t}(\varphi_{kj}\varphi_{ij})\,dx\,dt\\
&=\frac{q}{2}\sum_{i,j,k=1}^{n}\iint\limits_{\Omega}\psi a'(S_m(x,T))\varphi_{kj}(x,T)\varphi_{ij}(x,T)\,dx\\
&\quad-\sum_{i,j,k=1}^{n}\iint\limits_{Q}\psi a''(S_m)|\nabla U_m|^{q}\varphi_{kj}\varphi_{ij}\,dx\,dt.
\end{aligned} \tag{4.3.8}$$

As $a'(S)=p(1+S)^{p-1}\ge 0$, $a''(S)=p(p-1)(1+S)^{p-2}\le 0$, one deduces that $I_2\ge 0$.
Introducing the notation

$$X_{kj}=D_kU_mD_j(D_kU_m),$$

for I_3, we get

$$\begin{aligned}
I_3&=(q-2)\sum_{i,j,k=1}^{n}\iint\limits_{Q}\psi a(S_m)|\nabla U_m|^{q-4}D_kU_mD_j(D_kU_m)D_iU_mD_i(D_jU_m)\,dx\,dt\\
&=(q-2)\sum_{i,j,k=1}^{n}\iint\limits_{Q}\psi a(S_m)|\nabla U_m|^{q-4}X_{kj}X_{ij}\,dx\,dt\ge 0.
\end{aligned} \tag{4.3.9}$$

From the expression for I_6, I_7, I_{10} and I_{11}, it is clear that

$$\begin{aligned}
I_6+I_{10}&=q\sum_{i,j,k=1}^{n}\iint\limits_{Q}D_i\psi a'(S_m)\int_0^t|\nabla U_m|^{q-2}D_kU_mD_j(D_kU_m)\,d\tau\,|\nabla U_m|D_iU_mD_jU_m\,dx\,dt\\
&-q\sum_{i,j,k=1}^{n}\iint\limits_{Q}D_j\psi a'(S_m)\int_0^t|\nabla U_m|^{q-2}D_kU_mD_i(D_kU_m)\,d\tau\,|\nabla U_m|D_iU_mD_jU_m\,dx\,dt=0,
\end{aligned} \tag{4.3.10}$$

$$\begin{aligned}
I_7+I_{11}&=(q-2)\sum_{i,j,k=1}^{n}\iint\limits_{Q}D_i\psi a(S_m)|\nabla U_m|^{q-4}D_kU_mD_j(D_kU_m)D_iU_mD_jU_m\,dx\,dt\\
&-(q-2)\sum_{i,j,k=1}^{n}\iint\limits_{Q}D_j\psi a(S_m)|\nabla U_m|^{q-4}D_kU_mD_i(D_kU_m)D_iU_mD_jU_m\,dx\,dt=0.
\end{aligned} \tag{4.3.11}$$

Consider now the rest terms of the quantity $J_2^{(3)}$. We have

$$\begin{aligned} I_5+I_8+I_9+I_{12} &= 2\sum_{i,j=1}^{n}\iint_Q D_i\psi a(S_m)|\nabla U_m|^{q-2}D_j(D_iU_m)D_jU_m\,dx\,dt \\ &\quad - 2\sum_{i,j=1}^{n}\iint_Q D_j\psi a(S_m)|\nabla U_m|^{q-2}D_i(D_iU_m)D_jU_m\,dx\,dt \\ &= 2\sum_{i,j=1}^{n}\iint_Q \frac{D_i\psi}{\sqrt{\psi}}\sqrt{\psi}\,a(S_m)|\nabla U_m|^{q-2}D_j(D_iU_m)D_jU_m\,dx\,dt \\ &\quad - 2\sum_{i,j=1}^{n}\iint_Q \frac{D_i\psi}{\sqrt{\psi}}\sqrt{\psi}\,a(S_m)|\nabla U_m|^{q-2}D_i(D_iU_m)D_jU_m\,dx\,dt. \end{aligned}$$

Using the ε-inequality

$$|bc|\le\frac{\varepsilon^2}{2}\,b^2+\frac{1}{2\varepsilon^2}\,c^2,$$

we finally arrive at

$$\begin{aligned} I_5+I_8+I_9+I_{12}\ge\ &-\frac{\varepsilon^2}{2}\sum_{i,j=1}^{n}\iint_Q \psi a(S_m)|\nabla U_m|^{q-2}|D_j(D_iU_m)|^2\,dx\,dt \\ &-\frac{C}{2\varepsilon^2}\sum_{j=1}^{n}\iint_Q a(S_m)|\nabla U_m|^{q-2}|D_jU_m|^2\,dx\,dt \\ &-\frac{\varepsilon^2}{2}\sum_{i,j=1}^{n}\iint_Q \psi a(S_m)|\nabla U_m|^{q-2}|D_i(D_iU_m)|^2\,dx\,dt \\ &-\frac{C}{2\varepsilon^2}\sum_{j=1}^{n}\iint_Q a(S_m)|\nabla U_m|^{q-2}|D_jU_m|^2\,dx\,dt \\ &-\varepsilon^2\sum_{i,j=1}^{n}\iint_Q \psi a(S_m)|\nabla U_m|^{q-2}|D_j(D_iU_m)|^2\,dx\,dt \\ &-\frac{C}{\varepsilon^2}\iint_Q a(S_m)|\nabla U_m|^{q}\,dx\,dt. \end{aligned} \tag{4.3.12}$$

Collecting (4.3.4), (4.3.7) and (4.3.12), we have

$$\begin{aligned} I_1+I_4+I_5+I_8+I_9+I_{12}+J_2^{(1)} &\ge 2\sum_{i,j=1}^{n}\iint_Q \psi a(S_m)|\nabla U_m|^{q-2}|D_i(D_jU_m)|^2\,dx\,dt \\ &\quad -\varepsilon^2\sum_{i,j=1}^{n}\iint_Q \psi a(S_m)|\nabla U_m|^{q-2}|D_j(D_iU_m)|^2\,dx\,dt \\ &\quad -\frac{C}{\varepsilon^2}\iint_Q a(S_m)|\nabla U_m|^{q}\,dx\,dt+\lambda\iint_Q a(S_m)|\nabla U_m|^{q}\,dx\,dt \\ &\ge (2-\varepsilon^2)\sum_{i,j=1}^{n}\iint_Q \psi a(S_m)|\nabla U_m|^{q-2}|D_i(D_jU_m)t|^2\,dx\,dt \\ &\quad +\Big(\lambda-\frac{C}{\varepsilon^2}\Big)\iint_Q a(S_m)|\nabla U_m|^{q}\,dx\,dt. \end{aligned}$$

Choosing now ε sufficiently small, and λ sufficiently large such that

$$2-\varepsilon^2=C>0,\ \ \lambda-\frac{C}{\varepsilon^2}=C(\lambda)>0,$$

taking into account the identity

$$|D_iU_m|^{\frac{q-2}{2}}D_j(D_iU_m)=\frac{2}{q}\,D_j\big(|D_iU_m|^{\frac{q-2}{2}}D_iU_m\big),$$

we obtain

$$\begin{aligned}
&I_1+I_4+I_5+I_8+I_9+I_{12}+J_2^{(1)}\\
&\quad\geq C\sum_{i,j=1}^{n}\iint_Q \psi a(S_m)\Big(\sum_{k=1}^{n}|D_kU_m|^2\Big)^{\frac{q-2}{2}}|D_i(D_jU_m)|^2\,dx\,dt+C(\lambda)\iint_Q a(S_m)|\nabla U_m|^q\,dx\,dt\\
&\quad\geq C\sum_{i,j=1}^{n}\iint_Q \psi a(S_m)|D_iU_m|^{q-2}|D_i(D_jU_m)|^2\,dx\,dt+C(\lambda)\iint_Q a(S_m)|\nabla U_m|^q\,dx\,dt\\
&\quad=\frac{4C}{q^2}\sum_{i,j=1}^{n}\iint_Q \psi a(S_m)\big[D_j(|D_iU_m|^{\frac{q-2}{2}}D_iU_m)\big]^2\,dx\,dt+C(\lambda)\iint_Q a(S_m)|\nabla U_m|^q\,dx\,dt. \qquad (4.3.13)
\end{aligned}$$

Note that in (4.3.13), $C(\lambda)\to\infty$ as $\lambda\to\infty$.

Taking into account relations (4.3.3), (4.3.6), (4.3.8)–(4.3.11), (4.3.13), we finally arrive at

$$\begin{aligned}
&\iint_Q\Big(\frac{\partial U_m}{\partial t}+A(U_m)\Big)BU_m\,dx\,dt\\
&\qquad\geq C\iint_Q\Big|\frac{\partial U_m}{\partial t}\Big|^2\,dx\,dt+C(\lambda)\iint_Q a(S_m)|\nabla U_m|^q\,dx\,dt\\
&\qquad\quad+C\iint_Q a'(S_m)|\nabla U_m|^{2q}\,dx\,dt-C\iint_Q (T-t)a''(S_m)|\nabla U_m|^{3q}\,dx\,dt\\
&\qquad\quad+C\sum_{i,j=1}^{n}\iint_Q (T-t)a(S_m)\Big[\frac{\partial}{\partial t}\big(|D_iU_m|^{\frac{q-2}{2}}D_iU_m\big)\Big]^2\,dx\,dt\\
&\qquad\quad+(q-2)\iint_Q (T-t)a(S_m)|\nabla U_m|^{q-4}\Big(\sum_{k=1}^{n}D_iU_m\,\frac{\partial}{\partial t}(D_iU_m)\Big)^2\,dx\,dt\\
&\qquad\quad+\frac{q}{2}\sum_{i,j,k=1}^{n}\iint_\Omega \psi a'(S_m(x,T))\varphi_{kj}(x,T)\varphi_{ij}(x,T)\,dx\\
&\qquad\quad-\sum_{i,j,k=1}^{n}\iint_Q \psi a''(S_m)|\nabla U_m|^q\varphi_{kj}\varphi_{ij}\,dx\,dt\\
&\qquad\quad+(q-2)\sum_{i,j,k=1}^{n}\iint_Q \psi a(S_m)|\nabla U_m|^{q-4}X_{kj}X_{ij}\,dx\,dt\\
&\qquad\quad+C\sum_{i,j,k=1}^{n}\iint_Q \psi a(S_m)\big[D_j(|D_iU_m|^{\frac{q-2}{2}}D_iU_m)\big]^2\,dx\,dt.
\end{aligned}$$

From this estimation follows the validity of Lemma 4.3.1. □

We also have [54, 71]

$$\Big|\sum_{j=1}^{n} c_{mj}\iint\limits_{Q} fB\omega_j\,dx\,dt\Big| = \Big|\iint\limits_{Q} fBU_m\,dx\,dt\Big| \le C\Big[\iint\limits_{Q}\Big(\Big|\frac{\partial U_m}{\partial t}\Big|^2 + |\nabla \overline{U}_m|^2\Big)\,dx\,dt\Big]^{\frac{1}{2}}. \tag{4.3.14}$$

Next, choosing

$$\iint\limits_{Q}\Big[\Big|\frac{\partial U_m}{\partial t}\Big|^2 + |\nabla U_m|^q\Big]\,dx\,dt$$

sufficiently large, due to (4.3.2) and (4.3.14) it is not difficult to establish the validity of the inequality

$$\iint\limits_{Q}\Big(\frac{\partial U_m}{\partial t} + A(U_m) - f\Big)BU_m\,dx\,dt \ge 0. \tag{4.3.15}$$

From estimation (4.3.15), taking into account the Brouwer fixed-point lemma (see, e.g., [54], or Lemma 4.1.1) follows the solvability of system (4.3.1) with regards to the coefficients c_{mk}.

On the basis of Lemma 4.3.1, we get

$$\begin{aligned}\iint\limits_{Q}\Big|\frac{\partial U_m}{\partial t}\Big|^2\,dx\,dt &+ \iint\limits_{Q} a(S_m)|\nabla U_m|^q\,dx\,dt + \iint\limits_{Q} a'(S_m)|\nabla U_m|^{2q}\,dx\,dt\\ &+ \sum_{i=1}^{n}\iint\limits_{Q}(T-t)a(S_m)\Big[\frac{\partial}{\partial t}\Big(t\Big|\frac{\partial U_m}{\partial x_i}\Big|^{\frac{q-2}{2}}\frac{\partial U_m}{\partial x_i}\Big)\Big]^2\,dx\,dt\\ &+ \sum_{i,j=1}^{n}\iint\limits_{Q}\psi a(S_m)\Big[\frac{\partial}{\partial x_j}\Big(\Big|\frac{\partial U_m}{\partial x_i}\Big|^{\frac{q-2}{2}}\frac{\partial U_m}{\partial x_i}\Big)\Big]^2\,dx\,dt \le C.\end{aligned} \tag{4.3.16}$$

From a priori estimations (4.3.16), we can conclude that

$$U_m \text{ are bounded in } L_q(0,T;\overset{\circ}{W}{}^1_q(\Omega)), \tag{4.3.17}$$

$$\frac{\partial U_m}{\partial t} \text{ are bounded in } L_2(Q), \tag{4.3.18}$$

$$\sqrt{T-t}\,\frac{\partial}{\partial t}\big(|D_iU_m|^{\frac{q-2}{2}}D_iU_m\big) \text{ are bounded in } L_2(Q), \tag{4.3.19}$$

$$\sqrt{\psi}\,D_j\big(|D_iU_m|^{\frac{q-2}{2}}D_iU_m\big) \text{ are bounded in } L_2(Q),\quad i,j=1,\dots,n. \tag{4.3.20}$$

Besides, we have the estimation

$$\iint\limits_{Q}\Big[1+\int\limits_0^t|\nabla U_m|^q\,d\tau\Big]^{p-1}|\nabla u_m|^{2q}\,dx\,dt \le C. \tag{4.3.21}$$

Let us obtain some additional a priori estimations for limiting transition in identity (4.3.1) as $m\to\infty$.

Using (4.3.21), we get the following statement.

Lemma 4.3.2. The sequence of approximate solutions $\{U_m\}$ is bounded in the space $L_{pq+q}(0,T;\overset{\circ}{W}{}^1_{pq+q}(\Omega))$, i.e.,

$$\iint\limits_{Q}|\nabla U_m|^{pq+q}\,dx\,dt \le C. \tag{4.3.22}$$

Proof. Applying the easily verifiable inequalities

$$\int_0^t |\nabla U_m|^q \, d\tau \le C\Big[\int_0^t |\nabla U_m|^{2q} \, d\tau\Big]^{\frac{1}{2}}, \quad 1+\int_0^t |\nabla U_m|^q \, d\tau \le C\Big[1+\int_0^t |\nabla U_m|^{2q} \, d\tau\Big]^{\frac{1}{2}},$$

$$\Big[1+\int_0^t |\nabla U_m|^q \, d\tau\Big]^{p-1} \ge C\Big[1+\int_0^t |\nabla U_m|^{2q} \, d\tau\Big]^{\frac{p-1}{2}},$$

from (4.3.21), we obtain

$$\iint_Q \Big[1+\int_0^t |\nabla U_m|^{2q} \, d\tau\Big]^{\frac{p-1}{2}} |\nabla U_m|^{2q} \, dx\, dt \le C. \tag{4.3.23}$$

From (4.3.23), it is not difficult to conclude that

$$\int_\Omega \Big[1+\int_0^T |\nabla U_m|^{2q} \, d\tau\Big]^{\frac{p-1}{2}} \, dx \le C. \tag{4.3.24}$$

Indeed, introducing the notation

$$g_m(x,t) = 1+\int_0^t |\nabla U_m|^{2q} \, d\tau,$$

from (4.3.23), we have

$$\iint_Q \Big[1+\int_0^t |\nabla U_m|^{2q} \, d\tau\Big]^{\frac{p-1}{2}} |\nabla U_m|^{2q} \, dx\, dt = \iint_Q g_m^{\frac{p-1}{2}}(x,t)\, \frac{\partial g_m}{\partial t} \, dx\, dt$$

$$= \frac{2}{p+1}\iint_Q \frac{\partial g_m^{\frac{p+1}{2}}}{\partial t} \, dx\, dt = \frac{2}{p+1}\int_\Omega g_m^{\frac{p+1}{2}}(x,T)\, dx - \frac{2}{p+1}\int_\Omega g_m^{\frac{p+1}{2}}(x,0)\, dx$$

$$= \frac{2}{p+1}\int_\Omega \Big[1+\int_0^T |\nabla U_m|^{2q} \, dt\Big]^{\frac{p+1}{2}} dx - \frac{2}{p+1}|\Omega| \le C,$$

which evidently results in a priori estimation (4.3.24).

As

$$\iint_Q |\nabla U_m|^{pq+q} \, dx\, dt = \int_\Omega \Big[\int_0^T |\nabla U_m|^{pq+q} \, dt\Big] dx$$

$$\le C\int_\Omega \Big[\int_0^T |\nabla U_m|^{(pq+q)\frac{2}{p+1}} \, dt\Big]^{\frac{p+1}{2}} dx \le C\int_\Omega \Big[\int_0^T |\nabla U_m|^{2q} \, dt\Big]^{\frac{p+1}{2}} dx \le C,$$

we get the validity of inequality (4.3.22).

To ensure the possibility of limiting transition in nonlinear terms

$$a(S_m)|\nabla U_m|^{q-2} D_i U_m, \quad i=1,\dots,n,$$

let us prove the following statement. □

Lemma 4.3.3. The set $\{a(S_m)|\nabla U_m|^{q-2}D_iU_m\}$ is bounded in the space $L^*_{pq+q}(Q)=L_{\frac{pq+q}{pq+q-1}}(Q)$, i.e.,

$$\sum_{i=1}^{n}\iint\limits_{Q}\left|a(S_m)|\nabla U_m|^{q-2}D_iU_m\right|^{\frac{pq+q}{pq+q-1}}\,dx\,dt\le C. \tag{4.3.25}$$

Proof. We have

$$\begin{aligned}
&\iint\limits_{Q}\left|a(S_m)|\nabla U_m|^{q-2}D_iU_m\right|^{\frac{pq+q}{pq+q-1}}\,dx\,dt\\
&\quad=\iint\limits_{Q}\left[1+\int\limits_0^t|\nabla U_m|^q\,d\tau\right]^{p\,\frac{pq+q}{pq+q-1}}|\nabla U_m|^{(q-2)\,\frac{pq+q}{pq+q-1}}|D_iU_m|^{\frac{pq+q}{pq+q-1}}\,dx\,dt\\
&\quad\le\iint\limits_{Q}\left[1+\int\limits_0^t|\nabla U_m|^{2q}\,d\tau\right]^{p\,\frac{pq+q}{2(pq+q-1)}}|\nabla U_m|^{(q-2)\,\frac{pq+q}{pq+q-1}}\left(|D_iU_m|^2\right)^{\frac{pq+q}{pq+q-1}}\,dx\,dt\\
&\quad\le C\iint\limits_{Q}\left[1+\int\limits_0^t|\nabla U_m|^{2q}\,d\tau\right]^{\frac{pq(p+1)}{2(pq+q-1)}}|\nabla U_m|^{(q-2)\,\frac{q(p+1)}{pq+q-1}}\left(|D_iU_m|^2\right)^{\frac{q(p+q)}{pq+q-1}}\,dx\,dt\\
&\quad=C\iint\limits_{Q}\left[1+\int\limits_0^t|\nabla U_m|^{2q}\,d\tau\right]^{\frac{pq(p+1)}{2(pq+q-1)}}|\nabla U_m|^{(q-1)\,\frac{q(p+1)}{pq+q-1}}\,dx\,dt\\
&\quad\le C\left\{\iint\limits_{Q}\left[1+\int\limits_0^t|\nabla U_m|^{2q}\,d\tau\right]^{\frac{pq(p+1)}{2(pq+q-1)}\,\frac{pq+q-1}{pq}}|\nabla U_m|^{(q-1)\,\frac{q(p+1)}{pq+q-1}}\,dx\,dt\right\}^{\frac{pq}{pq+q-1}}\\
&\qquad\times\left\{\iint\limits_{Q}|\nabla U_m|^{(q-1)\,\frac{q(p+1)}{pq+q-1}\,\frac{pq+q-1}{q-1}}\,dx\,dt\right\}^{\frac{q-1}{pq+q-1}}\\
&\quad=C\left\{\iint\limits_{Q}\left[1+\int\limits_0^t|\nabla U_m|^{2q}\,d\tau\right]^{\frac{p+1}{2}}\,dx\,dt\right\}^{\frac{pq}{pq+q+1}}\left\{\iint\limits_{Q}|\nabla U_m|^{pq+q}\,dx\,dt\right\}^{\frac{q-1}{pq+q-1}}\\
&\quad\le C\left\{\int\limits_{\Omega}\left[1+\int\limits_0^T|\nabla U_m|^{2q}\,d\tau\right]^{\frac{p+1}{2}}\,dx\,dt\right\}^{\frac{pq}{pq+q+1}}\left\{\iint\limits_{Q}|\nabla U_m|^{pq+q}\,dx\,dt\right\}^{\frac{q-1}{pq+q-1}}.
\end{aligned}$$

From this, taking into account (4.3.22) and (4.3.24), we get

$$\iint\limits_{Q}\left|a(S_m)|\nabla U_m|^{q-2}D_iU_m\right|^{\frac{pq+q}{pq+q-1}}\,dx\,dt\le C.$$

Thus (4.3.25) is proved. □

Now, it remains only to implement a limit transition as $m\to\infty$ in identity (4.3.1).

Proof of Theorem 4.3.1. From a priori estimations (4.3.19), (4.3.20), it follows that

$$|D_iu_m|^{\frac{q-2}{2}}D_iU_m\ \text{ are bounded in } W_2^1(Q_1),$$

where $Q_1=\Omega_1\times(0,T-\delta)$, δ is any sufficiently small positive number, and Ω_1 is an arbitrary domain such that $\overline{\Omega}_1\subset\Omega$. Since the imbedding $W_2^1(Q_1)\subset L_2(Q_1)$ is compact, from estimations (4.3.17)–(4.3.20), (4.3.25) we can choose the subsequence of the sequence $\{U_m\}$ (denote it again by $\{U_m\}$) such that

$$U_m \longrightarrow U \ \text{ weakly in } L_q(0,T;\overset{\circ}{W}{}_q^1(\Omega)), \tag{4.3.26}$$

$$\frac{\partial U_m}{\partial t} \longrightarrow \frac{\partial U}{\partial t} \ \text{ weakly in } L_2(Q), \tag{4.3.27}$$

$$|D_iU_m|^{\frac{q-2}{2}} D_iU_m \ \text{ converge almost everywhere in } Q, \tag{4.3.28}$$

$$\sqrt{T-t}\,\frac{\partial}{\partial t}\big(|D_iU_m|^{\frac{q-2}{2}} D_iU_m\big) \longrightarrow \xi_i \ \text{ weakly } L_2(Q), \tag{4.3.29}$$

$$\sqrt{\psi}\, D_j\big(|D_iU_m|^{\frac{q-2}{2}} D_iU_m\big) \longrightarrow \rho_{ij} \ \text{ weakly in } L_2(Q), \tag{4.3.30}$$

$$a(S_m)|D_iU_m|^{q-2} D_iU_m \longrightarrow \eta_i \ \text{ weakly in } L_{\frac{pq+q}{pq+q+1}}(Q). \tag{4.3.31}$$

Note that for the derivation of relation (4.3.28), we have used the diagonal process with regard for an arbitrariness of δ and Ω_1 [54, 71].

As far as the function $r \to |r|^{\frac{q-2}{2}} r$ is monotonic, from (4.3.28) it follows that

$$D_iU_m \to D_iU \ \text{ almost everywhere in } Q. \tag{4.3.32}$$

From (4.3.29), (4.3.30), we have [54, 71]

$$\xi_i = \sqrt{T-t}\,\frac{\partial}{\partial t}\big(|D_iU_m|^{\frac{q-2}{2}} D_iU_m\big), \quad \rho_{ij} = \sqrt{\psi}\, D_j\big(|D_iU_m|^{\frac{q-2}{2}} D_iU_m\big).$$

Let us now proceed to proving the main limiting relation

$$a(S_m)|\nabla U_m|^{q-2} D_iU_m \longrightarrow a(S)|\nabla U|^{q-2} DU.$$

Result (4.3.22) can be strengthened. Namely, let us show that the following statement is valid. □

Lemma 4.3.4. Galerkin's approximations $\{U_m\}$ converge strongly in the space $L_q(0,T;\overset{\circ}{W}{}_q^1(\Omega))$, i.e.,

$$D_iU_m \longrightarrow D_iU \ \text{ strongly in } L_q(Q), \ \ i = 1,\dots,n. \tag{4.3.33}$$

Proof. From estimation (4.3.22) and the Valle–Poussen Theorem 4.1.1 follows absolute equicontinuity of the integrals

$$\iint\limits_Q |\nabla U_m|^q \, dx\, dt, \ \ i = 1,\dots,n.$$

From the convergence of the sequence of measurable functions almost everywhere to some function there follows the convergence in measure of this sequence to the same function. From this, in turn, due to the Vitali Theorem 4.1.2 we have the following equality

$$\lim_{m\to\infty} \iint\limits_Q |D_iU_m|^q \, dx\, dt = \iint\limits_Q |D_iU_m|^q \, dx\, dt, \ \ i = 1,\dots,n,$$

or

$$\|D_iU_m\|_{L_q(Q)} \longrightarrow \|D_iU_m\|_{L_q(Q)}, \ \ i = 1,\dots,n. \tag{4.3.34}$$

Further, from (4.3.26), we have

$$D_iU_m \longrightarrow D_iU \ \text{ weakly in } L_q(Q), \ \ i = 1,\dots,n. \tag{4.3.35}$$

From (4.3.34) and (4.3.35), according to the Riesz theorem, we obtain the validity of relation (4.3.33).

Thus Lemma 4.3.4 is proved. □

Applying Lemma 4.3.4, it is not difficult to derive

$$|\nabla(U_m - U)| \longrightarrow 0 \ \text{ strongly in } L_q(Q),$$

i.e.,

$$\iint\limits_{Q} |\nabla(U_m - \nabla U)|^q \, dx\, dt \longrightarrow 0, \quad m \to \infty. \tag{4.3.36}$$

Consider the sequence of functions

$$\int\limits_0^T |\nabla U_m - \nabla U|^q \, dt. \tag{4.3.37}$$

From (4.3.36), it follows that sequence (4.3.37) converges to zero in $L_1(\Omega)$. Hence, it is possible to select a subsequence (denote it again by $\{U_m\}$) which for almost every $x \in \Omega$ (4.3.37) will converge to zero, i.e.,

$$\int\limits_0^T |\nabla U_m - \nabla U|^q \, dt \longrightarrow 0 \ \text{ in almost everywhere in } \Omega,$$

or

$$\nabla U_m \longrightarrow \nabla U \ \ L_q(0,T) \ \text{ for almost every } x \in \Omega.$$

Thus,

$$\int\limits_0^T |\nabla U_m|^q \, dt \longrightarrow \int\limits_0^T |\nabla U|^q \, dt \ \text{ almost everywhere in } \Omega.$$

As far as the limiting transition can be carried out on any subset $[0,t] \subset [0,T]$, we get

$$\int\limits_0^t |\nabla U_m|^q \, d\tau \longrightarrow \int\limits_0^t |\nabla U|^q \, d\tau \ \text{ almost everywhere in } Q. \tag{4.3.38}$$

From (4.3.33) and (4.3.38), using the continuity of the function $a = a(S)$, we have

$$a(S_m)|D_i U_m|^{q-2} D_i U_m \longrightarrow a(S)|D_i U|^{q-2} D_i U \ \text{ almost everywhere in } Q. \tag{4.3.39}$$

Applying now Lemma 4.3.4 and relation (4.3.39), we conclude that

$$a(S_m)|D_i U_m|^{q-2} D_i U_m \longrightarrow a(S)|D_i U|^{q-2} D_i U \ \text{ weakly in } L_{\frac{pq+q}{pq+q+1}}(Q). \tag{4.3.40}$$

In view of (4.3.27) and (4.3.40), taking the limit in (4.3.1) as $m \to \infty$, we obtain the identity

$$\iint\limits_{Q} \Big(\frac{\partial U}{\partial t} + A(U)\Big) B w_j \, dx\, dt = \iint\limits_{Q} f B w_j \, dx\, dt$$

for any $j \in N$.

Taking into account that $\{Bw_j\}$ form the "basis" in $L_{pq+q}(0,T; \overset{\circ}{W}{}^1_{pq+q}(\Omega))$ [54, 71], we come to (4.2.4) and thus, the existence of the solution of problem (4.2.1)–(4.2.3) is proved.

Proof of uniqueness. Let us now prove the uniqueness of the solution of problem (4.2.1)–(4.2.3). Assume that problem (4.2.1)–(4.2.3) has two solutions $\overline{U}$ and $\overline{\overline{U}}$. Introducing the difference $W(x,t) =$

$\overline{\overline{U}}(x,t) - \overline{U}(x,t)$, we have

$$\frac{\partial W}{\partial t} - \sum_{i=1}^{n} D_i\Big[a(\overline{\overline{S}})\nabla|\overline{\overline{U}}|^{q-2}D_i\overline{\overline{U}} - a(\overline{S})\nabla|\overline{U}|^{q-2}D_i\overline{U}\Big] = 0, \tag{4.3.41}$$

$$W(x,t) = 0, \ \ (x,t) \in \partial\Omega \times (0,T), \tag{4.3.42}$$

$$W(x,0) = 0, \ \ x \in \Omega, \tag{4.3.43}$$

$$\overline{S}(x,t) = \int_0^t |\nabla\overline{U}|^q \, d\tau, \quad \overline{\overline{S}}(x,t) = \int_0^t |\nabla\overline{\overline{U}}|^q \, d\tau.$$

Multiplying (4.3.41) by W, integrating the obtained identity on $Q_t = \Omega \times (0,t)$ and taking into account (4.3.42), (4.3.43), we have

$$\frac{1}{2}\int_\Omega W^2(x,t)\,dx + \sum_{i=1}^{n} \iint_{Q_t} \Big[a(\overline{\overline{S}})\nabla|\overline{\overline{U}}|^{q-2}D_i\overline{\overline{U}} - a(\overline{S})\nabla|\overline{U}|^{q-2}D_i\overline{U})\Big] D_iW \ dx\,dt = 0. \tag{4.3.44}$$

Following [50], it is not difficult to get

$$\sum_{i=1}^{n} \iint_{Q_t} \Big[a(\overline{\overline{S}})|\nabla\overline{\overline{U}}|^{q-2}D_i\overline{\overline{U}} - a(\overline{S})|\nabla\overline{U}|^{q-2}D_i\overline{U}\Big] D_i(\overline{\overline{U}} - \overline{U}) \ dx\,dt \geq 0.$$

From this, taking into account (4.3.44), we derive

$$\int_\Omega W^2(x,t)\,dx \leq 0$$

for arbitrary $t \in [0,T]$.

The latter inequality gives the uniqueness of the solution of problem (4.2.1)–(4.2.3). □

Remark 4.3.2. The theorem is also true for a more general right-hand side $f = f(x,t)$ and for the coefficients $a = a(x,t,S)$, depending on variables x, t and satisfying certain conditions. In particular, it is true for $f \in L_{\frac{pq+q}{pq+q-1}}(0,T; W^{-1}_{\frac{pq+q}{pq+q-1}}(\Omega))$, and the homogeneity of the initial and boundary conditions are not essential (see [54, 71]).

Remark 4.3.3. In the one-dimensional spatial case ($n = 1$) the system $\{w_j\}$ can be built by applying the following operator

$$Bw = -\frac{\partial}{\partial t}\Big[(T-t)\,\frac{\partial w}{\partial t}\Big] - \frac{\partial^2 w}{\partial x^2} + \lambda w.$$

Remark 4.3.4. If $q = 2$, i.e., in the absence of degeneration in equation (4.2.1), the following relations take place:

$$\sqrt{T-t}\,\frac{\partial^2 U}{\partial t \partial x_i} \in L_2(Q), \quad \sqrt{\psi}\,\frac{\partial^2 U}{\partial x_i^2} \in L_2(Q), \ \ i = 1, \dots, n.$$

Remark 4.3.5. Investigations carried out above hold also for the following nonlinear integro-differential equation

$$\frac{\partial U}{\partial t} - \sum_{i=1}^{n} D_i\Big[a_i\Big(\int_0^t |D_iU|^q \, d\tau\Big)|D_iU|^{q-2}D_iU\Big] = f(x,t),$$

where the functions $a_i = a_i(S)$ satisfy the same conditions as $a = a(S)$.

Remark 4.3.6. Investigations by the compactness method can be carried out also for the following nonlinear integro-differential equations of order $2m$:

$$\frac{\partial U}{\partial t} + (-1)^m \sum_{|\alpha|=m} D^\alpha \bigg[a_\alpha \bigg(\int_0^t |D^\alpha U|^q \, d\tau \bigg) |D^\alpha U|^{q-2} D^\alpha U \bigg] = f(x,t)$$

and

$$\frac{\partial U}{\partial t} + (-1)^m \sum_{|\alpha|=m} D^\alpha \bigg[a_\alpha \bigg(\int_0^t |D^m U|^q \, d\tau \bigg) |D^m U|^{q-2} D^\alpha U \bigg] = f(x,t),$$

where

$$|D^m U|^2 = \sum_{|\alpha|=m} |D^\alpha U|^2.$$

According to the operator scheme of the method of conditionally weakly closed operators, these equations have been studied in [50].

4.4 Asymptotic behavior of solutions for Volterra-type equations

In the present section, for one nonlinear integro-differential problem we obtain a priori estimations of solutions independent of T. From these estimations follows stabilization of the solution, as $t \to \infty$.

In the cylinder $Q = \Omega \times (0,\infty)$, we consider the following problem:

$$\frac{\partial U}{\partial t} - \sum_{i=1}^n D_i \bigg[a \bigg(\int_0^t |\nabla U|^2 \, d\tau \bigg) D_i U \bigg] = 0, \tag{4.4.1}$$

$$U(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,\infty), \tag{4.4.2}$$

$$U(x,0) = U_0(x), \quad x \in \Omega. \tag{4.4.3}$$

We will use the scheme of the work [43], in which the stabilization of a solution of the initial-boundary value problem for equations of barotropic viscous fluid is established.

Here and in the sequel, C will denote various positive constants, independent of t. The norm in $L_2(\Omega)$, as usual, will be denoted by $\| \cdot \|$.

Theorem 4.4.1. If $a(S) \geq a_0 = const > 0$, $a'(S) \geq 0$, $a''(S) \leq 0$, $U_0 \in W_2^2(0,1) \cap \overset{\circ}{W}{}_2^1(0,1)$, then for solution of problem (3.1.2)–(3.1.5) the following asymptotic expression holds:

$$\|U(\,\cdot\,,t)\|_{W_2^1(\Omega)} \longrightarrow 0, \quad t \to \infty.$$

Proof. Let us begin with the obtaining a priori estimations. Multiplying equation (4.4.1) scalarly by U, using formula of integration by parts and taking into account the boundary condition (4.4.2), we get

$$\frac{1}{2} \frac{d}{dt} \|U\|^2 + \sum_{i=1}^n \int_\Omega a(S) |D_i U|^2 \, dx = 0. \tag{4.4.4}$$

Multiplying (4.4.4) by $e^{\beta\tau}$, $\beta = const > 0$, we have

$$\int_0^t e^{\beta\tau} \frac{d}{d\tau} \|U\|^2 \, d\tau + 2 \sum_{i=1}^n \int_0^t \int_\Omega e^{\beta\tau} a(S) |D_i U|^2 \, dx \, d\tau = 0. \tag{4.4.5}$$

Application of the formula of integration by parts gives

$$\int_0^t e^{\beta\tau}\frac{d}{d\tau}\|U\|^2\,d\tau = e^{\beta\tau}\|U\|^2\Big|_0^t - \int_0^t e^{\beta\tau}\frac{d}{dt}\|U\|^2\,d\tau. \tag{4.4.6}$$

Using the Poincaré inequality

$$\|U\|^2 \le C(\Omega)\int_\Omega |\nabla U|^2\,dx,$$

from (4.4.6), we get

$$\int_0^t e^{\beta\tau}\frac{d}{d\tau}\|U\|^2\,d\tau \ge e^{\beta t}\|U\|^2 - \|U_0\|^2 - C\beta\int_0^t e^{\beta\tau}\int_\Omega |\nabla U|^2\,dx\,d\tau.$$

Taking into account the condition $a(S) \ge a_0$, we derive

$$\int_0^t e^{\beta\tau}\frac{d}{d\tau}\|U\|^2\,d\tau \ge e^{\beta t}\|U\|^2 - \|U_0\|^2 - \frac{C\beta}{a_0}\sum_{i=1}^n\int_0^t\!\!\int_\Omega e^{\beta\tau}a(S)|D_iU|^2\,dx\,d\tau.$$

Thus, from the identities (4.4.6) and (4.4.6), we obtain

$$e^{\beta t}\|U\|^2 - \|U_0\|^2 + \Big(2 - \frac{C\beta}{a_0}\Big)\sum_{i=1}^n\int_0^t\!\!\int_\Omega e^{\beta\tau}a(S)|D_iU|^2\,dx\,d\tau \le 0.$$

From this, if $2 - C\beta/a_0 \ge 0$, i.e., choosing $\beta \le 2a_0/C$, we arrive at

$$e^{\beta t}\|U\|^2 \le \|U_0\|^2,$$

or, finally,

$$\|U\| \le e^{-\frac{\beta t}{2}}\|U_0\|.$$

Thus, for the solution of problem (4.4.1)–(4.4.3) there takes place the stabilization as $t \to \infty$, and the rate of stabilization in the norm of the space $L_2(\Omega)$ is exponential.

Let us now investigate asymptotic behavior of the solution of problem (4.4.1)–(4.4.3) in the norm of the space $W_2^1(\Omega)$ [37].

Suppose that $a = a(S)$ satisfies additional restrictions $a'(S) \ge 0$ and $a''(S) \le 0$.

We integrate (4.4.4) on $(0, t)$. Taking into account the initial condition (4.4.3), we have

$$\|U\|^2 + a(S)\int_0^t\int_\Omega |\nabla U|^2\,dx\,d\tau \le \|U_0\|^2,$$

whence we get a priori estimations

$$\|U\| \le C, \quad \int_0^t\!\!\int_\Omega |\nabla U|^2\,dx\,d\tau \le C. \tag{4.4.7}$$

For an arbitrary i, we have

$$\int_0^t\left|\frac{d}{d\tau}\int_\Omega |D_iU|^2\,dx\right|d\tau = 2\int_0^t\left|\int_\Omega D_iU\frac{d}{d\tau}(D_iU)\,dx\right|d\tau$$

$$\le \int_0^t\!\!\int_\Omega |D_iU|^2\,dx\,d\tau + \int_0^t\!\!\int_\Omega \left|\frac{d}{d\tau}(D_iU)\right|^2\,dx\,d\tau. \tag{4.4.8}$$

Estimate the second integral on the right-hand side of inequality (4.4.8). To this end, let us differentiate (4.4.1) with respect to t and multiply the obtained equality scalarly by $\partial U/\partial t$,

$$\int\limits_{\Omega} \frac{\partial^2 U}{\partial t^2}\,\frac{\partial U}{\partial t}\,dx - \int\limits_{\Omega} \frac{\partial U}{\partial t} \sum_{i=1}^{n} D_i\Big[D_i U a'(S)|\nabla U| + a(S)\,\frac{\partial}{\partial t}\,(D_i U)\Big]\,dx = 0.$$

Applying the formula of integration by parts and taking into account the boundary conditions (4.4.2), we have

$$\frac{d}{dt}\,\Big\|\frac{\partial U}{\partial t}\Big\|^2 + 2\sum_{i=1}^{n}\int\limits_{\Omega} \frac{\partial^2 U}{\partial x_i \partial t}\,\Big(D_i U a'(S)|\nabla U|^2 + a(S)\,\frac{\partial}{\partial t}\,(D_i U)\Big)\,dx = 0,$$

or

$$\frac{d}{dt}\,\Big\|\frac{\partial U}{\partial t}\Big\|^2 + 2\sum_{i=1}^{n}\int\limits_{\Omega} a(S)\Big|\frac{\partial}{\partial t}\,(D_i U)\Big|^2\,dx + 2\sum_{i=1}^{n}\int\limits_{\Omega} a'(S)\,\frac{\partial}{\partial t}\,(D_i U) D_i U |D_k U|^2\,dx = 0.$$

We integrate the obtained identity on $(0,t)$,

$$\begin{aligned}\Big\|\frac{\partial U}{\partial t}\Big\|^2 - \Big\|\frac{\partial U}{\partial t}\Big\|^2_{t=0} &+ 2\sum_{i=1}^{n}\int\limits_0^t\!\!\int\limits_{\Omega} a(S)\Big|\frac{\partial}{\partial \tau}\,(D_i U)\Big|^2\,dx\,d\tau \\ &+ \sum_{i,k=1}^{n}\int\limits_0^t\!\!\int\limits_{\Omega} a'(S)|D_k U|^2\Big|\frac{\partial}{\partial \tau}\,(D_i U)^2\Big|\,dx\,d\tau = 0.\end{aligned}$$

Taking into account the boundary condition (4.4.2) and the restrictions on $a = a(S)$, we have

$$\Big\|\frac{\partial U}{\partial t}\Big\|^2 + 2a_0\sum_{i=1}^{n}\int\limits_0^t\!\!\int\limits_{\Omega}\Big|\frac{\partial}{\partial \tau}\,(D_i U)\Big|^2\,dx\,d\tau + J \le 0, \tag{4.4.9}$$

where

$$\begin{aligned}J &= \sum_{i,k=1}^{n}\int\limits_0^t\!\!\int\limits_{\Omega} a'(S)|D_k U|^2\Big|\frac{\partial}{\partial \tau}\,(D_i U)^2\Big|\,dx\,d\tau \\ &= \sum_{i,k=1}^{n}\int\limits_{\Omega} a'(S)|D_k U|^2|D_i U|^2\,dx - \sum_{i,k=1}^{n}\int\limits_{\Omega} a'(S)|D_k U|^2|D_i U|^2\,dx\Big|_{t=0} \\ &\quad - \sum_{i,j,k=1}^{n}\int\limits_0^t\!\!\int\limits_{\Omega} a''(S)|D_i U|^2|D_j U|^2|D_k U|^2\,dx\,d\tau \\ &\quad - \sum_{i,k=1}^{n}\int\limits_0^t\!\!\int\limits_{\Omega} a'(S)|D_i U|^2|D_j U|^2\Big|\frac{\partial}{\partial \tau}\,(D_k U)^2\Big|\,dx\,d\tau \ge C - J.\end{aligned}$$

From this we get $J \ge C/2$, and from inequality (4.4.9), we have

$$\Big\|\frac{\partial U}{\partial t}\Big\|^2 + 2a_0\sum_{i=1}^{n}\int\limits_0^t\!\!\int\limits_{\Omega}\Big|\frac{\partial}{\partial \tau}\,(D_i U)\Big|^2\,dx\,d\tau \le C.$$

Thus, the following a priori estimations hold:

$$\int\limits_0^t\int\limits_{\Omega}\Big|\frac{\partial}{\partial \tau}\,(D_i U)\Big|^2\,dx\,d\tau \le C, \quad i = 1,\dots,n.$$

From relations (4.4.7), (4.4.8) and (4.4.10), we get

$$\int_0^t \int_\Omega |D_i U|^2 \, dx \, d\tau \le C, \quad \int_0^t \left| \frac{d}{d\tau} \int_\Omega |D_i U|^2 \, dx \right| d\tau \le C, \quad i = 1, \dots, n.$$

The obtained a priori estimations hold for an arbitrary t. Therefore, we arrive at

$$\int_0^t \int_\Omega |\nabla U|^2 \, dx \, d\tau \le C, \quad \int_0^t \left| \frac{d}{d\tau} \int_\Omega |\nabla U|^2 \, dx \right| d\tau \le C, \tag{4.4.10}$$

and from (4.4.10), using Lemma 4.1.4, we finally get

$$\|U(\,\cdot\,,t)\|_{W_2^1(\Omega)} \longrightarrow 0, \quad t \to \infty.$$

Thus, stabilization of the solution of problem (4.4.1)–(4.4.3) is proved in the norm of the space $W_2^1(\Omega)$ as well in the case $a(S) \ge a_0 = const > 0$, $a'(S) \ge 0$, $a''(S) \le 0$. □

4.5 Some comments on the unique solvability and asymptotic behavior of solutions for averaged equations

In the domain $Q = \Omega \times (0,T)$, let us consider the following initial-boundary value problem for the following AIDE:

$$\frac{\partial U}{\partial t} - a\left(\int_\Omega \int_0^t |\nabla U|^q \, d\tau \right) \sum_{i=1}^n D_i\big(|\nabla U|^{q-2} D_i U\big) = f(x,t), \tag{4.5.1}$$

$$U(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{4.5.2}$$

$$U(x,0) = 0, \quad x \in \Omega. \tag{4.5.3}$$

The study of equation (4.5.1) does not fit the well-known theories of nonlinear operator equations, including the operator scheme with the so-called conditionally weakly closed operators proposed in [50]. This scheme includes, as a special case, equations with coercive monotone operators and is successfully used to determine the solvability of the problems considered in Section 4.3 [58]. The author noted in [50] that equations of type (4.5.1) require an independent scheme of investigation.

As we have noted in Chapter 1, the equations of type (4.5.1) simulate the processes of diffusion supposing the dependence of temperature only on time and independence of spatial variables. Thus, on the one hand, the equations simulate the process of a more simplified physical problem, which should, at the first glance, facilitate their study. But on the other hand, the appearance of an integral in the coefficients depending not only on time, but also on spatial variables, attaches a more complex nonlocal character to the operator. This fact, apparently, also complicates their mathematical research.

The compactness method applied in Section 4.3 with the use of Galerkin's method makes it possible to get the necessary a priori estimates. Moreover, in the case under consideration, just as in the one-dimensional variant of the Volterra-type IDE, the system $\{Bw_j\}$ can be constructed by using the operator (see Remark 4.3.3)

$$Bw_j = -\frac{\partial}{\partial t}\left[(T-t)\,\frac{\partial w}{\partial t}\right] - \Delta w + \lambda w.$$

For Galerkin's approximations U_m of problem (4.5.1)–(4.5.3), we obtain the following a priori estimations

$$\iint\limits_Q \Big|\frac{\partial U_m}{\partial t}\Big|^2\,dx\,dt+\iint\limits_Q a(S_m)|\nabla U_m|^q\,dx\,dt+\iint\limits_Q a'(S_m)|\nabla U_m|^{2q}\,dx\,dt$$
$$+\sum_{i=1}^n\iint\limits_Q (T-t)a(S_m)\Big[\frac{\partial}{\partial t}\Big(\Big|\frac{\partial U_m}{\partial x_i}\Big|^{\frac{q-2}{2}}\,\frac{\partial U_m}{\partial x_i}\Big)\Big]^2\,dx\,dt$$
$$+\sum_{i,j=1}^n\iint\limits_Q a(S_m)\Big[\frac{\partial}{\partial x_j}\Big(\Big|\frac{\partial U_m}{\partial x_i}\Big|^{\frac{q-2}{2}}\,\frac{\partial U_m}{\partial x_i}\Big)\Big]^2\,dx\,dt\le C.$$

So, the unique solvability of problem (4.5.1)–(4.5.3) can be obtained by applying a similar approach as in Section 4.4.

Using the schemes of investigations of asymptotic behavior of solutions that are given in previous chapters, it is possible to get a rate of convergence, as well. Many authors study this property for IDEs type models investigated in our works (see, e.g., [65–67, 74] and the references therein).

Chapter 5

The unique solvability and additive Rothe-type semi-discrete schemes for two nonlinear multi-dimensional parabolic integro-differential problems

Chapter 5 is devoted to the construction and study of the decomposition type semi-discrete scheme for nonlinear multi-dimensional IDEs of parabolic type considered in Chapters 1–4. The existence and uniqueness of a solution of the first type initial-boundary value problem are given, as well. The studied by us equations are certain generalizations of integro-differential models based on the well-known system of Maxwell equations arising in mathematical simulation of an electromagnetic field penetration into a medium. Chapter 5 consists of two sections. Section 5.1 has four subsections dealing with the unique solvability and decomposition method for one particular case of the Volterra-type nonlinear multi-dimensional integro-differential parabolic equation. Some results of numerical experiments are given in this section as well. In Section 5.2, some comments on the unique solvability and on the decomposition algorithm are shortly given for one particular case of a nonlinear multi-dimensional AIDE.

5.1 The unique solvability and additive scheme for the Volterra-type equation

5.1.1 Introduction

The relevance of partial IDEs can be motivated by a number of their applications in practice and in many fields of science. The main characteristic feature of models considered in Chapters 1–4 are associated with the appearance of nonlinear terms depending on time and spatial integrals. This circumstance complicates the study and requires other considerations than those required normally for solving the local differential problems. So, the integro-differential models under consideration may be interpreted as, e.g., generalized models of the problems occurring in the theory of nonlinear parabolic equations (see, e.g., [54, 71]). The Volterra-type models considered in the previous chapters for the scalar one-dimensional spatial case were first investigated in [20] and [28]. The scalar multi-dimensional spatial case was first studied in [17]. Later on, these types of integro-differential models were considered in a number of papers (see, e.g., [24, 32, 33, 39] and the references therein).

The literature devoted to the questions of the existence, uniqueness and regularity of solutions to the above types of equations is ample. Asymptotic behavior of solutions is discussed in [24, 32, 39] and in a number of other works, as well. Note also that many works are devoted to the numerical resolution of the Volterra-type one-dimensional equations (see, e.g., Chapter 2, [33,39] and the references therein). Many authors study the Rothe scheme, the semi-discrete scheme with a spatial variable, finite element and finite difference approximations for integro-differential models (see, e.g., [4, 8–11, 18, 19, 22, 30, 33, 35, 37–42, 44, 51, 54, 56, 61, 65–67, 74]).

Of special importance is investigation of decomposition analogues for the above-mentioned multi-dimensional integro-differential models. At present there are several effective algorithms for solving the multi-dimensional problems (see, e.g., [1, 2, 57, 64, 72] and the references therein).

This section is devoted to the global existence and uniqueness of solutions of the initial-boundary value problem. Great attention is paid to the investigation of a semi-discrete additive average scheme. We will focus our attention to the particular case of the Volterra-type multi-dimensional integro-differential equation.

This section is organized as follows. Subsection 5.1.2 presents the formulation of the problem and some of its properties. Especially, the existence and uniqueness of the solution of the stated problem are studied therein. Main attention is paid to the construction and investigation of semi-discrete additive averaged scheme. This question is discussed in Subsection 5.1.3. Some comments on the numerical implementations and the results of numerical experiments are given in Subsection 5.1.4.

5.1.2 Formulation of the problem and the unique solvability

Let Ω be the bounded domain in the n-dimensional Euclidean space R^n, with a sufficiently smooth boundary $\partial\Omega$. In the domain $Q = \Omega \times (0,T)$ of the variables $(x,t) = (x_1, x_2, \dots, x_n, t)$ let us consider the following first type initial-boundary value problem:

$$\frac{\partial U}{\partial t} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left[\left(1 + \int_0^t \left|\frac{\partial U}{\partial x_i}\right|^2 d\tau\right)\frac{\partial U}{\partial x_i}\right] = f(x,t), \quad (x,t) \in Q, \tag{5.1.1}$$

$$U(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T], \tag{5.1.2}$$

$$U(x,0) = 0, \quad x \in \overline{\Omega}, \tag{5.1.3}$$

where T is a fixed positive constant, f is a given function of its arguments.

Since problem (5.1.1)–(5.1.3) is similar to that considered in [32] and in Chapter 4, we can follow the same procedure used there. Using the modified version of Galerkin's method and the compactness arguments [54, 71], the following statement can be proved for problem (5.1.1)–(5.1.3).

Theorem 5.1.1. If

$$f, \ \frac{\partial f}{\partial t}, \ \sqrt{\psi}\,\frac{\partial f}{\partial x_i} \in L_2(Q), \quad f(x,0) = 0,$$

then there exists the unique solution U of problem (5.1.1)–(5.1.3) satisfying the properties

$$U \in L_4(0,T,\overset{\circ}{W}{}_4^1(\Omega)), \quad \frac{\partial U}{\partial t} \in L_2(Q),$$

$$\sqrt{\psi}\,\frac{\partial^2 U}{\partial x_i \partial x_j} \in L_2(Q), \quad \sqrt{T-t}\,\frac{\partial^2 U}{\partial t \partial x_i} \in L_2(Q), \quad i,j = 1,\dots,n,$$

where $\psi \in C^\infty(\Omega)$, $\psi(x) > 0$, $x \in \Omega$; $\psi = \partial\psi/\partial\nu = 0$, $x \in \partial\Omega$, ν is an outer normal of $\partial\Omega$.

Proof. Let us consider the following weak formulation of problem (5.1.1)–(5.1.3):

$$\int_0^T\!\!\int_\Omega \left(\frac{\partial U}{\partial t}\,V + \sum_{i=1}^{n}\left(1 + \int_0^t \left|\frac{\partial U}{\partial x_i}\right|^2 d\tau\right)\frac{\partial U}{\partial x_i}\,\frac{\partial V}{\partial x_i}\right) dx\,dt = \int_0^T\!\!\int_\Omega fV\,dx\,dt \tag{5.1.4}$$

for all $V \in L_4(0,T;\overset{\circ}{W}{}_4^1(\Omega))$. We seek for an approximate solution of the problem in the form

$$u_m(x,t) = \sum_{k=1}^{m} c_{mk} w_k(x,t),$$

where the unknown coefficients c_{mk} satisfy the following system of nonlinear algebraic equations:

$$\int_0^T\int_\Omega \left(\frac{\partial u_m}{\partial t} Bw_k + \sum_{i=1}^n \left(1+\int_0^t \left|\frac{\partial u_m}{\partial x_i}\right|^2 d\tau\right)\frac{\partial u_m}{\partial x_i}\,\frac{\partial (Bw_k)}{\partial x_i}\right) dx\,dt$$
$$= \int_0^T\int_\Omega f Bw_k \,dx\,dt,\quad k=1,\dots,m. \tag{5.1.5}$$

Here, $\{Bw_j\}$ is the complete system in the space $L_4(0,T;\overset{\circ}{W}{}_4^1(\Omega))$ (see, e.g., Chapter 4 and [54, 71]) and

$$Bw = -\frac{\partial}{\partial t}\left[(T-t)\,\frac{\partial w}{\partial t}\right] - \psi\sum_{i=1}^n \frac{\partial^2 w}{\partial x_i^2} + \lambda w,\quad \lambda = const > 0.$$

After some straightforward computations (for more details see Chapter 4 and [17, 32]), one establishes the relation

$$\int_0^T\int_\Omega \left(\frac{\partial u_m}{\partial t}\, Bu_m + \sum_{i=1}^n \left(1+\int_0^t \left(\frac{\partial u_m}{\partial x_i}\right)^2 d\tau\right)\frac{\partial u_m}{\partial x_i}\,\frac{\partial (Bu_m)}{\partial x_i}\right) dx\,dt$$
$$\geq C\iint_Q \left|\frac{\partial u_m}{\partial t}\right|^2 dx\,dt + C\iint_Q \left(1+\int_0^t \left(\frac{\partial u_m}{\partial x_i}\right)^2 d\tau\right) dx\,dt$$
$$+ C\sum_{i=1}^n \iint_Q (T-t)\left(1+\int_0^t \left(\frac{\partial u_m}{\partial x_i}\right)^2 d\tau\right)\left(\frac{\partial^2 u_m}{\partial t\partial x_i}\right)^2 dx\,dt$$
$$+ C\sum_{i,j=1}^n \iint_Q \psi\left(1+\int_0^t \left(\frac{\partial u_m}{\partial x_i}\right)^2 d\tau\right)\left(\frac{\partial^2 u_m}{\partial x_j\partial x_i}\right)^2 dx\,dt. \tag{5.1.6}$$

Throughout this subsection, the constants used are not necessarily to be the same at different occurrences.

It is not difficult to estimate the right-hand side of (5.1.5)

$$\left|\sum_{k=1}^m c_{mj}\int_0^T\int_\Omega f Bw_j \,dx\,dt\right| = \left|\int_0^T\int_\Omega f Bu_m\,dx\,dt\right|$$
$$= \left| -\iint_Q f\,\frac{\partial}{\partial t}\left[(T-t)\,\frac{\partial u_m}{\partial t}\right] dx\,dt - \sum_{i=1}^n \iint_Q f\psi\,\frac{\partial^2 u_m}{\partial x_i^2}\,dx\,dt + \lambda\iint_Q f u_m\,dx\,dt\right|$$
$$\leq \left|\iint_Q (T-t)\,\frac{\partial f}{\partial t}\,\frac{\partial u_m}{\partial t}\,dx\,dt\right| + \left|\sum_{i=1}^n \iint_Q \psi\,\frac{\partial f}{\partial x_i}\,\frac{\partial u_m}{\partial x_i}\,dx\,dt\right|$$
$$+ \left|\sum_{i=1}^n \iint_Q f\,\frac{\partial \psi}{\partial x_i}\,\frac{\partial u_m}{\partial x_i}\,dx\,dt\right| + \lambda\left|\iint_Q f u_m\,dx\,dt\right|$$
$$\leq \varepsilon\iint_Q \left|\frac{\partial u_m}{\partial t}\right|^2 dx\,dt + \frac{C}{\varepsilon}\iint_Q \left|\frac{\partial f}{\partial t}\right|^2 dx\,dt + \varepsilon\sum_{i=1}^n \iint_Q \psi\left|\frac{\partial u_m}{\partial x_i}\right|^2 dx\,dt$$
$$+ \frac{C}{\varepsilon}\sum_{i=1}^n \iint_Q \psi\left|\frac{\partial f}{\partial x_i}\right|^2 dx\,dt + \varepsilon\sum_{i=1}^n \iint_Q \psi\left|\frac{\partial u_m}{\partial x_i}\right|^2 dx\,dt + \frac{C}{\varepsilon}\lambda\iint_Q f^2\,dx\,dt, \tag{5.1.7}$$

where ε is a positive constant of the ε-inequality.

As a consequence, choosing ε sufficiently small and λ sufficiently large, after simple transformations, estimations (5.1.6) and (5.1.7) easily yield

$$\begin{aligned}\iint\limits_Q \Big|\frac{\partial u_m}{\partial t}\Big|^2\,dx\,dt &+ \iint\limits_Q \bigg(1+\int\limits_0^t \Big(\frac{\partial u_m}{\partial x_i}\Big)^2 d\tau\bigg)\,dx\,dt \\ &+\sum_{i=1}^n \iint\limits_Q (T-t)\bigg(1+\int\limits_0^t \Big(\frac{\partial u_m}{\partial x_i}\Big)^2 d\tau\bigg)\Big(\frac{\partial^2 u_m}{\partial t\partial x_i}\Big)^2\,dx\,dt \\ &+\sum_{i,j=1}^n \iint\limits_Q \psi\bigg(1+\int\limits_0^t \Big(\frac{\partial u_m}{\partial x_i}\Big)^2 d\tau\bigg)\Big(\frac{\partial^2 U_m}{\partial x_j\partial x_i}\Big)^2\,dx\,dt \le C. \qquad (5.1.8)\end{aligned}$$

From the a priori estimations (5.1.8), the desired result now follows by using the method proposed in [17, 32] and used in Chapter 4 for proving the suitable existence results for models of type (5.1.1)–(5.1.3).

Let us now establish the second part of Theorem 5.1.1, namely, a uniqueness property of the solution of problem (5.1.1)–(5.1.3). We will have a more detailed discussion here than in Chapter 4 for proving the analogous fact.

Let problem (5.1.1)–(5.1.3) have two solutions $\overline{U}$ and $\overline{\overline{U}}$. Introduce the following notation: $Z(x,t)=\overline{\overline{U}}(x,t)-\overline{U}(x,t)$. Subtracting (5.1.1)–(5.1.3) for $\overline{U}$ from (5.1.1)–(5.1.3) for $\overline{\overline{U}}$ we get

$$\frac{\partial Z}{\partial t}-\sum_{i=1}^n\frac{\partial}{\partial x_i}\bigg[\bigg(1+\int\limits_0^t\Big(\frac{\partial\overline{\overline{U}}}{\partial x_i}\Big)^2d\tau\bigg)\frac{\partial\overline{\overline{U}}}{\partial x_i}-\bigg(1+\int\limits_0^t\Big(\frac{\partial\overline{U}}{\partial x_i}\Big)^2d\tau\bigg)\frac{\partial\overline{U}}{\partial x_i}\bigg]=0,\quad (x,t)\in Q, \qquad (5.1.9)$$

$$Z(x,t)=0,\quad (x,t)\in\partial\Omega\times[0,T], \qquad (5.1.10)$$

$$Z(x,0)=0,\quad x\in\overline{\Omega}. \qquad (5.1.11)$$

We multiply (5.1.9) by Z and integrate the obtained identity on $\Omega\times(0,t)$. Taking into account (5.1.10), (5.1.11), we have

$$\begin{aligned}&\frac12\int\limits_\Omega Z^2(x,t)\,dx \\ &\quad+\sum_{i=1}^n\int\limits_\Omega\int\limits_0^t\bigg[\bigg(1+\int\limits_0^t\Big(\frac{\partial\overline{\overline{U}}}{\partial x_i}\Big)^2d\tau\bigg)\frac{\partial\overline{\overline{U}}}{\partial x_i}-\bigg(1+\int\limits_0^t\Big(\frac{\partial\overline{U}}{\partial x_i}\Big)^2d\tau\bigg)\frac{\partial\overline{U}}{\partial x_i}\bigg]\frac{\partial Z}{\partial x_i}\,dx\,d\tau=0. \qquad (5.1.12)\end{aligned}$$

Note that

$$\begin{aligned}&\bigg[\bigg(1+\int\limits_0^t\Big(\frac{\partial\overline{\overline{U}}}{\partial x_i}\Big)^2d\tau\bigg)\frac{\partial\overline{\overline{U}}}{\partial x_i}-\bigg(1+\int\limits_0^t\Big(\frac{\partial\overline{U}}{\partial x_i}\Big)^2d\tau\bigg)\frac{\partial\overline{U}}{\partial x_i}\bigg]\Big(\frac{\partial\overline{\overline{U}}}{\partial x_i}-\frac{\partial\overline{U}}{\partial x_i}\Big) \\ &\qquad=\frac12\bigg[2+\int\limits_0^t\Big(\frac{\partial\overline{\overline{U}}}{\partial x_i}\Big)^2d\tau+\int\limits_0^t\Big(\frac{\partial\overline{U}}{\partial x_i}\Big)^2d\tau\bigg]\Big[\frac{\partial\overline{\overline{U}}}{\partial x_i}-\frac{\partial\overline{U}}{\partial x_i}\Big]^2 \\ &\qquad\quad+\frac12\bigg[\int\limits_0^t\Big(\frac{\partial\overline{\overline{U}}}{\partial x_i}\Big)^2d\tau-\int\limits_0^t\Big(\frac{\partial\overline{U}}{\partial x_i}\Big)^2d\tau\bigg]\bigg[\Big(\frac{\partial\overline{\overline{U}}}{\partial x_i}\Big)^2-\Big(\frac{\partial\overline{U}}{\partial x_i}\Big)^2\bigg] \\ &\qquad\ge\frac12\bigg[\int\limits_0^t\Big(\frac{\partial\overline{\overline{U}}}{\partial x_i}\Big)^2d\tau-\int\limits_0^t\Big(\frac{\partial\overline{U}}{\partial x_i}\Big)^2d\tau\bigg]\bigg[\Big(\frac{\partial\overline{\overline{U}}}{\partial x_i}\Big)^2-\Big(\frac{\partial\overline{U}}{\partial x_i}\Big)^2\bigg]. \qquad (5.1.13)\end{aligned}$$

Introducing the notation

$$W(x,t) = \sum_{i=1}^{n} \int_0^t \left[\left(\frac{\partial \overline{\overline{U}}}{\partial x_i} \right)^2 - \left(\frac{\partial \overline{U}}{\partial x_i} \right)^2 \right] d\tau,$$

from (5.1.12) and (5.1.13), we get

$$\int_\Omega Z^2(x,t)\,dx + \int_0^t \int_\Omega W(x,t)\,\frac{\partial W(x,t)}{\partial t}\,dx\,d\tau \le 0$$

or

$$\int_\Omega Z^2(x,t)\,dx + \frac{1}{2} \int_\Omega W^2(x,t)\,dx \le 0.$$

The latter estimation implies that $Z \equiv 0$ and hence, the uniqueness of the solution of problem (5.1.1)–(5.1.3). Thus, Theorem 5.1.1 is complete. ☐

Using the scheme of investigation as, e.g., in [24, 32, 39], it is not difficult to get the exponential stabilization of solution as $t \to \infty$ for equation (5.1.1) with $f(x,t) \equiv 0$ and homogeneous boundary (5.1.2) and nonhomogeneous initial conditions (5.1.3).

5.1.3 Semi-discrete additive scheme

On $[0,T]$, let us introduce a net with mesh points denoted by $t_j = j\tau$, $j = 0,1,\dots,N$, with $\tau = 1/N$. Coming back to problem (5.1.1)–(5.1.3), let us construct the following additive average Rothe-type semi-discrete scheme:

$$\begin{gathered} \eta_i\,\frac{u_i^{j+1} - u^j}{\tau} = \frac{\partial}{\partial x_i} \left[\left(1 + \tau \sum_{k=1}^{j+1} \left(\frac{\partial u_i^k}{\partial x_i} \right)^2 \right) \frac{\partial u_i^{j+1}}{\partial x_i} \right] + f_i^{j+1}, \\ u_i^0 = u^0 = 0, \quad i = 1,\dots,n; \quad j = 0,1,\dots,N-1, \end{gathered} \tag{5.1.14}$$

with homogeneous boundary conditions, where $u_i^j(x)$, $j = 0,1,\dots,N$, is the solution of problem (5.1.14), and introduce the following notation:

$$u^j(x) = \sum_{i=1}^{n} \eta_i u_i^j(x), \quad \sum_{i=1}^{n} \eta_i = 1, \ \ \eta_i > 0, \quad \sum_{i=1}^{n} f_i^{j+1}(x) = f^{j+1}(x) = f(x,t_{j+1}),$$

where u^j denotes approximation of the exact solution U of problem (5.1.1)–(5.1.3) for t_j.

The object of this subsection is to prove one main statement of this chapter. Here we use a usual scalar product $(\,\cdot\,,\,\cdot\,)$ and the norm $\|\cdot\|$ of the space $L_2(\Omega)$.

Theorem 5.1.2. If problem (5.1.1)–(5.1.3) has a sufficiently smooth solution, then the functions u^m defined by the solutions of problems (5.1.14) converge to the solution of problem (5.1.1)–(5.1.3) and the estimate

$$\|U^m - u^m\| = O(\tau^{\frac{1}{2}}), \ \ m = 1,2,\dots,N,$$

holds.

Proof. Let us introduce the notations:

$$z^k = U^k - u^k, \quad z_i^k = U^k - u_i^k.$$

For the exact solution of problem (5.1.1)–(5.1.3) we have

$$\eta_i\,\frac{U^{j+1} - U^j}{\tau} = \eta_i \sum_{\ell=1}^{n} \frac{\partial}{\partial x_\ell} \left[\left(1 + \tau \sum_{k=1}^{j+1} \left(\frac{\partial U^k}{\partial x_\ell} \right)^2 \right) \frac{\partial U^{j+1}}{\partial x_\ell} \right] + \eta_i f^{j+1} + O(\tau).$$

After subtracting (5.1.1), from the above relation we get

$$\eta_i\Big(\frac{U^{j+1}-U^j}{\tau}-\frac{u_i^{j+1}-u^j}{\tau}\Big)=\eta_i\sum_{\ell=1}^n\frac{\partial}{\partial x_\ell}\bigg[\bigg(1+\tau\sum_{k=1}^{j+1}\Big(\frac{\partial U^k}{\partial x_\ell}\Big)^2\bigg)\frac{\partial U^{j+1}}{\partial x_\ell}\bigg]$$
$$-\frac{\partial}{\partial x_i}\bigg[\bigg(1+\tau\sum_{k=1}^{j+1}\Big(\frac{\partial u_i^k}{\partial x_i}\Big)^2\bigg)\frac{\partial u_i^{j+1}}{\partial x_i}\bigg]+\eta_i f^{j+1}-f_i^{j+1}+O(\tau).$$

Thus, introducing the notation

$$\frac{z_i^{j+1}-z^j}{\tau}=z_{i\bar t}^{j+1},$$

we have

$$\eta_i z_{i\bar t}^{j+1}=\frac{\partial}{\partial x_i}\bigg[\bigg(1+\tau\sum_{k=0}^{j+1}\Big(\frac{\partial U^k}{\partial x_i}\Big)^2\bigg)\frac{\partial U^{j+1}}{\partial x_i}\bigg]-\frac{\partial}{\partial x_i}\bigg[\bigg(1+\tau\sum_{k=0}^{j+1}\Big(\frac{\partial U^k}{\partial x_i}\Big)^2\bigg)\frac{\partial U^{j+1}}{\partial x_i}\bigg]$$
$$+\eta_i\sum_{\ell=1}^n\frac{\partial}{\partial x_\ell}\bigg[\bigg(1+\tau\sum_{k=1}^{j+1}\Big(\frac{\partial U^k}{\partial x_\ell}\Big)^2\bigg)\frac{\partial U^{j+1}}{\partial x_\ell}\bigg]$$
$$-\frac{\partial}{\partial x_i}\bigg[\bigg(1+\tau\sum_{k=1}^{j+1}\Big(\frac{\partial u_i^k}{\partial x_i}\Big)^2\bigg)\frac{\partial u_i^{j+1}}{\partial x_i}\bigg]+\eta_i f^{j+1}-f_i^{j+1}+O(\tau).$$

Here we add and subtract the first and the second terms in the right-hand side.

Using (5.1.1) and (5.1.14), we obtain the following problem:

$$\eta_i z_{i\bar t}^{j+1}=\frac{\partial}{\partial x_i}\bigg[\bigg(1+\tau\sum_{k=1}^{j+1}\Big(\frac{\partial U^k}{\partial x_i}\Big)^2\bigg)\frac{\partial U^{j+1}}{\partial x_i}-\bigg(1+\tau\sum_{k=1}^{j+1}\Big(\frac{\partial u_i^k}{\partial x_i}\Big)^2\bigg)\frac{\partial u_i^{j+1}}{\partial x_i}\bigg]+\psi_i^{j+1}(x),\qquad(5.1.15)$$
$$z_i^0=0,$$

with the homogeneous boundary conditions and where

$$\psi_i^{j+1}(x)=-\frac{\partial}{\partial x_i}\bigg[\bigg(1+\tau\sum_{k=1}^{j+1}\Big(\frac{\partial U^k}{\partial x_i}\Big)^2\bigg)\frac{\partial U^{j+1}}{\partial x_i}\bigg]$$
$$+\eta_i\sum_{\ell=1}^n\frac{\partial}{\partial x_\ell}\bigg[\bigg(1+\tau\sum_{k=1}^{j+1}\Big(\frac{\partial U^k}{\partial x_\ell}\Big)^2\bigg)\frac{\partial U^{j+1}}{\partial x_\ell}\bigg]+\eta_i f^{j+1}(x)-f_i^{j+1}(x)+O(\tau)$$
$$=\overline{\psi}_i^{j+1}(x)+O(\tau).$$

Using the assumptions on f_i^{j+1} and η_i, we have the identity of sum-approximation

$$\sum_{i=1}^n\overline{\psi}_i^{j+1}(x)=-\sum_{i=1}^n\frac{\partial}{\partial x_i}\bigg[\bigg(1+\tau\sum_{k=1}^{j+1}\Big(\frac{\partial U^k}{\partial x_i}\Big)^2\bigg)\frac{\partial U^{j+1}}{\partial x_i}\bigg]$$
$$+\sum_{i=1}^n\eta_i\sum_{\ell=1}^n\frac{\partial}{\partial x_\ell}\bigg[\bigg(1+\tau\sum_{k=1}^{j+1}\Big(\frac{\partial U^k}{\partial x_\ell}\Big)^2\bigg)\frac{\partial U^{j+1}}{\partial x_\ell}\bigg]$$
$$+\sum_{i=1}^n\eta_i f^{j+1}(x)-\sum_{i=1}^n f_i^{j+1}(x)=0.\qquad(5.1.16)$$

So,

$$\sum_{i=1}^n\psi_i^{j+1}(x)=O(\tau).$$

Multiplying (5.1.15) scalarly by $2\tau z_i^{j+1}$, we obtain

$$2\tau\eta_i(z_{i\bar t}^{j+1}, z_i^{j+1}) + 2\tau\left(\left(1+\tau\sum_{k=1}^{j+1}\left(\frac{\partial U^k}{\partial x_i}\right)^2\right)\frac{\partial U^{j+1}}{\partial x_i} - \left(1+\tau\sum_{k=1}^{j+1}\left(\frac{\partial u_i^k}{\partial x_i}\right)^2\right)\frac{\partial u_i^{j+1}}{\partial x_i}, \frac{\partial z_i^{j+1}}{\partial x_i}\right) - 2\tau(\psi_i^{j+1}, z_i^{j+1}) = 0. \tag{5.1.17}$$

It can be easily checked that

$$\begin{aligned}&\left(\left(1+\tau\sum_{k=1}^{j+1}\left(\frac{\partial U^k}{\partial x_i}\right)^2\right)\frac{\partial U^{j+1}}{\partial x_i} - \left(1+\tau\sum_{k=1}^{j+1}\left(\frac{\partial u_i^k}{\partial x_i}\right)^2\right)\frac{\partial u_i^{j+1}}{\partial x_i}, \frac{\partial z_i^{j+1}}{\partial x_i}\right)\\
&= \frac{1}{2}\left[\left(2+\tau\sum_{k=1}^{j+1}\left(\frac{\partial U^k}{\partial x_i}\right)^2 + \tau\sum_{k=1}^{j+1}\left(\frac{\partial u_i^k}{\partial x_i}\right)^2, \left(\frac{\partial z_i^{j+1}}{\partial x_i}\right)^2\right)\right.\\
&\quad\left. + \left(\tau\sum_{k=1}^{j+1}\left[\left(\frac{\partial U^k}{\partial x_i}\right)^2 - \left(\frac{\partial u_i^k}{\partial x_i}\right)^2\right], \left(\frac{\partial U^{j+1}}{\partial x_i}\right)^2 - \left(\frac{\partial u_i^{j+1}}{\partial x_i}\right)^2\right)\right]\\
&\geq \frac{1}{2}\left(\tau\sum_{k=1}^{j+1}\left[\left(\frac{\partial U^k}{\partial x_i}\right)^2 - \left(\frac{\partial u_i^k}{\partial x_i}\right)^2\right], \left(\frac{\partial U^{j+1}}{\partial x_i}\right)^2 - \left(\frac{\partial u_i^{j+1}}{\partial x_i}\right)^2\right).\end{aligned}$$

From (5.1.17), for the error we get

$$2\tau\eta_i(z_{i\bar t}^{j+1}, z_i^{j+1}) + \tau\left(\tau\sum_{k=1}^{j+1}\left[\left(\frac{\partial U^k}{\partial x_i}\right)^2 - \left(\frac{\partial u_i^k}{\partial x_i}\right)^2\right], \left(\frac{\partial U^{j+1}}{\partial x_i}\right)^2 - \left(\frac{\partial u_i^{j+1}}{\partial x_i}\right)^2\right) \leq 2\tau(\psi_i^{j+1}, z_i^{j+1}).$$

Using the identities

$$z_i^{j+1} = z^j + \tau z_{i\bar t}^{j+1}, \quad 2\tau(z_{i\bar t}^{j+1}, z_i^{j+1}) = \|z_i^{j+1}\|^2 + \tau^2\|z_{i\bar t}^{j+1}\|^2 - \|z^j\|^2,$$

after simple transformations, from the latter inequality, we have

$$\begin{aligned}&\eta_i\left(\|z_i^{j+1}\|^2 + \tau^2\|z_{i\bar t}^{j+1}\|^2\right)\\
&\quad + \frac{1}{2}\left\|\tau\sum_{k=1}^{j+1}\left[\left(\frac{\partial U^k}{\partial x_i}\right)^2 - \left(\frac{\partial u_i^k}{\partial x_i}\right)^2\right]\right\|^2 + \frac{\tau^2}{2}\left\|\left(\frac{\partial U^{j+1}}{\partial x_i}\right)^2 - \left(\frac{\partial u_i^{j+1}}{\partial x_i}\right)^2\right\|^2\\
&\qquad \leq \eta_i\|z^j\|^2 + \frac{1}{2}\left\|\tau\sum_{k=1}^{j}\left[\left(\frac{\partial U^k}{\partial x_i}\right)^2 - \left(\frac{\partial u_i^k}{\partial x_i}\right)^2\right]\right\|^2 + 2\tau\left(\psi_i^{j+1}, z^j + \tau z_{i\bar t}^{j+1}\right).\end{aligned}$$

Summing-up this equality from 1 to n, we arrive at

$$\begin{aligned}&\sum_{i=1}^n\eta_i\left(\|z_i^{j+1}\|^2 + \tau^2\|z_{i\bar t}^{j+1}\|^2\right) + \frac{1}{2}\sum_{i=1}^n\left\|\tau\sum_{k=0}^{j+1}\left[\left(\frac{\partial U^k}{\partial x_i}\right)^2 - \left(\frac{\partial u_i^k}{\partial x_i}\right)^2\right]\right\|^2\\
&\quad + \frac{\tau^2}{2}\sum_{i=1}^n\left\|\left(\frac{\partial U^{j+1}}{\partial x_i}\right)^2 - \left(\frac{\partial u_i^{j+1}}{\partial x_i}\right)^2\right\|^2\\
&\leq \sum_{i=1}^n\eta_i\|z^j\|^2 + \frac{1}{2}\sum_{i=1}^n\left\|\tau\sum_{k=1}^{j}\left[\left(\frac{\partial U^k}{\partial x_i}\right)^2 - \left(\frac{\partial u_i^k}{\partial x_i}\right)^2\right]\right\|^2 + 2\tau\sum_{i=1}^n(\psi_i^{j+1}, z^j)\\
&\quad + 2\tau\sum_{i=1}^n(\psi_i^{j+1}, \tau z_{i\bar t}^{j+1}).\end{aligned} \tag{5.1.18}$$

Note that

$$\sum_{i=1}^{n}\eta_i z_i^{j+1}=\sum_{i=1}^{n}\eta_i(U^{j+1}-u_i^{j+1})=z^{j+1},\quad \sum_{i=1}^{n}\eta_i\|z^j\|^2=\|z^j\|^2,$$
$$\sum_{i=1}^{n}\eta_i\|z_i^{j+1}\|^2\ge\Big\|\sum_{i=1}^{n}\eta_i z_i^{j+1}\Big\|^2=\|z^{j+1}\|^2.$$

Using these relations, identity of the sum-approximation (5.1.16) and the Schwarz inequality, from (5.1.18) we get

$$\begin{aligned}\|z^{j+1}\|^2&+\sum_{i=1}^{n}\eta_i\tau^2\|z_{i\bar t}^{j+1}\|^2+\frac12\sum_{i=1}^{n}\Big\|\tau\sum_{k=1}^{j+1}\Big[\Big(\frac{\partial U^k}{\partial x_i}\Big)^2-\Big(\frac{\partial u_i^k}{\partial x_i}\Big)^2\Big]\Big\|^2\\&\le\|z^j\|^2+\frac12\sum_{i=1}^{n}\Big\|\tau\sum_{k=1}^{j}\Big[\Big(\frac{\partial U^k}{\partial x_i}\Big)^2-\Big(\frac{\partial u_i^k}{\partial x_i}\Big)^2\Big]\Big\|^2\\&\quad+2\tau(O(\tau),z^j)+\tau^2\|\psi^{j+1}\|^2+\sum_{i=1}^{n}\eta_i\tau^2\|z_{i\bar t}^{j+1}\|^2.\end{aligned}$$

Here,

$$\|\psi^{j+1}\|^2=\sum_{i=1}^{n}\eta_i^{-1}\|\psi_i^{j+1}\|^2.$$

Using the boundedness of $\|\psi^{j+1}\|$, we find

$$\|z^{j+1}\|^2+\frac12\sum_{i=1}^{n}\Big\|\tau\sum_{k=1}^{j+1}\Big[\Big(\frac{\partial U^k}{\partial x_i}\Big)^2-\Big(\frac{\partial u_i^k}{\partial x_i}\Big)^2\Big]\Big\|^2 \tag{5.1.19}$$

$$\le\|z^j\|^2+\frac12\sum_{i=1}^{n}\Big\|\tau\sum_{k=1}^{j}\Big[\Big(\frac{\partial U^k}{\partial x_i}\Big)^2-\Big(\frac{\partial u_i^k}{\partial x_i}\Big)^2\Big]\Big\|^2+2\tau(O(\tau),z^j)+O(\tau^2). \tag{5.1.20}$$

Summing-up (5.1.20) with respect to j from 0 to $m-1$, we get

$$\begin{aligned}\|z^m\|^2+\frac{\tau}{2}\sum_{i=1}^{n}\Big\|\Big(\frac{\partial U^m}{\partial x_i}\Big)^2-\Big(\frac{\partial u_i^m}{\partial x_i}\Big)^2\Big\|^2&\le2\tau\sum_{j=0}^{m-1}(O(\tau),z^j)+C\tau\\&\le\tau\sum_{j=0}^{m-1}\big[O(\tau^2)+\|z^j\|^2\big]+O(\tau)\\&\le\tau\sum_{j=0}^{m-1}\|z^j\|^2+C\tau.\end{aligned} \tag{5.1.21}$$

The desired result of Theorem 5.1.2 now follows from (5.1.21) by the standard discrete Gronwall's lemma. □

5.1.4 Numerical implementation remarks

The study of the operator splitting techniques has a long history and has been pursued with various methods. Since alternating-direction methods and fractional step methods reduce the time-stepping of multi-dimensional problems to locally one-dimensional computations, those methods have been applied in the numerical simulation of many practically important problems.

Starting with the basic works (see, e.g., [14, 15]), the methods of constructing effective algorithms for the numerical solution of the multi-dimensional problems of the mathematical physics and a range of problems solvable with the help of those algorithms are essentially extended. At present, there are some effective algorithms for solving the multi-dimensional problems of the mathematical physics (see, e.g., [1, 57, 64, 72] and the references therein). Those algorithms belong mainly to the methods of splitting-up or sum-approximation according to their approximating properties.

Several test examples are carried out using semi-discrete additive scheme (5.1.14). Here we consider one of such test examples: consider the case $n = 2$, $\Omega = (0, 1) \times (0, 1)$ and choose the right-hand side of equation (5.1.1), thus the exact solution is given by

$$U(x_1, x_2, t) = x_1 x_2 (1 - x_1)(1 - x_2)(1 + t^2).$$

Parameters used here are $T = 1$, $\tau = 0.004$, and for the spatial discretization we used $h_1 = h_2 = 0.02$. Results of the test experiment for exact and numerical solution are given in Figures 5.1–5.4.

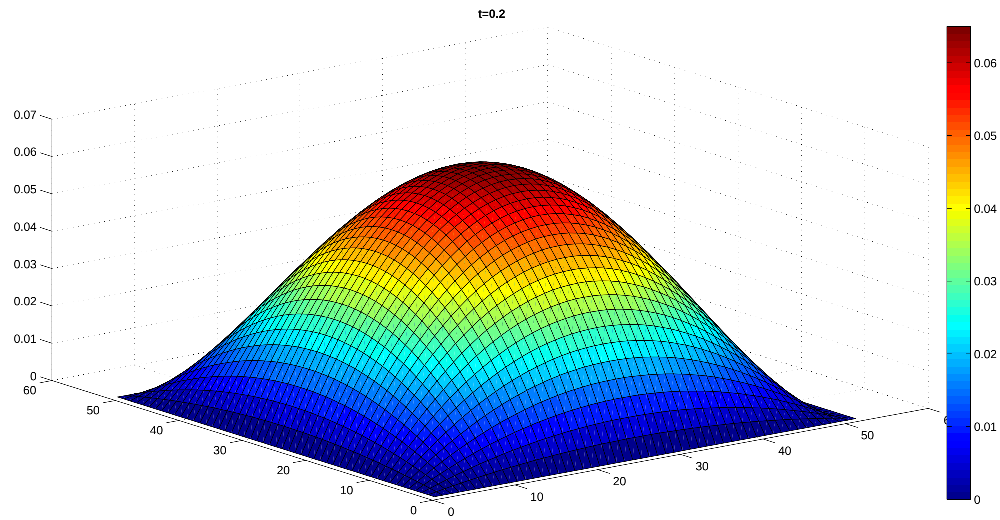

Figure 5.1. The exact solution for $t = 0.2$.

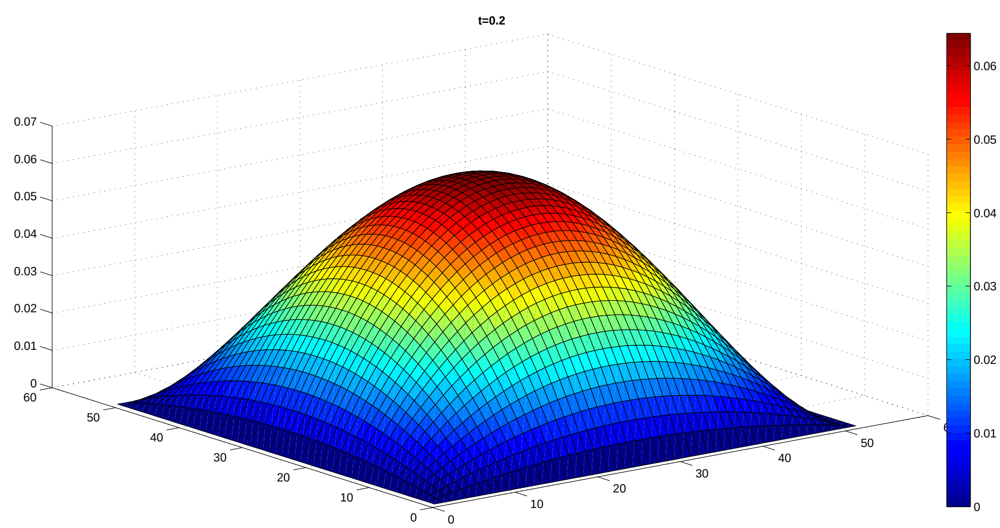

Figure 5.2. The numerical solution for $t = 0.2$.

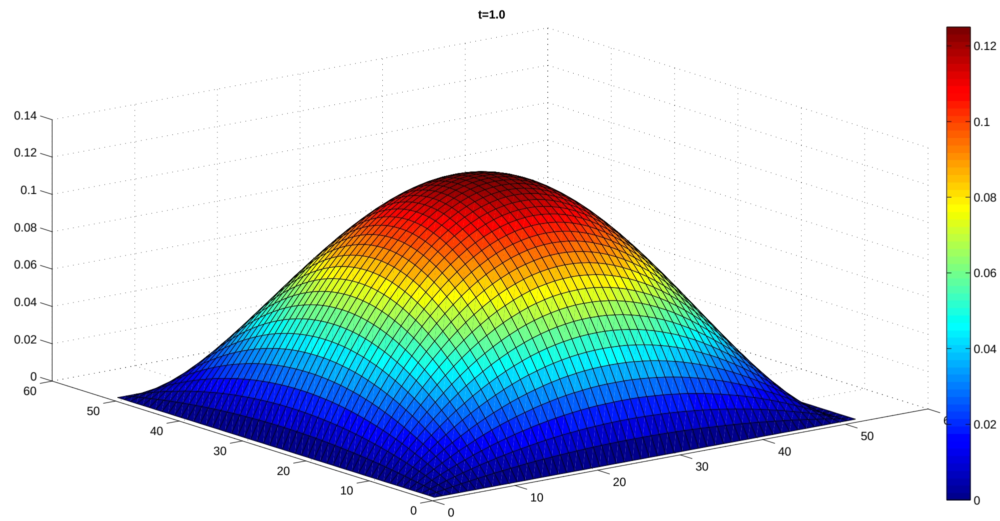

Figure 5.3. The exact solution for $t = 1.0$.

For the errors analysis the maximum of the absolute values of errors between the exact and numerical solutions for different time levels are shown in Figures 5.5 and 5.6. Absolute values of errors are given in Table 5.1, as well.

Table 5.1. Absolute values of errors.

t	Absolute values of errors between exact and numerical solutions
0.2	0.0010559595
0.4	0.0014377328
0.6	0.0021740239
0.8	0.0034338814
1.0	0.0053835245

We have carried out several other numerical experiments and observed the same situation.

In our next experiment we have taken the zero right-hand side and the initial solution given by

$$U(x_1, x_2, 0) = x_1 x_2 (1 - x_1)(1 - x_2)\big(e^{x_2 - x_1} - \cos(8\pi(x_1 - x_2))\big).$$

In this case, we know that the solution should decay in time. The parameters τ and h are as before. In Table 5.2, we put the values of numerical solution for the initial and five different time levels. It is clear that the numerical solution approaches zero, as it was expected according to the theoretical result given at the end of Subsection 5.1.2.

We will try to make some comments on the results we have obtained. The additive averaged semi-discrete scheme for nonlinear multi-dimensional IDE of parabolic type is studied. The investigated equation is a certain generalization of the integro-differential model which is based on the well-known system of Maxwell equations arising under the mathematical simulation of electromagnetic field penetration into a medium. The existence and uniqueness of the solution of the first type initial-boundary value problem are given. A long time behavior of the solution is fixed. For the

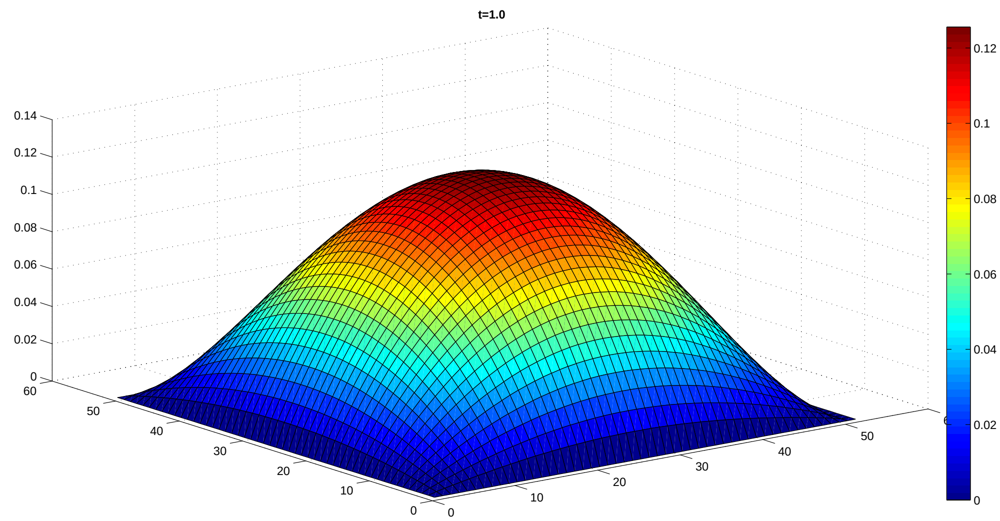

Figure 5.4. The numerical solution for $t = 1.0$.

Table 5.2. Asymptotic behavior of solution.

t	Numerical solution
0.0	0.12868
0.4	1.42398e-003
0.8	2.94541e-005
1.2	6.09236e-007
2.0	2.60654e-010

two-dimensional case, the numerical experiments agreeing with the theoretical findings, are presented. Numerical experiments illustrating asymptotic behavior as $t \to \infty$ of solutions are given, as well.

5.2 Some comments on the unique solvability and additive Rothe-type scheme for one nonlinear averaged integro-differential parabolic problem

The main attention of this section is paid to the Rothe-type additive averaged scheme for the averaged type multi-dimensional integro-differential scalar equation.

Let Ω be the bounded domain in the n-dimensional Euclidean space R^n, with a sufficiently smooth boundary $\partial\Omega$. In the domain $Q = \Omega \times (0,T)$ of variables $(x,t) = (x_1, x_2, \ldots, x_n, t)$, let us consider the following first type initial-boundary value problem:

$$\frac{\partial U}{\partial t} - \sum_{i=1}^{n} \left(1 + \int_{\Omega}\int_{0}^{t} \left| \frac{\partial U}{\partial x_i} \right|^2 dx\, d\tau \right) \frac{\partial^2 U}{\partial x_i^2} = f(x,t), \quad (x,t) \in Q, \tag{5.2.1}$$

$$U(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T], \tag{5.2.2}$$

$$U(x,0) = 0, \quad x \in \overline{\Omega}, \tag{5.2.3}$$

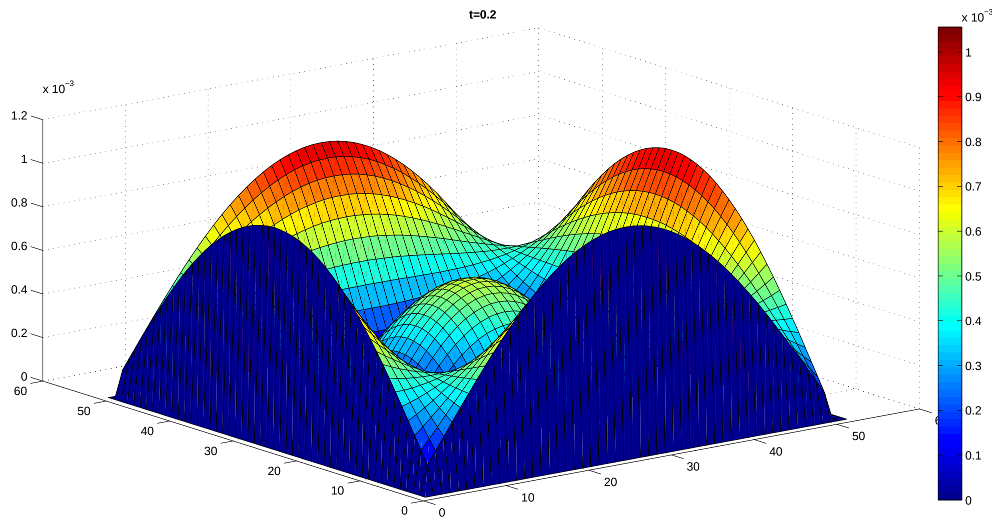

Figure 5.5. The differences between the exact and numerical solutions for $t = 0.2$.

where T is a fixed positive constant, f is a given function of its arguments.

Since problem (5.2.1)–(5.2.3) is similar to that considered in [32], where unique solvability and asymptotic behavior of (5.2.1) type models are discussed, we can follow the same procedure used there (see also Chapters 1, 3, 4). Using the modified version of Galerkin's method and the compactness arguments (Chapter 4 and [54, 71]), we can prove the following statement.

Theorem 5.2.1. If

$$f \in W_2^1(Q), \quad f(x,0) = 0,$$

then there exists the unique solution U of problem (5.2.1)–(5.2.3) satisfying the properties

$$U \in L_4(0,T;\overset{\circ}{W}{}_4^1(\Omega)) \cap L_2(0,T;W_2^2(\Omega)), \quad \frac{\partial U}{\partial t} \in L_2(Q),$$
$$\sqrt{T-t}\,\frac{\partial^2 U}{\partial t \partial x_i} \in L_2(Q), \quad i = 1,2,\dots,n.$$

Using the scheme of investigation as in Chapters 3 and 4, it is not difficult to get the exponential stability of solution as $t \to \infty$ for equation (5.2.1) with $f(x,t) \equiv 0$ and homogeneous boundary (5.2.2) and nonhomogeneous initial (5.2.3) type conditions.

For problem (5.2.1)–(5.2.3), let us construct the following additive averaged Rothe-type scheme:

$$\eta_i \,\frac{u_i^{j+1} - u^j}{\tau} = \left(1 + \tau \sum_{k=1}^{j+1} \int_\Omega \left|\frac{\partial u_i^k}{\partial x_i}\right|^2 dx\right) \frac{\partial^2 u_i^{j+1}}{\partial x_i^2} + f_i^{j+1},$$
$$u_i^0 = u^0 = 0, \quad i = 1,\dots,n; \quad j = 0,1,\dots,J-1, \tag{5.2.4}$$

with the homogeneous boundary conditions, where $u_i^j(x)$, $j = 1,2,\dots,N$, is the solution of problem (5.2.4), and the following notations are introduced:

$$u^j(x) = \sum_{i=1}^n \eta_i u_i^j(x), \quad \sum_{i=1}^n \eta_i = 1, \quad \eta_i > 0, \quad \sum_{i=1}^n f_i^{j+1}(x) = f^{j+1}(x) = f(x,t_{j+1}),$$

where u^j denotes approximation of the exact solution U of problem (5.2.1)–(5.2.3) for t_j. We use usual norm $\|\cdot\|$ of the space $L_2(\Omega)$.

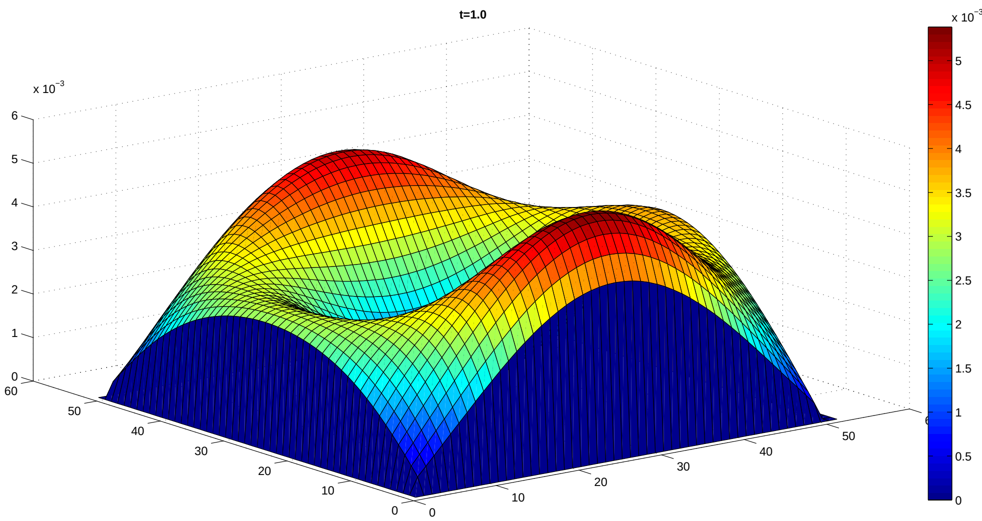

Figure 5.6. The differences between the exact and numerical solutions for $t = 1.0$.

Theorem 5.2.2. If problem (5.2.1)–(5.2.3) has a sufficiently smooth solution, then the solution of problem (5.2.4) converges to that of problem (5.2.1)–(5.2.3), and the following estimate is true:

$$\|U^j - u^j\| = O(\tau^{\frac{1}{2}}), \quad j = 1, 2, \ldots, N.$$

It is very important to construct and investigate the above-studied type of models for more general type of nonlinearities as well.

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