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INTERACTION PROBLEMS OF ACOUSTIC WAVES AND ELECTRO-MAGNETO-ELASTIC STRUCTURES

Abstract. In the paper, is consider a three-dimensional model of fluid-solid acoustic interaction when an electro-magneto-elastic body occupying a bounded region Ω^+ is embedded in an unbounded fluid domain $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega^+}$. In this case in the domain Ω^+ is a five-dimensional electro-magneto-elastic field (the displacement vector with three components, electric potential and magnetic potential), while in the unbounded domain Ω^- is a scalar acoustic pressure field. The physical kinematic and dynamic relations mathematically are described by appropriate boundary and transmission conditions. In the paper, less restrictions are considered on matrix differential operator of electro-magneto-elasticity and asymptotic classes are introduced. In particular, corresponding characteristic polynomial of the matrix differential operator can have multiple real zeros. With the help of the potential method and theory of pseudodifferential equations, for above mentioned fluid-solid acoustic interaction mathematical problems the uniqueness and existence theorems are proved in Sobolev–Slobodetskii spaces.

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Key words and phrases. Boundary-transmission problems, fluid-solid interaction, potential method, pseudodifferential equations, Helmholtz equation, steady state oscillations, Jones modes, Jones eigenfrequencies.

რეზიუმე. ნაშრომში განხილულია სითხისა და სხეულის აკუსტიკური ურთიერთქმედების სამგანზომილებიანი მოდელი, როდესაც ელექტრო-მაგნეტო-დრეკადი სხეულს უკავია Ω^+ შემოსაზღვრული არე, რომელიც ჩადგმულია $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega^+}$ შემოუსაზღვრელ არეში. ამ შემთხვევაში შემოსაზღვრულ Ω^+ არეში არის ხუთგანზომილებიანი ელექტრო-მაგნეტო-დრეკადი ველი (გადაადგილების ვექტორის სამი კომპონენტი, ელექტრული პოტენციალი და მაგნიტური პოტენციალი), ხოლო Ω^- შემოუსაზღვრელ არეში - აკუსტიკური წნევის სკალარული ველი. ფიზიკური კინემატიკური და დინამიკური ურთიერთქმედებები მათემატიკურად აღწერილია შესაბამისი სასაზღვრო და ტრანსმისიის პირობებით. ნაშრომში მოთხოვნილია ნაკლები შეზღუდვები ელექტრო-მაგნეტო-დრეკადობის დიფერენციალურ ოპერატორზე და შემოღებულია შესაბამისი ასიმპტოტური კლასები. კერძოდ, მატრიცული დიფერენციალური ოპერატორის შესაბამის მახასიათებელ პოლინომს შეიძლება გააჩნდეს ჯერადი ნამდვილი ნულები. პოტენციალთა მეთოდისა და ფსევდოდიფერენციალურ განტოლებათა თეორიის გამოყენებით დამტკიცებულია ზემოთ აღნიშნული სითხისა და სხეულის აკუსტიკური ურთიერთქმედების მათემატიკური ამოცანების ამონახსნების ერთადერთობისა და არსებობის თეორემები სობოლევ-სლობოდეცკის სივრცეებში.

1 Formulation of the problems

1.1 Introduction

Interaction problems of different dimensional fields of this type appear in mathematical models of electro-magneto transducers. Further examples of similar models are related to phased array microphones, ultrasound equipment, inkjet droplet actuators, sonar transducers, bioimaging, immunochemistry, and acousto-biotherapeutics (see [38, 39]).

Due to the rapidly increasing use of composite materials in modern industrial and technological processes on the one hand, and in biology and medicine on the other hand, mathematical modeling related to complex composite structures and their mathematical analysis became very important from the theoretical and practical points of view in recent years.

The Dirichlet, Neumann and mixed type interaction problems of acoustic waves and piezoelectric structures are studied in [9, 11, 12].

Similar interaction problems for the classical model of elasticity has been investigated by a number of authors. An exhaustive information concerning theoretical and numerical results, for the case when the both interacting media are isotropic, can be found in [1–4, 15, 17–19, 26, 27, 31]. The cases when the elastic body is homogeneous and anisotropic, and the fluid is isotropic, has been considered in [25, 35, 36]. In this case, one has a three-dimensional elastic field, the displacement vector with three components in the bounded domain Ω^+, and a scalar pressure field in the unbounded domain Ω^-.

In our case, in the domain Ω^+ we have an additional electric and magnetic fields which essentially complicate the investigation of the transmission problems in question. In contrast to the classical elasticity, the differential operator of electro-magneto-elasticity is not self-adjoint and is not positive-definite.

We consider less restrictions on the matrix differential operator of electro-magneto-elasticity by introducing asymptotic classes $M_{m_1,m_2,m_3}(\mathbf{P})$, where $\mathbf{P}$ is determinant of the electro-magneto-elasticity matrix operator, in particular, we allow for the corresponding characteristic polynomial of the matrix differential operator to have multiple real zeros. This class is generalization of the Sommerfeld-Kupradze class.

We investigate the above problems with the use of the boundary integral equations method and the theory of pseudodifferential equations on manifolds and prove the existence and uniqueness theorems in Sobolev–Slobodetskii spaces.

1.2 Piezoelectric field

Let Ω^+ be a bounded three-dimensional domain in $\mathbb{R}^3$ with a compact C^∞-smooth boundary $S = \partial\Omega^+$ and let $\Omega^- := \mathbb{R}^3 \setminus \overline{\Omega^+}$. Assume that the domain Ω^+ is filled with an anisotropic homogeneous piezoelectro-magnetic material.

The basic equations of steady state oscillations of piezoelectro-magneticity for anisotropic homogeneous media are written as follows:

$$\begin{gathered}
c_{ijkl}\partial_i\partial_l u_k + \rho_1\omega^2\delta_{jk}u_k + e_{lij}\partial_l\partial_i\varphi + q_{lij}\partial_i\partial_l\psi + F_j = 0, \quad j = 1, 2, 3, \\
-e_{ikl}\partial_i\partial_l u_k + \varepsilon_{il}\partial_i\partial_l\varphi + a_{il}\partial_i\partial_l\psi + F_4 = 0, \\
-q_{ikl}\partial_i\partial_l u_k + a_{il}\partial_i\partial_l\varphi + \mu_{il}\partial_i\partial_l\psi + F_5 = 0,
\end{gathered}$$

or in the matrix form

$$A(\partial,\omega)U + F = 0 \ \text{ in } \Omega^+,$$

where $U = (u,\varphi,\psi)^\top$, $u = (u_1,u_2,u_3)^\top$ is the displacement vector, $\varphi = u_4$ is the electric potential, $\psi = u_5$ is the magnetic potential and $F = (F_1,F_2,F_3,F_4,F_5)^\top$ is a given vector-function. The three-dimensional vector (F_1,F_2,F_3) is the mass force density, while $-F_4$ is the electric charge density, $-F_5$

is the electric current density, and $A(\partial,\omega)$ is the matrix differential operator,

$$
\begin{gathered}
A(\partial,\omega)=[A_{jk}(\partial,\omega)]_{5\times 5}, \qquad (1.1)\\
A_{jk}(\partial,\omega)=c_{ijkl}\partial_i\partial_l+\rho_1\omega^2\delta_{jk},\quad A_{j4}(\partial,\omega)=e_{lij}\partial_l\partial_i,\quad A_{j5}(\partial,\omega)=q_{lij}\partial_l\partial_i,\\
A_{4k}(\partial,\omega)=-e_{ikl}\partial_i\partial_l,\quad A_{44}(\partial,\omega)=\varepsilon_{il}\partial_i\partial_l,\quad A_{45}(\partial,\omega)=a_{il}\partial_i\partial_l,\\
A_{5k}(\partial,\omega)=-q_{ikl}\partial_i\partial_l,\quad A_{54}(\partial,\omega)=a_{il}\partial_i\partial_l,\quad A_{55}(\partial,\omega)=\mu_{il}\partial_i\partial_l,
\end{gathered}
$$

$j,k=1,2,3$, where $\omega\in\mathbb{R}$ is a frequency parameter, ρ_1 is the density of the piezoelectro-magnetic material, c_{ijlk}, e_{ikl}, q_{ikl}, ε_{il}, μ_{il}, a_{il} are elastic, piezoelectric, piezomagnetic, dielectric, magnetic permeability and electromagnetic coupling constants, respectively, δ_{jk} is the Kronecker symbol and summation over repeated indices is meant from 1 to 3, if not stated otherwise. These constants satisfy the standard symmetry conditions

$$
c_{ijkl}=c_{jikl}=c_{klij},\ \ e_{ijk}=e_{ikj},\ \ q_{ijk}=q_{ikj},\ \ \varepsilon_{ij}=\varepsilon_{ji},\ \ \mu_{jk}=\mu_{kj},\ \ a_{jk}=a_{kj},\ \ i,j,k,l=1,2,3.
$$

Moreover, from physical considerations related to positiveness of the internal energy, it follows that the quadratic forms $c_{ijkl}\xi_{ij}\xi_{kl}$ and $\varepsilon_{ij}\eta_i\eta_j$ are positive definite:

$$
c_{ijkl}\xi_{ij}\xi_{kl}\geq c_0\xi_{ij}\xi_{ij}\ \ \forall\,\xi_{ij}=\xi_{ji}\in\mathbb{R}, \qquad (1.2)
$$

$$
\varepsilon_{ij}\eta_i\eta_j\geq c_2|\eta|^2,\ \ q_{ij}\eta_i\eta_j\geq c_3|\eta|^2,\ \ \mu_{ij}\eta_i\eta_j\geq c_1|\eta|^2\ \ \forall\,\eta=(\eta_1,\eta_2,\eta_3)\in\mathbb{R}^3, \qquad (1.3)
$$

where c_0, c_1, c_2 and c_3 are positive constants.

More careful analysis related to the positive definiteness of the potential energy insures that the matrix

$$
\Lambda:=\begin{pmatrix}[\varepsilon_{kj}]_{3\times 3} & [a_{kj}]_{3\times 3}\\ [a_{kj}]_{3\times 3} & [\mu_{kj}]_{3\times 3}\end{pmatrix}_{6\times 6}
$$

is positive definite, i.e.,

$$
\varepsilon_{kj}\zeta'_k\overline{\zeta'_j}+a_{kj}\big(\zeta'_k\overline{\zeta''_j}+\overline{\zeta'_k}\zeta''_j\big)+\mu_{kj}\zeta''_k\overline{\zeta''_j}\geq c_4\big(|\zeta'|^2+|\zeta''|^2\big)\ \ \forall\,\zeta',\zeta''\in\mathbb{C}^3, \qquad (1.4)
$$

where c_4 some positive constant.

The principal homogeneous symbol matrix of the operator $A(\partial,\omega)$ has the following form:

$$
A^{(0)}(\xi)=\begin{pmatrix}[-c_{ijlk}\xi_i\xi_l]_{3\times 3} & [-e_{lij}\xi_l\xi_i]_{3\times 1} & [-q_{lij}\xi_l\xi_i]_{3\times 1}\\ [e_{ikl}\xi_i\xi_l]_{1\times 3} & -\varepsilon_{il}\xi_i\xi_l & -a_{il}\xi_i\xi_l\\ [q_{ikl}\xi_i\xi_l]_{1\times 3} & -a_{il}\xi_i\xi_l & -\mu_{il}\xi_i\xi_l\end{pmatrix}_{5\times 5}.
$$

With the help of inequalities (1.2) and (1.3) it can be easily shown that

$$
-\operatorname{Re}A^{(0)}(\xi)\zeta\cdot\zeta\geq c|\zeta|^2|\xi|^2\ \ \forall\,\zeta\in\mathbb{C}^4,\ \ \forall\,\xi\in\mathbb{R}^3,\ \ c=const>0,
$$

implying that $A(\partial,\omega)$ is a strongly elliptic, formally nonselfadjoint differential operator.

Here and in the sequel, $a\cdot b$ denotes the scalar product of two vectors $a,b\in\mathbb{C}^N$, $a\cdot b:=\sum\limits_{k=1}^{N}a_k\overline{b}_k$.

In the theory of electro-magneto-elasticity, the components of the three-dimensional mechanical stress vector acting on a surface element with a normal $n=(n_1,n_2,n_3)$ have the form

$$
\sigma_{ij}n_i:=c_{ijlk}n_i\partial_l u_k+e_{lij}n_i\partial_l\varphi+q_{lij}n_i\partial_l\psi,\ \ j=1,2,3,
$$

while the normal component of the electric displacement vector $D=(D_1,D_2,D_3)^\top$ and the normal component of the magnetic induction vector $B=(B_1,B_2,B_3)^\top$ read as

$$
\begin{aligned}
-D_in_i&=-e_{ikl}n_i\partial_l u_k+\varepsilon_{il}n_i\partial_l\varphi+a_{il}n_i\partial_l\psi,\\
-B_in_i&=-q_{ikl}n_i\partial_l u_k+a_{il}n_i\partial_l\varphi+\mu_{il}n_i\partial_l\psi.
\end{aligned}
$$

Let us introduce the boundary matrix differential operator

$$
\begin{gathered}
T(\partial,n)=[T_{jk}(\partial,n)]_{5\times 5},\\
T_{jk}(\partial,n)=c_{ijlk}n_i\partial_l,\quad T_{j4}(\partial,n)=e_{lij}n_i\partial_l,\quad T_{j5}(\partial,n)=q_{lij}n_i\partial_l,\\
T_{4k}(\partial,n)=-e_{ikl}n_i\partial_l,\quad T_{44}(\partial,n)=\varepsilon_{il}n_i\partial_l,\quad T_{45}(\partial,n)=a_{il}n_i\partial_l,\\
T_{5k}(\partial,n)=-q_{ikl}n_i\partial_l,\quad T_{54}(\partial,n)=a_{il}n_i\partial_l,\quad T_{55}(\partial,n)=\mu_{il}n_i\partial_l,
\end{gathered}
$$

$j,k=1,2,3$. For a vector $U=(u,\varphi,\psi)^\top$, we have

$$
T(\partial,n)U=(\sigma_{1j}n_j,\sigma_{2j}n_j,\sigma_{3j}n_j,-D_in_i,\,-B_in_i)^\top. \tag{1.5}
$$

The components of the vector TU given by (1.5) have the following physical sense: the first three components correspond to the mechanical stress vector in the theory of electro-magneto-elasticity, while the fourth one is the normal component of the electric displacement vector and the fifth one is the normal component of the magnetic induction vector.

In Green's formulae, one also has the following boundary operator associated with the adjoint differential operator $A^*(\partial,\omega)=A^\top(-\partial,\omega)=A^\top(\partial,\omega)$,

$$
\widetilde{T}(\partial,n)=[\widetilde{T}_{jk}(\partial,n)]_{5\times 5},
$$

where

$$
\begin{gathered}
\widetilde{T}_{jk}(\partial,n)=T_{jk}(\partial,n),\quad \widetilde{T}_{j4}(\partial,n)=-T_{j4}(\partial,n),\quad \widetilde{T}_{j5}(\partial,n)=-T_{j5}(\partial,n),\\
\widetilde{T}_{4k}(\partial,n)=-T_{4k}(\partial,n),\quad \widetilde{T}_{44}(\partial,n)=T_{44}(\partial,n),\quad \widetilde{T}_{45}(\partial,n)=T_{45}(\partial,n),\\
\widetilde{T}_{5k}(\partial,n)=-T_{5k}(\partial,n),\quad \widetilde{T}_{54}(\partial,n)=T_{54}(\partial,n),\quad \widetilde{T}_{55}(\partial,n)=T_{55}(\partial,n),
\end{gathered}
$$

$j,k=1,2,3$.

1.3 Green's formulae for electro-magneto-elastic vector fields

For arbitrary vector-functions $U=(u_1,u_2,u_3,u_4,u_5)^\top\in[C^2(\overline{\Omega^+})]^5$ and $V=(v_1,v_2,v_3,v_4,v_5)^\top\in[C^2(\overline{\Omega^+})]^5$, we have the following Green's formulae (see [6]):

$$
\begin{gathered}
\int\limits_{\Omega^+}\big[A(\partial,\omega)U\cdot V+E(U,\overline{V})\big]\,dx=\int\limits_S\{TU\}^+\cdot\{V\}^+\,dS,\\
\int\limits_{\Omega^+}\big[A(\partial,\omega)U\cdot V-U\cdot A^*(\partial,\omega)V\big]\,dx=\int\limits_S\big[\{TU\}^+\cdot\{V\}^+-\{U\}^+\cdot\{\widetilde{T}V\}^+\big]\,dS,
\end{gathered}
$$

where

$$
\begin{aligned}
E(U,\overline{V})=c_{ijlk}\partial_iu_j\partial_l\overline{v}_k-\rho_1\omega^2u\cdot v+e_{lij}(\partial_lu_4\partial_i\overline{v}_j-\partial_iu_j\partial_l\overline{v}_4)\\
+q_{lij}(\partial_lu_5\partial_i\overline{v}_j-\partial_iu_j\partial_l\overline{v}_5)+\varepsilon_{jl}\partial_ju_4\partial_l\overline{v}_4+a_{jl}(\partial_lu_4\partial_j\overline{v}_5-\partial_ju_5\partial_l\overline{v}_4)+\mu_{jl}\partial_ju_5\partial_l\overline{v}_5
\end{aligned}
$$

with $u=(u_1,u_2,u_3)^\top$ and $v=(v_1,v_2,v_3)^\top$. The symbol $\{\,\cdot\,\}^+$ denotes the one-sided limits (the trace operator) on S from Ω^+. Note that by the standard limiting procedure, the above Green's formulae can be generalized to the vector-functions $U\in[H^1(\Omega^+)]^5$ and $V\in[H^1(\Omega^+)]^5$ with $A(\partial,\omega)U\in[L_2(\Omega^+)]^5$ and $A^*(\partial,\omega)V\in[L_2(\Omega^+)]^5$.

With the help of these Green's formulae, we can define a generalized trace vector $\{T(\partial,n)U\}^+\in[H^{-1/2}(S)]^5$ for a function $U\in[H^1(\Omega^+)]^5$ with $A(\partial,\omega)U\in[L_2(\Omega^+)]^5$:

$$
\big\langle\{T(\partial,n)U\}^+,\{V\}^+\big\rangle_S:=\int\limits_{\Omega^+}\big[A(\partial,\omega)U\cdot V+E(U,\overline{V})\big]\,dx,
$$

where $V\in[H^1(\Omega^+)]^5$ is an arbitrary vector-function.

Here and in what follows, the symbol $\langle\,\cdot\,,\,\cdot\,\rangle_S$ denotes the duality between the mutually adjoint function spaces $[H^{-1/2}(S)]^N$ and $[H^{1/2}(S)]^N$, which extends the usual L_2 scalar product

$$\langle f,g\rangle_S=\int\limits_S\sum_{j=1}^N f_j\overline{g}_j\,dS \ \text{ for } \ f,g\in[L_2(S)]^N.$$

1.4 Scalar acoustic pressure field and Green's formulae

We assume that the exterior domain Ω^- is filled with a homogeneous isotropic inviscid fluid medium with the constant density ρ_2. Further, let the propagation of acoustic wave in Ω^- be described by a complex-valued scalar function (scalar field) w, being a solution of the homogeneous Helmholtz equation

$$\Delta\mathrm{w}+\rho_2\omega^2\mathrm{w}=0 \ \text{ in } \Omega^-, \tag{1.6}$$

where $\Delta=\sum\limits_{j=1}^3\frac{\partial^2}{\partial x_j^2}$ is the Laplace operator and $\omega>0$. The function $\mathrm{w}(x)=P^{sc}(x)$ is the pressure of a scattered acoustic wave.

We say that a solution w to the Helmholtz equation (1.6) belongs to the class $Som_p(\Omega^-)$, $p=1,2$, if w satisfies the classical Sommerfeld radiation condition

$$\frac{\partial\mathrm{w}(x)}{\partial|x|}+i(-1)^p\sqrt{\rho_2}\,\omega\mathrm{w}(x)=O(|x|^{-2}) \ \text{ as } \ |x|\to\infty. \tag{1.7}$$

Note that if a solution w of the Helmholtz equation (1.6) in Ω^- satisfies the Sommerfeld radiation condition (1.7), then (see [43])

$$\mathrm{w}(x)=O(|x|^{-1}) \ \text{ as } \ |x|\to\infty.$$

Let Ω be a domain in $\mathbb{R}^3$ with a compact simply connected boundary $\partial\Omega\in C^\infty$.

We denote by $H^s(\Omega)$ ($H^s_{loc}(\Omega)$) and $H^s(\partial\Omega)$ $s\in\mathbb{R}$, the L_2 based Sobolev–Slobodetskii (Bessel potential) spaces in Ω and on the closed manifold $\partial\Omega$.

Respectively, we denote by $H^s_{comp}(\Omega)$ the subspace of $H^s(\Omega)$ ($H^s_{loc}(\Omega)$) consisting of functions with compact supports.

If M is a smooth proper submanifold of a manifold $\partial\Omega$, then we denote by $\widetilde{H}^s(M)$ the following subspace of $H^s(\partial\Omega)$:

$$\widetilde{H}^s(M):=\big\{g:\ g\in H^s(\partial\Omega),\ \operatorname{supp}g\subset\overline{M}\big\},$$

while $H^s(M)$ denotes the space of restrictions to M of functions from $H^s(\partial\Omega)$,

$$H^s(M):=\big\{r_Mf:\ f\in H^s(\partial\Omega)\big\},$$

where r_M is the restriction operator to M.

Let $\mathrm{w}_1\in H^1_{loc}(\Omega^-)\cap Som_p(\Omega^-)$, $p=1,2,$, $\Delta\mathrm{w}_1\in L_{2,loc}(\Omega^-)$, $\mathrm{w}_2\in H^1_{comp}(\overline{\Omega^-})$, then the following Green's first formula holds:

$$\int\limits_{\Omega^-}(\Delta+k^2)\mathrm{w}_1\overline{\mathrm{w}}_2\,dx+\int\limits_{\Omega^-}\nabla\mathrm{w}_1\nabla\overline{\mathrm{w}}_2\,dx-k^2\int\limits_{\Omega^-}\mathrm{w}_1\overline{\mathrm{w}}_2\,dx=-\big\langle\{\partial_n\mathrm{w}_1\}^-,\{\mathrm{w}_2\}^-\big\rangle_S, \tag{1.8}$$

where $n=(n_1,n_2,n_3)$ is the exterior unit normal vector to S directed outward with respect to the domain Ω^+, and $\partial_n=\frac{\partial}{\partial n}$ denotes the normal derivative.

1.5 Formulation of the Dirichlet and Neumann type interaction problems for steady state oscillation equations

Now we formulate the fluid-solid interaction problems. We assume that $S=\partial\Omega^+=\partial\Omega^-\in C^\infty$.

Dirichlet type problem (D_ω)**:** Find a vector-function $U = (u, u_4, u_5)^\top = (u, \varphi, \psi)^\top \in [H^1(\Omega^+)]^5$ and a scalar function $\mathrm{w} \in H^1_{loc}(\Omega^-) \cap Som_1(\Omega^-)$ satisfying the differential equations

$$A(\partial, \omega)U = 0 \ \text{ in } \Omega^+, \tag{1.9}$$

$$\Delta\mathrm{w} + \rho_2\omega^2\mathrm{w} = 0 \text{in } \Omega^-, \tag{1.10}$$

the transmission conditions

$$\{u \cdot n\}^+ = b_1\{\partial_n \mathrm{w}\}^- + f_0 \ \text{ on } S, \tag{1.11}$$

$$\big\{[T(\partial, n)U]_j\big\}^+ = b_2\{\mathrm{w}\}^- n_j + f_j \ \text{ on } S, \ \ j = 1, 2, 3, \tag{1.12}$$

and the Dirichlet boundary conditions

$$\{\varphi\}^+ = f_1^{(D)} \ \text{ on } S, \tag{1.13}$$

$$\{\psi\}^+ = f_2^{(D)} \ \text{ on } S, \tag{1.14}$$

where b_1 and b_2 are the given complex constants satisfying the conditions

$$b_1 b_2 \neq 0 \ \text{ and } \ \mathrm{Im}[\bar{b}_1 b_2] = 0, \tag{1.15}$$

and $f_0 \in H^{-1/2}(S)$, $f_j \in H^{-1/2}(S)$, $j = 1, 2, 3$, $f_1^{(D)} \in H^{1/2}(S)$, $f_2^{(D)} \in H^{1/2}(S)$.

Neumann type problem (N_ω)**:** Find a vector-function $U = (u, u_4, u_5) = (u, \varphi, \psi)^\top \in [H^1(\Omega^+)]^5$ and a scalar function $\mathrm{w} \in H^1_{loc}(\Omega^-) \cap Som_1(\Omega^-)$ satisfying the differential equations (1.9), (1.10), the transmission conditions (1.11), (1.12) and the Neumann boundary conditions

$$\big\{[T(\partial, n)U]_4\big\}^+ = f_1^{(N)} \ \text{ on } S, \tag{1.16}$$

$$\big\{[T(\partial, n)U]_5\big\}^+ = f_2^{(N)} \ \text{ on } S, \tag{1.17}$$

where b_1 and b_2 are the given complex constants satisfying conditions (1.15), and $f_0 \in H^{-1/2}(S)$, $f_j \in H^{-1/2}(S)$, $j = 1, 2, 3$, $f_1^{(N)} \in H^{-1/2}(S)$, $f_2^{(N)} \in H^{-1/2}(S)$.

The transmission conditions (1.11), (1.12) are called the *kinematic and dynamic* conditions. For an interaction problem of fluid and electro-magneto-elastic body

$$\begin{gathered} b_1 = [\rho_2\omega^2]^{-1}, \quad b_2 = -1, \quad f_0(x) \equiv f_0^{inc}(x) = [\rho_2\omega^2]^{-1}\partial_n P^{inc}(x), \\ f_j = -P^{inc}(x)\, n_j(x), \ \ j = 1, 2, 3, \end{gathered} \tag{1.18}$$

where P^{inc} is an incident plane wave,

$$P^{inc}(x) = e^{id\cdot x}, \quad d = \omega\sqrt{\rho_2}\,\eta, \ \ \eta \in \mathbb{R}^3, \ \ |\eta| = 1.$$

2 The uniqueness of solutions of the problems (D_ω) and (N_ω)

2.1 Jones modes and Jones eigenfrequencies

We denote by $J_D(\Omega^+)$ the set of values of the frequency parameter $\omega > 0$ for which the following boundary value problem

$$A(\partial, \omega)U = 0 \ \text{ in } \Omega^+, \tag{2.1}$$

$$\{u \cdot n\}^+ = 0 \ \text{ on } S, \tag{2.2}$$

$$\big\{[T(\partial, n)U]_j\big\}^+ = 0 \ \text{ on } S, \ \ j = 1, 2, 3, \tag{2.3}$$

$$\{\varphi\}^+ = 0 \ \text{ on } S, \tag{2.4}$$

$$\{\psi\}^+ = 0 \ \text{ on } S, \tag{2.5}$$

has a nontrivial solution $U = (u, \varphi, \psi)^\top \in [H^1(\Omega^+)]^5$ (cf. [25]).

We denote by $J_N(\Omega^+)$ the set of values of the frequency parameter $\omega > 0$ for which the following boundary value problem

$$A(\partial, \omega)U = 0 \ \text{ in } \Omega^+, \tag{2.6}$$

$$\{u \cdot n\}^+ = 0 \ \text{ on } S, \tag{2.7}$$

$$\big\{[T(\partial, n)U]\big\}^+ = 0 \ \text{ on } S, \tag{2.8}$$

has a nontrivial solution $U = (u, \varphi, \psi)^\top \in [H^1(\Omega^+)]^5$ (cf. [25]).

Nontrivial solutions of problems (2.1)–(2.5) and (2.6)–(2.8) will be referred as *Jones modes*, while the corresponding values of ω are called *Jones eigenfrequencies*, as they were first discussed by D. S. Jones [25] in a related context (a thin layer of ideal fluid between an elastic body and a surrounding elastic exterior). For example, Jones eigenfrequencies exist for any axisymmetric body, such bodies can sustain torsional oscillations in which only the azimuthal component of displacement is nonzero. However, we do not expect Jones eigenfrequencies to exist for an arbitrary body. The spaces of Jones modes corresponding to ω we denote by $X_{D,\omega}(\Omega^+)$ and $X_{N,\omega}(\Omega^+)$, respectively.

Let $J^*_D(\Omega^+)$ be the set of values of the frequency parameter $\omega > 0$ for which the following boundary value problem

$$A^*(\partial, \omega)V = 0 \ \text{ in } \Omega^+, \tag{2.9}$$

$$\{v \cdot n\}^+ = 0 \ \text{ on } S, \tag{2.10}$$

$$\big\{[\widetilde{T}(\partial, n)V]_j\big\}^+ = 0 \ \text{ on } S, \ \ j = 1, 2, 3, \tag{2.11}$$

$$\{v_4\}^+ = 0 \ \text{ on } S, \tag{2.12}$$

$$\{v_5\}^+ = 0 \ \text{ on } S \tag{2.13}$$

has a nontrivial solution $V = (v, v_4, v_5)^\top \in [H^1(\Omega^+)]^5$.

Let $J^*_N(\Omega^+)$ be the set of values of the frequency parameter $\omega > 0$ for which the following boundary value problem

$$A^*(\partial, \omega)V = 0 \ \text{ in } \Omega^+, \tag{2.14}$$

$$\{v \cdot n\}^+ = 0 \ \text{ on } S, \tag{2.15}$$

$$\big\{[\widetilde{T}(\partial, n)V]\big\}^+ = 0 \ \text{ on } S \tag{2.16}$$

has a nontrivial solution $V = (v, v_4, v_5)^\top \in [H^1(\Omega^+)]^5$.

The spaces of Jones modes corresponding to ω for the differential operator $A^*(\partial, \omega)$ we denote by $X^*_{D,\omega}(\Omega^+)$, and $X^*_{N,\omega}(\Omega^+)$, respectively.

It can be shown that $J_D(\Omega^+)$ is at most countable, while $J_N(\Omega^+) \equiv \mathbb{R}$, since for an arbitrary nonzero constants c_1 and c_2, the vector $(0, 0, 0, c_1, c_2)^\top$ is a Jones eigenvector: $(0, 0, 0, c_1, c_2)^\top \in X_{N,\omega}(\Omega^+)$ for arbitrary ω. The same is true for $J^*_D(\Omega^+)$ and $J^*_N(\Omega^+)$. Note that for each ω the corresponding spaces of Jones modes $X_{D,\omega}(\Omega^+)$, $X_{N,\omega}(\Omega^+)$, $X^*_{D,\omega}(\Omega^+)$ and $X^*_{N,\omega}(\Omega^+)$ are of a finite dimension.

2.2 The uniqueness theorems for the problems (D_ω) and (N_ω)

Theorem 2.1. *Let a pair (U, w) be a solution of the homogeneous problem (D_ω) and $\omega > 0$. Then $\mathrm{w} = 0$ in Ω^- and either $U = 0$ in Ω^+ if $\omega \notin J_D(\Omega^+)$ or $U \in X_{D,\omega}(\Omega^+)$ if $\omega \in J_D(\Omega^+)$.*

Proof. Let us write Green's formula for the Helmholtz equation in the domain $\Omega_R := \Omega^- \cap B(0, R)$, where $\overline{\Omega^+} \subset B(0, R)$ with $B(0, R)$ being the ball of radius R and centered at the origin,

$$\int\limits_{\Omega_R} \big[(\Delta + \rho_2\omega^2)\mathrm{w}\overline{\mathrm{w}} - \mathrm{w}(\Delta + \rho_2\omega^2)\overline{\mathrm{w}}\big]\, dx = \int\limits_{S(0,R)} \partial_n \mathrm{w}\overline{\mathrm{w}}\, dS - \int\limits_{S(0,R)} \partial_n \overline{\mathrm{w}}\, \mathrm{w}\, dS - \big\langle \{\partial_n \mathrm{w}\}^-, \{\mathrm{w}\}^- \big\rangle_S + \big\langle \{\partial_n \overline{\mathrm{w}}\}^-, \{\overline{\mathrm{w}}\}^- \big\rangle_S, \tag{2.17}$$

where $S(0,R) = \partial B(0,R)$ is the boundary of the ball $B(0,R)$.

We have also the following Green's formula for the operator $A(\partial,\omega)$ in the domain Ω^+:

$$\int\limits_{\Omega^+} \Big[[A(\partial,\omega)U]_j \overline{u}_j + [\overline{A(\partial,\omega)U}]_4 u_4 + [\overline{A(\partial,\omega)U}]_5 u_5 + \mathcal{E}(U,\overline{U}) \Big]\, dx$$
$$= \big\langle \{TU\}_j^+, \{u_j\}^+ \big\rangle_S + \big\langle \{\overline{TU}\}_4^+, \{\overline{u}_4\}^+ \big\rangle_S + \big\langle \{\overline{TU}\}_5^+, \{\overline{u}_5\}^+ \big\rangle_S, \quad (2.18)$$

where $\mathcal{E}(U,\overline{U}) = c_{ijlk}\partial_i u_j \partial_l \overline{u}_k - \rho_1\omega^2|u|^2 + \varepsilon_{il}\partial_i u_4 \partial_l \overline{u}_4 + \mu_{jl}\partial_j u_5 \partial_l \overline{u}_5$. Clearly, $\operatorname{Im}\mathcal{E}(U,\overline{U}) = 0$ for an arbitrary vector-function U.

With the help of (1.9), (1.10), (1.13), and (1.14), we obtain from (2.17) and (2.18) the following equalities:

$$\int\limits_{S(0,R)} \partial_n \mathrm{w}\overline{\mathrm{w}}\, dS - \int\limits_{S(0,R)} \partial_n \overline{\mathrm{w}}\mathrm{w}\, dS - \big\langle \{\partial_n \mathrm{w}\}^-, \{\mathrm{w}\}^- \big\rangle_S + \big\langle \{\partial_n \overline{\mathrm{w}}\}^-, \{\overline{\mathrm{w}}\}^- \big\rangle_S = 0, \quad (2.19)$$

$$\operatorname{Im} \big\langle \{[TU]_j\}^+, \{u_j\}^+ \big\rangle_S = 0. \quad (2.20)$$

The homogeneous transmission conditions yield

$$\big\langle \{[TU]_j\}^+, \{u_j\}^+ \big\rangle_S = \big\langle b_2\{\mathrm{w}\}^- n_j, \{u_j\}^+ \big\rangle_S = b_2\overline{b}_1 \big\{\partial_n \overline{\mathrm{w}}\big\}^-, \{\overline{\mathrm{w}}\}^- \big\rangle_S. \quad (2.21)$$

Since $\operatorname{Im}[\overline{b}_1 b_2] = 0$, from (2.20) and (2.21) it follows that

$$\operatorname{Im} \big\langle \{\partial_n \overline{\mathrm{w}}\}^-, \{\overline{\mathrm{w}}\}^- \big\rangle_S = 0,$$

and from (2.19) we derive that

$$\operatorname{Im} \int\limits_{S(0,R)} \partial_n \overline{\mathrm{w}}\mathrm{w}\, dS = 0. \quad (2.22)$$

Taking into account the Sommerfeld radiation condition, from (2.22) we conclude that

$$\lim_{R\to\infty} \int\limits_{S(0,R)} |\mathrm{w}|^2\, dS = 0.$$

Using the Rellich-Vekua lemma, we find that $\mathrm{w} = 0$ in the domain Ω^- (see [13, 43]). Then from the homogeneous boundary conditions it follows that the vector-function $U = (u,\varphi,\psi)^\top$ solves problem (2.1)–(2.4), i.e., either $U = 0$ in Ω^+ if $\omega \notin J_D(\Omega^+)$ or $U \in X_{D,\omega}(\Omega^+)$ if $\omega \in J_D(\Omega^+)$, which completes the proof. □

The following assertions can be proved quite analogously.

Theorem 2.2. *Let a pair (U,w) be a solution of the homogeneous problem (N_ω). Then $U \in X_{N,\omega}(\Omega^+)$ and $\mathrm{w} = 0$ in Ω^-.*

Remark 2.3. Let a pair $(V,\mathrm{w}) \in [H^1(\Omega^+)]^5 \times [H^1_{loc}(\Omega^-) \cap Som_2(\Omega^-)]$ be a solution of the homogeneous problem

$$A^*(\partial,\omega)V = 0 \ \text{ in } \Omega^+,$$
$$(\Delta + \rho_2\omega^2)\mathrm{w} = 0 \ \text{ in } \Omega^-,$$
$$\{v\cdot n\}^+ + \overline{b}_2^{\,-1}\{\partial_n \mathrm{w}\}^- = 0 \ \text{ on } S,$$
$$\big\{[\widetilde{T}(\partial,n)V]_j\big\}^+ + \overline{b}_1^{\,-1}\{\mathrm{w}\}^- n_j = 0 \ \text{ on } S, \quad j = 1,2,3,$$
$$\{v_4\}^+ = 0 \ \text{ on } S,$$
$$\{v_5\}^+ = 0 \ \text{ on } S,$$

where b_1 and b_2 are the given complex constants satisfying the conditions (1.15).

Then $\mathrm{w} = 0$ in Ω^- and either $V = 0$ in Ω^+ if $\omega \notin J_D^*(\Omega^+)$ or $V \in X^*_{D,\omega}(\Omega^+)$ if $\omega \in J_D^*(\Omega^+)$.

Remark 2.4. Let a pair $(V,\mathrm{w}) \in [H^1(\Omega^+)]^5 \times [H^1_{loc}(\Omega^-) \cap Som_2(\Omega^-)]$ be a solution of the homogeneous problem

$$\begin{gathered}
A^*(\partial,\omega)V = 0 \ \text{ in } \Omega^+, \\
(\Delta + \rho_2\omega^2)\mathrm{w} = 0 \ \text{ in } \Omega^-, \\
\{v\cdot n\}^+ + \overline{b}_2^{\,-1}\{\partial_n \mathrm{w}\}^- = 0 \ \text{ on } S, \\
\big\{[\widetilde{T}(\partial,n)V]_j\big\}^+ + \overline{b}_1^{\,-1}\{\mathrm{w}\}^- n_j = 0 \ \text{ on } S, \quad j=1,2,3, \\
\big\{[\widetilde{T}(\partial,n)V]_4\big\}^+ = 0 \ \text{ on } S, \\
\big\{[\widetilde{T}(\partial,n)V]_5\big\}^+ = 0 \ \text{ on } S,
\end{gathered}$$

where b_1 and b_2 are the given complex constants satisfying conditions (1.15).

Then $V \in X^*_{N,\omega}(\Omega^+)$ and $\mathrm{w} = 0$ in Ω^-.

3 Layer potentials

3.1 Potentials associated with the Helmholtz equation

Let us introduce the single and double layer potentials,

$$\begin{aligned}
V_\omega(g)(x) &:= \int_S \gamma(x-y,\omega)g(y)\,d_yS, \quad x \notin S, \\
W_\omega(f)(x) &:= \int_S \partial_{n(y)}\gamma(x-y,\omega)f(y)\,d_yS, \quad x \notin S,
\end{aligned}$$

where

$$\gamma(x,\omega) := -\frac{\exp(i\sqrt{\rho_2}\,\omega|x|)}{4\pi|x|}$$

is the fundamental solution of the Helmholtz equation (1.6). These potentials satisfy the Sommerfeld radiation condition, i.e., belong to the class $Som_1(\Omega^-)$.

For these potentials the following theorems are valid (see [13, 37]).

Theorem 3.1. *Let $g \in H^{-1/2}(S)$, $f \in H^{1/2}(S)$. Then on the manifold S the following jump relations hold:*

$$\begin{gathered}
\{V_\omega(g)\}^\pm = \mathcal{H}_\omega(g), \quad \{W_\omega(f)\}^\pm = \pm 2^{-1}f + \mathcal{K}^*_\omega(f), \\
\{\partial_n V_\omega(g)\}^\pm = \mp 2^{-1}g + \mathcal{K}_\omega(g), \quad \{\partial_n W_\omega(f)\}^+ = \{\partial_n W_\omega(f)\}^- =: \mathcal{L}_\omega(f),
\end{gathered}$$

where $\mathcal{H}_\omega$, $\mathcal{K}^*_\omega$ and $\mathcal{K}_\omega$ are integral operators with the weakly singular kernels,

$$\begin{aligned}
\mathcal{H}_\omega(g)(z) &:= \int_S \gamma(z-y,\omega)g(y)\,d_yS, \quad z \in S, \\
\mathcal{K}^*_\omega(f)(z) &:= \int_S \partial_{n(y)}\gamma(z-y,\omega)f(y)\,d_yS, \quad z \in S, \\
\mathcal{K}_\omega(g)(z) &:= \int_S \partial_{n(z)}\gamma(z-y,\omega)g(y)\,d_yS, \quad z \in S,
\end{aligned}$$

while $\mathcal{L}_\omega$ is a singular integro-differential operator (pseudodifferential operator) of order 1.

Theorem 3.2. *The operators*

$$\mathcal{N} := -2^{-1}I_1 + \mathcal{K}^*_\omega + \mu\mathcal{H}_\omega : H^{1/2}(S) \to H^{1/2}(S), \tag{3.1}$$

$$\mathcal{M} := \mathcal{L}_\omega + \mu\big(2^{-1}I_1 + \mathcal{K}_\omega\big) : H^{1/2}(S) \to H^{-1/2}(S), \tag{3.2}$$

are invertible provided that $\operatorname{Im}\mu \neq 0$. Here I_1 is the scalar identity operator.

The mapping properties of the above potentials and the boundary integral operators are described in Appendix.

3.2 Fundamental solution and potentials of the steady state oscillation equations of electro-magneto-elasticity

Let us consider the equation

$$\Phi_A(\xi,\omega) := \det A(i\xi,\omega) = \det \begin{pmatrix} [c_{ijlk}\xi_i\xi_l - \rho_1\omega^2\delta_{jk}]_{3\times 3} & [e_{lij}\xi_l\xi_i]_{3\times 1} & [q_{lij}\xi_l\xi_i]_{3\times 1} \\ [-e_{ikl}\xi_i\xi_l]_{1\times 3} & \varepsilon_{il}\xi_i\xi_l & a_{il}\xi_i\xi_l \\ [-q_{ikl}\xi_i\xi_l]_{1\times 3} & a_{il}\xi_i\xi_l & \mu_{il}\xi_i\xi_l \end{pmatrix}_{5\times 5} = 0, \quad (3.3)$$

$$\xi \in \mathbb{R}^3 \setminus \{0\}, \ \ \omega \in \mathbb{R}, \ \ i,j,k,l = 1,2,3,$$

where $\Phi_A(\xi,\omega)$ is the characteristic polynomial of the operator $A(\partial,\omega)$. The origin is an isolated zero of (3.3).

We are interested in the real zeros of the function $\Phi_A(\xi,\omega)$, $\ \xi \in \mathbb{R}^3 \setminus \{0\}$.

Denote

$$\lambda := \frac{\rho_1\omega^2}{|\xi|^2}, \quad \widehat{\xi} := \frac{\xi}{|\xi|} \ \text{ for } \ |\xi| \neq 0,$$

$$B(\lambda,\widehat{\xi}) := \begin{pmatrix} [c_{ijkl}\widehat{\xi}_i\widehat{\xi}_l - \lambda\delta_{jk}]_{3\times 3} & [A_{j4}(\widehat{\xi})]_{3\times 1} & [A_{j5}(\widehat{\xi})]_{3\times 1} \\ [-A_{j4}(\widehat{\xi})]_{1\times 3} & \varepsilon_{il}\widehat{\xi}_i\widehat{\xi}_l & a_{il}\widehat{\xi}_i\widehat{\xi}_l \\ [-A_{j5}(\widehat{\xi})]_{1\times 3} & a_{il}\widehat{\xi}_i\widehat{\xi}_l & \mu_{il}\widehat{\xi}_i\widehat{\xi}_l \end{pmatrix}_{5\times 5}.$$

Then (3.3) can be rewritten as

$$\Psi(\lambda,\widehat{\xi}) := \det B(\lambda,\widehat{\xi}) = 0. \quad (3.4)$$

This is a cubic equation in λ with real coefficients.

Theorem 3.3. *Equation* (3.4) *possesses three real positive roots* $\lambda_1(\widehat{\xi})$, $\lambda_2(\widehat{\xi})$, $\lambda_3(\widehat{\xi})$.

Proof. Let $\widehat{\xi} \in \Sigma_1 := \{x \in \mathbb{R}^3 : \ |x| = 1\}$ and $\Psi(\lambda,\widehat{\xi}) = 0$. Then there is a non-trivial vector $\eta \in \mathbb{C}^5 \setminus \{0\}$ such that $B(\lambda,\widehat{\xi})\,\eta = 0$, i.e.,

$$(c_{ijkl}\widehat{\xi}_i\widehat{\xi}_l - \lambda\delta_{jk})\eta_k + e_{lij}\widehat{\xi}_l\widehat{\xi}_i\eta_4 + q_{lij}\widehat{\xi}_l\widehat{\xi}_i\eta_5 = 0, \ \ j = 1,2,3, \quad (3.5)$$

$$-e_{ikl}\widehat{\xi}_i\widehat{\xi}_l\eta_k + \varepsilon_{il}\widehat{\xi}_i\widehat{\xi}_l\eta_4 + a_{il}\widehat{\xi}_i\widehat{\xi}_l\eta_5 = 0, \quad (3.6)$$

$$-q_{ikl}\widehat{\xi}_i\widehat{\xi}_l\eta_k + a_{il}\widehat{\xi}_i\widehat{\xi}_l\eta_4 + \mu_{il}\widehat{\xi}_i\widehat{\xi}_l\eta_5 = 0, \quad (3.7)$$

Multiply the first three equations by $\overline{\eta}_j$, the complex conjugate of the fourth equation by η_4, the complex conjugate of the fifth equation by η_5 and sum them to obtain

$$\begin{aligned} &c_{ijkl}\widehat{\xi}_i\widehat{\xi}_l\eta_k\overline{\eta}_j - \lambda|\eta'|^2 + e_{lij}\widehat{\xi}_l\widehat{\xi}_i\eta_4\overline{\eta}_j + q_{lij}\widehat{\xi}_l\widehat{\xi}_i\eta_5\overline{\eta}_j \\ &\quad - e_{ijl}\widehat{\xi}_i\widehat{\xi}_l\overline{\eta}_j\eta_4 + \varepsilon_{il}\widehat{\xi}_i\widehat{\xi}_l|\eta_4|^2 + a_{il}\widehat{\xi}_i\widehat{\xi}_l\overline{\eta}_5\eta_4 - q_{ijl}\widehat{\xi}_i\widehat{\xi}_l\overline{\eta}_j\eta_5 + a_{il}\widehat{\xi}_i\widehat{\xi}_l\overline{\eta}_4\eta_5 + \mu_{il}\widehat{\xi}_i\widehat{\xi}_l|\eta_5|^2 = 0, \end{aligned} \quad (3.8)$$

where $\eta' = (\eta_1,\eta_2,\eta_3)$.

Due to the symmetry property of the coefficients e_{lij} and q_{lij},

$$e_{lij}\widehat{\xi}_l\widehat{\xi}_i\eta_4\overline{\eta}_j = e_{ijl}\widehat{\xi}_i\widehat{\xi}_l\overline{\eta}_j\eta_4, \quad q_{lij}\widehat{\xi}_l\widehat{\xi}_i\eta_5\overline{\eta}_j = q_{ijl}\widehat{\xi}_i\widehat{\xi}_l\overline{\eta}_j\eta_5.$$

Therefore, we derive from (3.8) that

$$c_{ijkl}\widehat{\xi}_i\widehat{\xi}_l\eta_k\overline{\eta}_j - \lambda|\eta'|^2 + \varepsilon_{il}\widehat{\xi}_i\widehat{\xi}_l|\eta_4|^2 + \mu_{il}\widehat{\xi}_i\widehat{\xi}_l|\eta_5|^2 + 2\,\mathrm{Re}\,a_{il}\widehat{\xi}_i\widehat{\xi}_l\overline{\eta}_5\eta_4 = 0. \quad (3.9)$$

Next, we note that $c_{ijkl}\widehat{\xi}_i\widehat{\xi}_l\eta_k\overline{\eta}_j = c_{ijkl}\overline{\varkappa}_{ij}\varkappa_{kl} \geq \delta_0\varkappa_{kl}\overline{\varkappa}_{kl} \geq 0$ with $\varkappa_{kl} = 2^{-1}(\widehat{\xi}_l\eta_k + \widehat{\xi}_k\eta_l)$.

Moreover, due to the strict inequalities $\varepsilon_{il}\widehat{\xi}_i\widehat{\xi}_l \geq \delta_1 > 0$, $\mu_{il}\widehat{\xi}_i\widehat{\xi}_l \geq \delta_2 > 0$, and (1.4), it follows that $|\eta'| \neq 0$, since otherwise from (3.9) we get $\eta_4 = 0$, which contradicts the inclusion $\eta = (\eta', \eta_4, \eta_5) \in \mathbb{C}^5 \setminus \{0\}$. Therefore, from (3.9) we finally conclude that $\lambda > 0$. $\square$

Denote the roots of equation (3.4) by λ_1, λ_2, λ_3. Clearly, the equation of the surface $S_{\omega,j}$, $j=1,2,3$, in the spherical coordinates reads as

$$r=r_j(\theta,\varphi)=\frac{\sqrt{\rho_1}\omega}{\sqrt{\lambda_j(\widehat{\xi})}}\,,$$

where $\xi_1=r\cos\varphi\sin\theta$, $\xi_2=r\sin\varphi\sin\theta$, $\xi_3=r\cos\theta$ with $0\le\varphi\le 2\pi$, $0\le\theta\le\pi$, $r=|\xi|$.

We also have the following identity:

$$\Phi_A(\xi,\omega)=\det A(i\xi,\omega)=\Phi_A(\widehat{\xi},0)\,r^4\prod_{j=1}^{3}\big(r^2-r_j^2(\widehat{\xi})\big)=\Phi_A(\widehat{\xi},0)\,r^4\prod_{j=1}^{3}P_j(\xi).$$

It can easily be shown that the vector

$$n(\xi)=(-1)^j|\nabla\Phi_A(\xi,\omega)|^{-1}\nabla\Phi_A(\xi,\omega),\ \ \xi\in S_{\omega,j},$$

is an external unit normal vector to $S_{\omega,j}$ at the point ξ.

Further, we assume that the following conditions are fulfilled (cf. [10,33,41,42]):

(i) if $\Phi_A(\xi,\omega)=\Phi_A(\widehat{\xi},0)\,r^4P_1(\xi)P_2(\xi)P_3(\xi)$, then $\nabla_\xi(P_1(\xi)P_2(\xi)P_3(\xi))\neq 0$ at real zeros $\xi\in\mathbb{R}^3\setminus\{0\}$ of the polynomial (3.3), or

if $\Phi_A(\xi,\omega)=\Phi_A(\widehat{\xi},0)\,r^4P_1^2(\xi)P_2(\xi)$, then $\nabla_\xi(P_1(\xi)P_2(\xi))\neq 0$ at real zeros $\xi\in\mathbb{R}^3\setminus\{0\}$ of the polynomial (3.3), or

if $\Phi_A(\xi,\omega)=\Phi_A(\widehat{\xi},0)\,r^4P_1^3(\xi)$, then $\nabla_\xi P_1(\xi)\neq 0$ at real zeros $\xi\in\mathbb{R}^3\setminus\{0\}$ of the polynomial (3.3);

(ii) the Gaussian curvature of the surface, defined by the real zeros of the polynomial $\Phi_A(\xi,\omega)$, $\xi\in\mathbb{R}^3\setminus\{0\}$, does not vanish anywhere.

It follows from the above conditions (i) and (ii) that the real zeros $\xi\in\mathbb{R}^3\setminus\{0\}$ of the polynomial $\Phi_A(\xi,\omega)$ form non-self-intersecting, closed, convex two-dimensional surfaces $S_{\omega,1},S_{\omega,2},S_{\omega,3}$, enclosing the origin. For an arbitrary unit vector $\eta=x/|x|$ with $x\in\mathbb{R}^3\setminus\{0\}$, there exists only one point on each $S_{\omega,j}$, namely, $\xi^j=(\xi_1^j,\xi_2^j,\xi_3^j)\in S_{\omega,j}$ such that the outward unit normal vector $n(\xi^j)$ to $S_{\omega,j}$ at the point ξ^j has the same direction as η, i.e., $n(\xi^j)=\eta$. In this case, we say that the points ξ^j, $j=1,2,3$, correspond to the vector η.

From (i), we see that the surfaces $S_{\omega,j}$ $j=1,2,3$, might have multiplicities.

We say that a vector-function $U=(u_1,u_2,u_3,u_4,u_5)^\top$ belongs to the class $M_{m_1,m_2,m_3}(\mathbf{P})$ if $U\in[C^\infty(\Omega^-)]^5$ and the relation

$$U(x)=\sum_{p=1}^{5}u^p(x)$$

holds, where u^p has the following uniform asymptotic expansion as $r=|x|\to\infty$:

$$u^p\sim\sum_{j=1}^{3}e^{-ir\xi^j}\Big\{d^p_{0,m_j}(\eta)r^{m_j-2}+\sum_{q=1}^{\infty}d^p_{q,m_j}(\eta)r^{m_j-2-q}\Big\},\ \ p=1,2,3, \tag{3.10}$$

$$u^4(x)=O(r^{-1}),\quad \partial_k u^4(x)=O(r^{-2}),\quad u^5(x)=O(r^{-1}),\quad \partial_k u^5(x)=O(r^{-2}),\ \ k=1,2,3,$$

here $\mathbf{P}=\det A(i\partial_x,\omega)$ and $d^p_{q,m_j}\in C^\infty,\ j=1,2,3$ (see [10]).

These conditions are generalization of Sommerfeld–Kupradze type radiation conditions in the anisotropic elasticity (cf. [28,33]).

From condition (i) it follows that our class $M_{m_1,m_2,m_3}(\mathbf{P})$ is $M_{1,1,1}(\mathbf{P})$ (when there is no multiplicity, i.e., surfaces do not coincide) or $M_{2,1}(\mathbf{P})$ (when two surfaces coincide) or $M_3(\mathbf{P})$ (when all three surfaces coincide).

The class $M_{1,1,1}(\mathbf{P})$ is a subset of the generalized Sommerfeld–Kupradze class.

We can show the following uniqueness theorems.

Theorem 3.4. *The homogeneous exterior Dirichlet boundary value problem*

$$A(\partial,\omega)U = 0 \ \text{ in } \Omega^-, \quad \{U\}^- = 0 \ \text{ on } S,$$

has only the trivial solution in the class $[H^1_{loc}(\Omega^-)]^5 \cap M_{m_1,m_2,m_3}(\mathbf{P})$.

Theorem 3.5. *The homogeneous exterior Dirichlet boundary value problem*

$$A^*(\partial,\omega)V = 0 \ \text{ in } \Omega^-, \quad \{V\}^- = 0 \ \text{ on } S,$$

has only the trivial solution in the class $[H^1_{loc}(\Omega^-)]^5 \cap M_{m_1,m_2,m_3}(\mathbf{P}^*)$, *where* $\mathbf{P}^* = \det A^*(\partial,\omega)$.

If surfaces $S_{\omega,j}$ $j = 1,2,3$, have no multiplicity, Theorems 3.4 and 3.5 are valid in generalized the Sommerfeld–Kupradze class (cf. [28]).

Denote by $\Gamma(x,\omega)$ the fundamental matrix of the operator $A(\partial,\omega)$. By means of the Fourier transform method and the limiting absorption principle, we can construct this matrix explicitly (see Ch. 1, Section 1, also see [42])

$$\Gamma(x,\omega) = \lim_{\varepsilon\to 0+} F^{-1}_{\xi\to x}\big[A^{-1}(i\xi,\omega+i\varepsilon)\big], \tag{3.11}$$

where F^{-1} is the inverse Fourier transform. The columns of the matrix $\Gamma(x,\omega)$ are infinitely differentiable in $\mathbb{R}^3\setminus\{0\}$ and belong to the class $M_{m_1,m_2,m_3}(\mathbf{P})$.

Further, we introduce the single and double layer potentials associated with the differential operator $A(\partial,\omega)$,

$$\mathbf{V}_\omega(g)(x) = \int_S \Gamma(x-y,\omega)g(y)\,d_yS, \quad x\in\Omega^\pm,$$

$$\mathbf{W}_\omega(f)(x) = \int_S \big[\widetilde{T}(\partial_y,n(y))\Gamma^\top(x-y,\omega)\big]^\top f(y)\,d_yS, \quad x\in\Omega^\pm,$$

where $g = (g_1,\dots,g_4)^\top$ and $f = (f_1,\dots,f_4)^\top$ are density vector-functions.

For a solution $U\in[H^1(\Omega^+)]^5$ to the homogeneous equation (1.9) in Ω^+ we have the integral representation

$$U = \mathbf{W}_\omega(\{U\}^+) - \mathbf{V}_\omega(\{TU\}^+) \ \text{ in } \Omega^+.$$

For these potentials the following theorem holds (see [6, 7]).

Theorem 3.6. *Let* $g\in[H^{-1+s}(S)]^4$ *and* $f\in[H^s(S)]^4$, $s>0$. *Then*

$$\begin{aligned}
\{\mathbf{V}_\omega(g)(z)\}^\pm &= \mathbf{H}_\omega(g)(z), \quad z\in S,\\
\{\mathbf{W}_\omega(f)(z)\}^\pm &= \pm 2^{-1}f(z) + \widetilde{\mathbf{K}}_\omega(f)(z), \quad z\in S,\\
\big\{T(\partial_y,n(y))\mathbf{V}_\omega(g)(z)\big\}^\pm &= \mp 2^{-1}g(z) + \mathbf{K}_\omega(g)(z), \quad z\in S,\\
\big\{T(\partial_z,n(z))\mathbf{W}_\omega(f)(z)\big\}^+ &= \big\{T(\partial_z,n(z))\mathbf{W}_\omega(f)(z)\big\}^- := \boldsymbol{L}_\omega(f)(z), \quad z\in S,
\end{aligned}$$

where $\mathbf{H}_\omega$ *is a weakly singular integral operator,* $\widetilde{\mathbf{K}}_\omega$ *and* $\mathbf{K}_\omega$ *are singular integral operators, while* $\boldsymbol{L}_\omega$ *is a pseudodifferential operator of order* 1,

$$\mathbf{H}_\omega(g)(z) := \int_S \Gamma(z-y,\omega)g(y)\,d_yS, \quad z\in S,$$

$$\widetilde{\mathbf{K}}_\omega(f)(z) := \int_S \big[\widetilde{T}(\partial_y,n(y))\Gamma^\top(z-y,\omega)\big]^\top f(y)\,d_yS, \quad z\in S,$$

$$\mathbf{K}_\omega(g)(z) := \int_S T(\partial_z,n(z))\Gamma(z-y,\omega)g(y)\,d_yS, \quad z\in S.$$

The mapping properties of these potentials and boundary integral operators are described in Appendix.

4 The Dirichlet and Neumann type interaction problems for pseudo-oscillation equations

In this section, we consider the Dirichlet and Neumann type interaction problems for the so-called pseudo-oscillation equations. These problems are intermediate auxiliary problems for investigation of interaction problems for the steady state oscillation equations.

4.1 Formulation of the problems

The matrix differential operator corresponding to the basic pseudo-oscillation equations of the electro-magneto-elasticity for anisotropic homogeneous media is written as follows:

$$\begin{gathered}A(\partial,\tau)=[A_{jk}(\partial,\tau)]_{5\times 5},\\ A_{jk}(\partial,\tau)=c_{ijkl}\partial_i\partial_l+\rho_1\tau^2\delta_{jk},\quad A_{j4}(\partial,\tau)=e_{lij}\partial_l\partial_i,\quad A_{j5}(\partial,\tau)=q_{lij}\partial_l\partial_i,\\ A_{4k}(\partial,\tau)=-e_{ikl}\partial_i\partial_l,\quad A_{44}(\partial,\tau)=\varepsilon_{il}\partial_i\partial_l,\quad A_{45}(\partial,\tau)=a_{il}\partial_i\partial_l,\\ A_{5k}(\partial,\tau)=-q_{ikl}\partial_i\partial_l,\quad A_{54}(\partial,\tau)=a_{il}\partial_i\partial_l,\quad A_{55}(\partial,\tau)=\mu_{il}\partial_i\partial_l,\end{gathered}$$

$j,k=1,2,3$, where τ is a purely imaginary complex parameter: $\tau=i\sigma$, $\sigma\neq 0$, $\sigma\in\mathbb{R}$.

Dirichlet type problem (D_τ)**:** Find a vector-function $U=(u,u_4,u_5)^\top\in[H^1(\Omega^+)]^5$ and a scalar function $\mathrm{w}\in H^1_{loc}(\Omega^-)\cap Som_1(\Omega^-)$ satisfying the differential equations

$$A(\partial,\tau)U=0\ \text{ in } \Omega^+, \tag{4.1}$$

$$\Delta\mathrm{w}+\rho_2\omega^2\mathrm{w}=0\ \text{ in } \Omega^-, \tag{4.2}$$

the transmission conditions

$$\{u\cdot n\}^+=b_1\{\partial_n\mathrm{w}\}^-+f_0\ \text{ on } S, \tag{4.3}$$

$$\{[TU]_j\}^+=b_2\{\mathrm{w}\}^-n_j+f_j\ \text{ on } S,\ \ j=1,2,3, \tag{4.4}$$

and the Dirichlet boundary conditions

$$\{u_4\}^+=f_1^{(D)}\ \text{ on } S, \tag{4.5}$$

$$\{u_5\}^+=f_2^{(D)}\ \text{ on } S, \tag{4.6}$$

where b_1 and b_2 are the given complex constants satisfying conditions (1.15), $f_0\in H^{-1/2}(S)$, $f_j\in H^{-1/2}(S)$, $j=1,2,3$, $f_1^{(D)}\in H^{1/2}(S)$, $f_2^{(D)}\in H^{1/2}(S)$.

Neumann type problem (N_τ)**:** Find a vector-function $U=(u,u_4,u_5)^\top\in[H^1(\Omega^+)]^5$ and a scalar function $\mathrm{w}\in H^1_{loc}(\Omega^-)\cap Som_1(\Omega^-)$ satisfying the differential equations (4.1) and (4.2), respectively, transmission conditions (4.3), (4.4), and the Neumann boundary conditions

$$\{[TU]_4\}^+=f_1^{(N)}\ \text{ on } S \text{ with } f_1^{(N)}\in H^{-1/2}(S), \tag{4.7}$$

$$\{[TU]_5\}^+=f_2^{(N)}\ \text{ on } S \text{ with } f_2^{(N)}\in H^{-1/2}(S). \tag{4.8}$$

4.2 Uniqueness theorems for problems (D_τ) and (N_τ)

Theorem 4.1. *Let $\tau=i\sigma$, $\sigma\neq 0$, $\sigma\in\mathbb{R}$. The homogeneous problem (D_τ) has only the trivial solution, while the general solution of the homogeneous problem (N_τ) is the vector $(0,0,0,c_1,c_2)$, where c_1 and c_2 are an arbitrary complex scalar constants.*

Proof. Let (U,w) be a solution of the homogeneous problem (D_τ).

Let us write Green's formula for the Helmholtz equation (4.2) in the domain $\Omega_R:=\Omega^-\cap B(0,R)$, where $\overline{\Omega^+}\subset B(0,R)$,

$$\int\limits_{\Omega_R} \big[(\Delta+\rho_2\omega^2)\mathrm{w}\overline{\mathrm{w}} - \mathrm{w}(\Delta+\rho_2\omega^2)\overline{\mathrm{w}}\big]\,dx = \int\limits_{S(0,R)} \partial_n \mathrm{w}\overline{\mathrm{w}}\,dS - \int\limits_{S(0,R)} \partial_n \overline{\mathrm{w}}\mathrm{w}\,dS - \big\langle\{\partial_n \mathrm{w}\}^-,\{\mathrm{w}\}^-\big\rangle_S + \big\langle\{\partial_n \overline{\mathrm{w}}\}^-,\{\overline{\mathrm{w}}\}^-\big\rangle_S. \quad (4.9)$$

Now write Green's formula for the operator $A(\partial,\tau)$ in the domain Ω^+,

$$\int\limits_{\Omega^+} \Big[[A(\partial,\tau)U]_j\overline{u}_j + [\overline{A(\partial,\tau)U}]_4 u_4 + [\overline{A(\partial,\tau)U}]_5 u_5 + \mathcal{E}(U,\overline{U})\Big]\,dx = \big\langle\{TU\}_j^+,\{u_j\}^+\big\rangle_S + \big\langle\{\overline{TU}\}_4^+,\{\overline{u}_4\}^+\big\rangle_S + \big\langle\{\overline{TU}\}_5^+,\{\overline{u}_5\}^+\big\rangle_S, \quad (4.10)$$

where $\mathcal{E}(U,\overline{U}) = c_{ijlk}\partial_i u_j \partial_l \overline{u}_k + \rho_1\sigma^2|u|^2 + \varepsilon_{il}\partial_i u_4 \partial_l \overline{u}_4 + \mu_{jl}\partial_j u_5 \partial_l \overline{u}_5$. Using (4.1), (4.2), and (4.5), from (4.9) and (4.10) we obtain the following equalities:

$$\int\limits_{S(0,R)} \partial_n \mathrm{w}\overline{\mathrm{w}}\,dS - \int\limits_{S(0,R)} \partial_n \overline{\mathrm{w}}\mathrm{w}\,dS - \big\langle\{\partial_n \mathrm{w}\}^-,\{\mathrm{w}\}^-\big\rangle_S + \big\langle\{\partial_n \overline{\mathrm{w}}\}^-,\{\overline{\mathrm{w}}\}^-\big\rangle_S = 0, \quad (4.11)$$

$$\operatorname{Im}\big\langle\{[TU]_j\}^+,\{u_j\}^+\big\rangle_S = 0, \quad j=1,2,3. \quad (4.12)$$

In view of the homogeneous transmission conditions, we get

$$\big\langle\{[TU]_j\}^+,\{u_j\}^+\big\rangle_S = \big\langle b_2\{\mathrm{w}\}^- n_j,\{u_j\}^+\big\rangle_S = b_2\overline{b}_1\big\langle\{\partial_n \overline{\mathrm{w}}\}^-,\{\overline{\mathrm{w}}\}^-\big\rangle_S. \quad (4.13)$$

Since $\operatorname{Im}[\overline{b}_1 b_2] = 0$, from (4.12) and (4.13) we get

$$\operatorname{Im}\big\langle\{\partial_n \overline{\mathrm{w}}\}^-,\{\overline{\mathrm{w}}\}^-\big\rangle_S = 0,$$

and from (4.11) we derive that

$$\operatorname{Im}\int\limits_{S(0,R)} \partial_n \overline{\mathrm{w}}\mathrm{w}\,dS = 0. \quad (4.14)$$

By the Sommerfeld radiation condition, from (4.14) we conclude that

$$\lim_{R\to\infty}\int\limits_{S(0,R)} |\mathrm{w}|^2\,dS = 0.$$

Using the Rellich–Vekua lemma, we find that $\mathrm{w}=0$ in the domain Ω^-.

Then from Green's formula (4.10) it follows that

$$\int\limits_{\Omega^+} \mathcal{E}(U,\overline{U})\,dx = 0. \quad (4.15)$$

Using (1.2) and (1.3), it is easy to see that for a complex vector $u=(u_1,u_2,u_3)^\top$ and a complex functions u_4, u_5,

$$c_{ijlk}\partial_i u_j \partial_l \overline{u}_k \geq 0, \quad \varepsilon_{jl}\partial_l u_4 \partial_j \overline{u}_4 \geq 0, \quad \mu_{jl}\partial_l u_5 \partial_j \overline{u}_5 \geq 0. \quad (4.16)$$

Taking into account (4.16), from (4.15) we obtain

$$\int\limits_{\Omega^+} \Big[c_{ijlk}\partial_i u_j \partial_l \overline{u}_k + \rho_1\sigma^2|u|^2 + \varepsilon_{jl}\partial_l u_4 \partial_j \overline{u}_4 + \mu_{jl}\partial_l u_5 \partial_j \overline{u}_5\Big]\,dx = 0, \quad (4.17)$$

implying that $u=0$ in Ω^+ and $u_4=c_1$, $u_5=c_2$ in Ω^+, where c_1, c_2 are arbitrary constants. Since $\{u_4\}^+ = \{u_5\}^+ = 0$ on S, we deduce that $u_4=u_5=0$ in the domain Ω^+.

Applying the same arguments, we can show that the general solution of the homogeneous problem (N_τ) is a vector $(0,0,0,c_1,c_2)^\top$, where c_1 and c_2 are arbitrary complex scalar constants. □

4.3 Fundamental solution and potentials for the pseudo-oscillation equations of piezoelectro-magneto-elasticity

The full symbol of the pseudo-oscillation operator $A(\partial,\tau)$ is elliptic provided $\tau=i\sigma$, $\sigma\neq 0$, $\sigma\in\mathbb{R}$, i.e.,

$$\det A(-i\xi,\tau)\neq 0 \ \ \forall\xi\in\mathbb{R}^3\setminus\{0\}.$$

Moreover, the entries of the inverse matrix $A^{-1}(-i\xi,\tau)$ are locally integrable functions decaying at infinity as $O(|\xi|^{-2})$. Therefore, we can construct the fundamental matrix $\Gamma(x,\tau)=[\Gamma_{kj}(x,\tau)]_{5\times 5}$ of the operator $A(\partial,\tau)$ by the Fourier transform technique,

$$\Gamma(x,\tau)=F^{-1}_{\xi\to x}[A^{-1}(-i\xi,\tau)]. \tag{4.18}$$

Note that in a neighbourhood of the origin the following estimates hold ($0<|x|<1$):

$$\big|\Gamma_{jk}(x,\tau)-\Gamma_{jk}(x,\omega)\big|\leq c(\tau,\omega), \tag{4.19}$$
$$\big|\partial_l\big[\Gamma_{jk}(x,\tau)-\Gamma_{jk}(x,\omega)\big]\big|\leq c(\tau,\omega)\ln|x|^{-1}, \tag{4.20}$$
$$\big|\partial^\alpha\big[\Gamma_{jk}(x,\tau)-\Gamma_{jk}(x,\omega)\big]\big|\leq c(\tau,\omega)|x|^{1-|\alpha|}, \quad j,k=\overline{1,5}, \tag{4.21}$$

where $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ is a multi-index with $|\alpha|=\alpha_1+\alpha_2+\alpha_3\geq 2$, while $c(\tau,\omega)$ is a positive constant depending on $\tau=i\sigma$ and ω with $\sigma,\omega\in\mathbb{R}\setminus\{0\}$ (cf. [33]).

Let us introduce the single and double layer pseudo-oscillation potentials

$$\mathbf{V}_\tau(h)=\int\limits_S\Gamma(x-y,\tau)h(y)\,d_yS,$$
$$\mathbf{W}_\tau(h)=\int\limits_S\big[\widetilde{T}(\partial_y,n(y))\Gamma^\top(x-y,\tau)\big]^\top h(y)\,d_yS,$$

where $h=(h_1,h_2,h_3,h_4,h_5)^\top$ is a density vector-function.

These pseudo-oscillation potentials have the following jump properties (see [6]).

Theorem 4.2. *Let $h^{(1)}\in[H^{-1+s}(S)]^5$, $h^{(2)}\in[H^s(S)]^5$, $s>0$. Then the following jump relations hold on S:*

$$\big\{\mathbf{V}_\tau(h^{(1)})(z)\big\}^\pm=\int\limits_S\Gamma(z-y,\tau)h^{(1)}(y)\,d_yS,$$
$$\big\{\mathbf{W}_\tau(h^{(2)})(z)\big\}^\pm=\pm 2^{-1}h^{(2)}(z)+\int\limits_S\big[\widetilde{T}(\partial_y,n(y))\Gamma^\top(z-y,\tau)\big]^\top h^{(2)}(y)\,d_yS,$$
$$\big\{T\mathbf{V}_\tau(h^{(1)})(z)\big\}^\pm=\mp 2^{-1}h^{(1)}(z)+\int\limits_S T(\partial_z,n(z))\Gamma(z-y,\tau)h^{(1)}(y)\,d_yS,$$
$$\big\{T\mathbf{W}_\tau(h^{(2)})(z)\big\}^+=\big\{T\mathbf{W}_\tau(h^{(2)})(z)\big\}^-.$$

Further, we introduce the boundary operators

$$\mathbf{H}_\tau(h)(z)=\int\limits_S\Gamma(z-y,\tau)h(y)\,d_yS,$$
$$\mathbf{K}_\tau(h)(z)=\int\limits_S T(\partial_z,n(z))\Gamma(z-y,\tau)h(y)\,d_yS,$$
$$\widetilde{\mathbf{K}}_\tau(h)(z)=\int\limits_S\big[\widetilde{T}(\partial_y,n(y))\Gamma^\top(z-y,\tau)\big]^\top h(y)\,d_yS,$$
$$\mathbf{L}_\tau(h)(z)=\big\{T\mathbf{W}_\tau(h)(z)\big\}^+=\big\{T\mathbf{W}_\tau(h)(z)\big\}^-.$$

Note that $\mathbf{H}_\tau$ is a weakly singular integral operator (pseudodifferential operator of order -1), $\mathbf{K}_\tau$ and $\widetilde{\mathbf{K}}_\tau$ are singular integral operators (pseudodifferential operator of order 0), and $\mathbf{L}_\tau$ is a pseudodifferential operator of order 1.

The mapping properties of these potentials are described in Appendix.

4.4 Existence of solutions of problem (D_τ)

By Theorem 6.4 (see Appendix) the operator $\mathbf{H}_\tau : [H^s(S)]^5 \to [H^{s+1}(S)]^5$ is invertible for all $s \in \mathbb{R}$ and we can look for a solution of problem (D_τ) in the following form

$$U = \mathbf{V}_\tau \mathbf{H}_\tau^{-1} g \ \text{ in } \Omega^+, \quad \mathrm{w} = (W_\omega + \mu V_\omega)h \ \text{ in } \Omega^-, \ \ \mu \in \mathbb{C}, \ \ \operatorname{Im}\mu \neq 0,$$

where $g = (\widetilde{g}, g_4, g_5)^\top \in [H^{1/2}(S)]^5$, $\widetilde{g} = (g_1, g_2, g_3)^\top$, $h \in H^{1/2}(S)$ are unknown densities. From Theorems 6.1, 6.3 and 6.4 (see Appendix) it follows that $U \in [H^1(\Omega^+)]^5$ and $\mathrm{w} \in H^1_{loc}(\Omega^-)$.

Transmission conditions (4.3), (4.4) and the Dirichlet type conditions (4.5), (4.6) lead to the following system of pseudodifferential equations with respect to the unknowns $\widetilde{g}$, g_4, g_5 and h:

$$\widetilde{g} \cdot n - b_1 \mathcal{M}(h) = f_0 \ \text{ on } S, \tag{4.22}$$

$$\big[(-2^{-1} I_5 + \mathbf{K}_\tau)\mathbf{H}_\tau^{-1} g\big]_j - b_2 n_j \mathcal{N}(h) = f_j \ \text{ on } S, \ \ j = 1,2,3, \tag{4.23}$$

$$g_4 = f_1^{(D)} \ \text{ on } S, \tag{4.24}$$

$$g_5 = f_2^{(D)} \ \text{ on } S, \tag{4.25}$$

where $\mathcal{N} = -2^{-1} I_1 + \mathcal{K}^*_\omega + \mu \mathcal{H}_\omega$, $\mathcal{M} = \mathcal{L}_\omega + \mu(2^{-1} I_1 + \mathcal{K}_\omega)$.

Here and in what follows, I_m stands for the $m \times m$ unit matrix.

The matrix operator generated by the left-hand side expressions in system (4.22)–(4.25) reads as

$$\mathcal{P}_{\tau,D} := \begin{pmatrix} [n]_{1\times 3} & 0 & 0 & -b_1\mathcal{M} \\ [\mathcal{A}_\tau^{jk}]_{3\times 3} & [\mathcal{A}_\tau^{j4}]_{3\times 1} & [\mathcal{A}_\tau^{j5}]_{3\times 1} & [-b_2 n_j \mathcal{N}]_{3\times 1} \\ [0]_{1\times 3} & I_1 & 0 & 0 \\ [0]_{1\times 3} & 0 & I_1 & 0 \end{pmatrix}_{6\times 6}, \quad j,k = 1,2,3,$$

where

$$\mathcal{A}_\tau := \big(-2^{-1} I_5 + \mathbf{K}_\tau\big)\mathbf{H}_\tau^{-1} = [\mathcal{A}_\tau^{jk}]_{5\times 5}, \ \ j,k = \overline{1,5}, \tag{4.26}$$

is the Steklov–Poincaré type operator on S. This operator is a strongly elliptic pseudodifferential operator of order 1 (see [6] for details).

By Theorems 6.2 and 6.4 (see Appendix), the operator $\mathcal{P}_{\tau,D}$ possesses the following mapping property:

$$\mathcal{P}_{\tau,D} : [H^{1/2}(S)]^6 \to [H^{-1/2}(S)]^5 \times H^{1/2}(S). \tag{4.27}$$

In view of (4.24) and (4.25), equations (4.22) and (4.23) can be rewritten in the following equivalent form as a system with respect to $\widetilde{g}$ and h:

$$\widetilde{g} \cdot n - b_1 \mathcal{M}(h) = f_0 \ \text{ on } S, \tag{4.28}$$

$$[\mathcal{A}_\tau(\widetilde{g}, 0, 0)^\top]_j - b_2 n_j \mathcal{N}(h) = F_j \ \text{ on } S, \ \ j = 1,2,3, \tag{4.29}$$

where $F_j := f_j - \mathcal{A}_\tau^{j4} f_1^{(D)} - \mathcal{A}_\tau^{j5} f_2^{(D)}$, $j = 1,2,3$.

Denote by $\mathcal{R}_{\tau,D}$ the operator corresponding to system (4.28), (4.29)

$$\mathcal{R}_{\tau,D} := \begin{pmatrix} [n]_{1\times 3} & -b_1\mathcal{M} \\ \widetilde{\mathcal{A}}_\tau & [-b_2 n_k \mathcal{N}]_{3\times 1} \end{pmatrix}_{4\times 4},$$

where $\widetilde{\mathcal{A}}_\tau := [\mathcal{A}_\tau^{jk}]_{3\times 3}$, $j,k = 1,2,3$.

Clearly, the operator

$$\mathcal{R}_{\tau,D} : [H^{1/2}(S)]^4 \to [H^{-1/2}(S)]^4 \tag{4.30}$$

is bounded.

Let us represent the operator $\mathcal{R}_{\tau,D}$ as the sum of two operators

$$\mathcal{R}_{\tau,D} = \mathcal{R}^{(1)}_{\tau,D} + \mathcal{R}^{(2)}_{\tau,D},$$

where

$$\mathcal{R}^{(1)}_{\tau,D} = \begin{pmatrix} [0]_{1\times 3} & -b_1\mathcal{M} \\ \widetilde{\mathcal{A}}_\tau & [0]_{3\times 1} \end{pmatrix}_{4\times 4}, \quad \mathcal{R}^{(2)}_{\tau,D} = \begin{pmatrix} [n]_{1\times 3} & 0 \\ [0]_{3\times 3} & [-b_2 n_k \mathcal{N}]_{3\times 1} \end{pmatrix}_{4\times 4}.$$

It is easy to see that the operator $\mathcal{N} : H^{1/2}(S) \to H^{-1/2}(S)$ is compact due to Theorem 3.2 and Rellich compact embedding theorem. Therefore, the operator $\mathcal{R}^{(2)}_{\tau,D} : [H^{1/2}(S)]^4 \to [H^{-1/2}(S)]^4$ is compact. Further, we show that the operator $\widetilde{\mathcal{A}}_\tau$ is Fredholm. Indeed,

$$\mathcal{A}_\tau : [H^{1/2}(S)]^5 \to [H^{-1/2}(S)]^5$$

is strongly elliptic pseudodifferential operator of order 1 (see [6]), i.e.,

$$\operatorname{Re} \mathfrak{S}(\mathcal{A}_\tau; x, \xi)\zeta \cdot \zeta \geq c|\xi|\,|\zeta|^2,$$

where c is a positive constant and $\mathfrak{S}(\mathcal{A}_\tau; x, \xi)$ with $x \in S$, $\xi \in \mathbb{R}^2 \setminus \{0\}$, is the principal homogeneous symbol of the operator $\mathcal{A}_\tau$ in some local coordinate system. Therefore, $\forall\, \xi \in \mathbb{R}^2 \setminus \{0\}$, $\forall\, \zeta' \in \mathbb{C}^3$ the following estimate holds:

$$\operatorname{Re} \mathfrak{S}(\widetilde{\mathcal{A}}_\tau; x, \xi)\zeta' \cdot \zeta' = \operatorname{Re} \mathfrak{S}(\mathcal{A}_\tau; x, \xi)(\zeta', 0)^\top \cdot (\zeta', 0)^\top \geq c|\xi||\zeta'|^2.$$

Thus $\widetilde{\mathcal{A}}_\tau$ is a strongly elliptic pseudodifferential operator of order 1. Therefore, by virtue of the general theory of elliptic pseudodifferential operators on a compact manifold without boundary (see [16, Ch. 19], [14, Ch. 5]), we conclude that

$$\widetilde{\mathcal{A}}_\tau : [H^{1/2}(S)]^3 \to [H^{-1/2}(S)]^3$$

is a Fredholm operator. From the strong ellipticity property it also follows that the index of the operator $\widetilde{\mathcal{A}}_\tau$ is zero (see [16, Ch. 6], [14, Ch. 2]). Taking into account Theorem 3.2, we find that the operator $\mathcal{R}^{(1)}_{\tau,D}$ is Fredholm with index zero. Therefore, operators (4.30) and, consequently, (4.27) are Fredholm with index zero.

Now we show that the operator $\mathcal{R}_{\tau,D}$ is injective. Let $(\widetilde{g}, h)^\top$ with $\widetilde{g} \in [H^{1/2}(S)]^3$ and $h \in H^{1/2}(S)$ be some solution of the homogeneous system

$$\mathcal{R}_{\tau,D}(\widetilde{g}, h)^\top = 0,$$

and set

$$\widetilde{U} = (\widetilde{u}, \widetilde{u}_4, \widetilde{u}_5)^\top = \mathbf{V}_\tau \mathbf{H}_\tau^{-1}(\widetilde{g}, 0, 0), \quad \widetilde{\mathrm{w}} = (W_\omega + \mu V_\omega)h, \quad \operatorname{Im}\mu \neq 0.$$

Evidently, $\widetilde{U}$ and $\widetilde{\mathrm{w}}$ solve the homogeneous problem (D_τ).

It follows from the uniqueness result for problem (D_τ) (see Theorem 4.1) that $\widetilde{U} = 0$ in Ω^+ and $\widetilde{\mathrm{w}} = 0$ in Ω^-. Then $\{\widetilde{U}\}^+ = (\widetilde{g}, 0, 0)^\top = 0$ on S. Since $\{\widetilde{\mathrm{w}}\}^- = \mathcal{N}(h) = 0$ and $\mathcal{N}$ is invertible operator, we obtain $h = 0$ on S. Consequently, the operators

$$\begin{aligned} &\mathcal{R}_{\tau,D} : [H^{1/2}(S)]^4 \to [H^{-1/2}(S)]^4, \\ &\mathcal{P}_{\tau,D} : [H^{1/2}(S)]^6 \to [H^{-1/2}(S)]^5 \times H^{1/2}(S) \end{aligned}$$

are invertible.

Therefore, system (4.22)–(4.25) is uniquely solvable. Thus the following assertion holds.

Theorem 4.3. *Let $\tau = i\sigma$, $\sigma \neq 0$, $\sigma \in \mathbb{R}$, and let $f_0 \in H^{-1/2}(S)$, $f_j \in H^{-1/2}(S)$, $j = 1, 2, 3$, and $f^{(D)} \in H^{1/2}(S)$. Then problem (D_τ) has a unique solution (U, w), $U \in [H^1(\Omega^+)]^5$, $\mathrm{w} \in H^1_{loc}(\Omega^-) \cap Som_1(\Omega^-)$, which can be represented as*

$$U = \mathbf{V}_\tau \mathbf{H}_\tau^{-1} g \ \text{ in } \Omega^+, \quad \mathrm{w} = (W_\omega + \mu V_\omega)h \ \text{ in } \Omega^-,$$

where the densities $g \in [H^{1/2}(S)]^5$ and $h \in H^{1/2}(S)$ are defined from the uniquely solvable system (4.22)–(4.25).

4.5 Existence of solutions of problem (N_τ)

As in the previous subsection, we can look for a solution of problem (N_τ) in the following form:

$$U = \mathbf{V}_\tau \mathbf{H}_\tau^{-1} g \ \text{ in } \Omega^+, \quad \mathrm{w} = (W_\omega + \mu V_\omega)h \ \text{ in } \Omega^-, \ \ \mu \in \mathbb{C}, \ \ \operatorname{Im} \mu \neq 0,$$

where $g = (\widetilde{g}, g_4, g_5)^\top \in [H^{1/2}(S)]^5$ and $h \in H^{1/2}(S)$ are unknown densities. From Theorems 6.1, 6.3 and 6.4 of Appendix it follows that $U \in [H^1(\Omega^+)]^5$ and $\mathrm{w} \in H^1_{loc}(\Omega^-)$.

Transmission conditions (4.3), (4.4), and the Neumann type condition (4.7) lead to the following system of pseudodifferential equations with respect to the unknowns g and h:

$$\widetilde{g} \cdot n - b_1 \mathcal{M}(h) = f_0 \ \text{ on } S, \tag{4.31}$$

$$[\mathcal{A}_\tau g]_j - b_2 n_j \mathcal{N}(h) = f_j \ \text{ on } S, \ \ j = 1, 2, 3, \tag{4.32}$$

$$[\mathcal{A}_\tau g]_4 = f_1^{(N)} \ \text{ on } S, \tag{4.33}$$

$$[\mathcal{A}_\tau g]_5 = f_2^{(N)} \ \text{ on } S, \tag{4.34}$$

where $\mathcal{N}$ and $\mathcal{M}$ are defined in (3.1) and (3.2), while $\mathcal{A}_\tau$ is defined in (4.26).

The operator generated by the left-hand side of the system (4.31)–(4.33) reads as

$$\mathcal{P}_{\tau,N} := \begin{pmatrix} [(n,0,0)]_{1\times 5} & -b_1\mathcal{M} \\ [\mathcal{A}_\tau^{jk}]_{3\times 5} & [-b_2 n_j \mathcal{N}]_{3\times 1} \\ [\mathcal{A}_\tau^{4j}]_{1\times 4} & [0]_{1\times 2} \\ [\mathcal{A}_\tau^{5j}]_{1\times 4} & [0]_{1\times 2} \end{pmatrix}_{6\times 6}, \quad j = 1, 2, 3, \ \ k = \overline{1, 5}.$$

The operator $\mathcal{P}_{\tau,N}$ possesses the following mapping property:

$$\mathcal{P}_{\tau,N} : [H^{1/2}(S)]^6 \to [H^{-1/2}(S)]^6.$$

From equation (4.31), we define h,

$$h = b_1^{-1} \mathcal{M}^{-1}(\widetilde{g} \cdot n) - b_1^{-1} \mathcal{M}^{-1} f_0,$$

and substitute this into equation (4.32). We obtain the system

$$[\mathcal{A}_\tau g]_j - b_2 b_1^{-1} n_j \mathcal{N} \mathcal{M}^{-1}(\widetilde{g} \cdot n) = F_j \ \text{ on } S, \ \ j = 1, 2, 3, \tag{4.35}$$

$$[\mathcal{A}_\tau g]_4 = f_1^{(N)} \ \text{ on } S, \tag{4.36}$$

$$[\mathcal{A}_\tau g]_5 = f_2^{(N)} \ \text{ on } S, \tag{4.37}$$

where $F_j = f_j - b_1^{-1} b_2 n_j \mathcal{N} \mathcal{M}^{-1} f_0$.

Denote by $\mathcal{R}_{\tau,N}$ the operator generated by the left-hand side of system (4.35)–(4.37),

$$\mathcal{R}_{\tau,N} = \begin{pmatrix} [C_\tau]_{3\times 3} & [\mathcal{A}_\tau^{j4}]_{3\times 1} & [\mathcal{A}_\tau^{j5}]_{3\times 1} \\ [\mathcal{A}_\tau^{4j}]_{1\times 3} & \mathcal{A}_\tau^{44} & \mathcal{A}_\tau^{45} \\ [\mathcal{A}_\tau^{5j}]_{1\times 3} & \mathcal{A}_\tau^{54} & \mathcal{A}_\tau^{55} \end{pmatrix}_{5\times 5},$$

where

$$[C_\tau]_{3\times 3} = [\mathcal{A}_\tau^{jk}]_{3\times 3} - b_2 b_1^{-1}[n_j\mathcal{N}]_{3\times 1}[\mathcal{M}^{-1}n_k]_{1\times 3}, \quad j,k = 1,2,3.$$

Note that the difference $\mathcal{A}_\tau - \mathcal{R}_{\tau,N} : [H^{1/2}(S)]^5 \to [H^{-1/2}(S)]^5$ is a compact operator.

Since the Steklov–Poincaré type operator $\mathcal{A}_\tau$ is strongly elliptic pseudodifferential operator of order 1, it follows that the operator $\mathcal{A}_\tau : [H^{1/2}(S)]^5 \to [H^{-1/2}(S)]^5$ is Fredholm with index zero. Hence the operators

$$\mathcal{R}_{\tau,N} : [H^{1/2}(S)]^5 \to [H^{-1/2}(S)]^5, \quad \mathcal{P}_{\tau,N} : [H^{1/2}(S)]^6 \to [H^{-1/2}(S)]^6$$

are Fredholm with index zero.

Now let us investigate the null space of the operator $\mathcal{P}_{\tau,N}$. Let $g \in [H^{1/2}(S)]^5$ and $h \in H^{1/2}(S)$ be solutions of the homogeneous system (4.31)–(4.33)

$$\mathcal{P}_{\tau,N}(g,h)^\top = 0,$$

and put

$$\widetilde{U} = (\widetilde{u}, \widetilde{u}_4, \widetilde{u}_5)^\top = \mathbf{V}_\tau \mathbf{H}_\tau^{-1} g, \quad \widetilde{\mathrm{w}} = (W_\omega + \mu V_\omega)h.$$

Evidently, $\widetilde{U}$ and $\widetilde{\mathrm{w}}$ solve the homogeneous problem (N_τ).

From the structure of a solution to the homogeneous problem (N_τ) (see Theorem 4.1) we have

$$\widetilde{U} = (0,0,0,c_1,c_2)^\top \text{ in } \Omega^+, \quad \widetilde{\mathrm{w}} = 0 \text{ in } \Omega^-,$$

where c_1 and c_2 are arbitrary constants. Then $\{\widetilde{U}\}^+ = (0,0,0,c_1,c_2)^\top = g$ on S, i.e. $g_1 = g_2 = g_3 = 0$, $g_4 = c_1$ and $g_5 = c_2$. Since $\{\mathrm{w}\}^- = \mathcal{N}h = 0$ on S, the invertibility of the operator $\mathcal{N}$ yields that $h = 0$ on S. Whence we obtain that if $\mathcal{P}_{\tau,N}(g,h)^\top = 0$, then $g = (0,0,0,c_1,c_2)^\top$ and $h = 0$.

Therefore, the dimension of the null space of the operator $\mathcal{P}_{\tau,N}$ equals to 2, $\dim\operatorname{Ker}\mathcal{P}_{\tau,N} = 2$. Thus $\dim\operatorname{Ker}\mathcal{P}^*_{\tau,N} = 2$, where $\mathcal{P}^*_{\tau,N} : [H^{1/2}(S)]^6 \to [H^{-1/2}(S)]^6$ is the operator adjoint to $\mathcal{P}_{\tau,N} : [H^{1/2}(S)]^6 \to [H^{-1/2}(S)]^6$.

Now we can formulate the following existence theorem.

Theorem 4.4. *Let $\tau = i\sigma$, $\sigma \neq 0$, $\sigma \in \mathbb{R}$, and let $f_0 \in H^{-1/2}(S)$, $f_j \in H^{-1/2}(S)$, $j = 1,2,3$, and $f_1^{(N)} \in H^{-1/2}(S)$, $f_2^{(N)} \in H^{-1/2}(S)$. Then problem (N_τ) is solvable if and only if the condition*

$$\langle f_0, \phi_1\rangle_S + \sum_{j=1}^{3}\langle f_j, \phi_{j+1}\rangle_S + \langle f_1^{(N)}, \phi_5\rangle_S + \langle f_2^{(N)}, \phi_6\rangle_S = 0 \tag{4.38}$$

is fulfilled, where $\phi = (\phi_1,\phi_2,\phi_3,\phi_4,\phi_5,\phi_6)^\top$ is a nontrivial solution of the homogeneous equation $\mathcal{P}^*_{\tau,N}\phi = 0$. If condition (4.38) holds, then solutions of problem (N_τ) are represented by the potentials

$$U = \mathbf{V}_\tau \mathbf{H}_\tau^{-1} g \text{ in } \Omega^+, \quad \mathrm{w} = (W_\omega + \mu V_\omega)h \text{ in } \Omega^-,$$

where the densities $g \in [H^{1/2}(S)]^5$ and $h \in H^{1/2}(S)$ are defined from system (4.31)–(4.35), and they are defined modulo the addend vector $(0,0,0,c_1,c_2)^\top$ with arbitrary complex constants c_1 and c_2.

5 Existence results for the steady state oscillation problems (D_ω) and (N_ω)

5.1 Existence of solution of the Dirichlet type problem (D_ω)

We look for a solution of problem (D_ω) in the form

$$U = \mathbf{V}_\omega g \text{ in } \Omega^+, \quad \mathrm{w} = (W_\omega + \mu V_\omega)h \text{ in } \Omega^-, \quad \mu \in \mathbb{C}, \ \operatorname{Im}\mu \neq 0,$$

where $g \in [H^{-1/2}(S)]^5$ and $h \in H^{1/2}(S)$ are unknown densities, and $\omega \in \mathbb{R}\setminus\{0\}$. From Theorems 6.1 and 6.3 of Appendix it follows that $U \in [H^1(\Omega^+)]^5$ and $\mathrm{w} \in H^1_{loc}(\Omega^-)$.

Transmission conditions (1.11), (1.12) and the Dirichlet boundary conditions (1.13), (1.14) lead to the following system of pseudodifferential equations with respect to the unknowns g and h:

$$[\mathbf{H}_\omega g]_l n_l - b_1\mathcal{M}(h) = f_0 \ \text{ on } S, \tag{5.1}$$

$$\big[(-2^{-1}I_4 + \mathbf{K}_\omega)g\big]_j - b_2 n_j\mathcal{N}(h) = f_j \ \text{ on } S, \ \ j = 1,2,3, \tag{5.2}$$

$$[\mathbf{H}_\omega g]_4 = f_1^{(D)} \ \text{ on } S, \tag{5.3}$$

$$[\mathbf{H}_\omega g]_5 = f_2^{(D)} \ \text{ on } S. \tag{5.4}$$

The operator generated by the left-hand side of system (5.1)–(5.4) reads as

$$Q_{\omega,D} = \begin{pmatrix} [n_l\mathbf{H}_\omega^{lk}]_{1\times 5} & -b_1\mathcal{M} \\ \big[(-2^{-1}I_5 + \mathbf{K}_\omega)^{jk}\big]_{3\times 5} & [-b_2 n_j\mathcal{N}]_{3\times 1} \\ [\mathbf{H}_\omega^{4k}]_{1\times 5} & 0 \\ [\mathbf{H}_\omega^{5k}]_{1\times 5} & 0 \end{pmatrix}_{6\times 6}, \quad j = \overline{1,3}, \ \ k = \overline{1,5}.$$

By Theorem 6.5, the operator

$$Q_{\omega,D} : [H^{-1/2}(S)]^5 \times H^{1/2}(S) \to [H^{-1/2}(S)]^4 \times [H^{1/2}(S)]^2$$

is bounded.

In view of estimates (4.19)-(4.21) it follows that the main parts of the operators $\mathbf{H}_\omega$ and $\mathbf{H}_\tau$ (as well as the main parts of the operators $\mathbf{K}_\omega$ and $\mathbf{K}_\tau$) are the same, implying that the operators

$$\mathbf{H}_\omega - \mathbf{H}_\tau : [H^{-1/2}(S)]^5 \to [H^{1/2}(S)]^5, \tag{5.5}$$

$$\mathbf{K}_\omega - \mathbf{K}_\tau : [H^{-1/2}(S)]^5 \to [H^{-1/2}(S)]^5 \tag{5.6}$$

are compact. Hence the operator

$$Q_{\omega,D} - Q_{\tau,D} : [H^{-1/2}(S)]^5 \times H^{1/2}(S) \to [H^{-1/2}(S)]^4 \times [H^{1/2}(S)]^2$$

is compact, where $Q_{\tau,D} := \mathcal{P}_{\tau,D}\mathcal{T}_\tau$ with

$$\mathcal{T}_\tau := \begin{pmatrix} \mathbf{H}_\tau & [0]_{4\times 1} \\ [0]_{1\times 4} & I_1 \end{pmatrix}_{5\times 5}. \tag{5.7}$$

Therefore, from the invertibility of the operators $\mathcal{P}_{\tau,D} : [H^{1/2}(S)]^6 \to [H^{-1/2}(S)]^5 \times H^{1/2}(S)$ and $\mathcal{T}_\tau : [H^{-1/2}(S)]^5 \times H^{1/2}(S) \to [H^{1/2}(S)]^6$ (see Section 4) the invertibility of the operator $Q_{\tau,D} : [H^{-1/2}(S)]^5 \times H^{1/2}(S) \to [H^{-1/2}(S)]^5 \times H^{1/2}(S)$ follows. In turn, this implies that the operator

$$Q_{\omega,D} : [H^{-1/2}(S)]^5 \times H^{1/2}(S) \to [H^{-1/2}(S)]^4 \times [H^{1/2}(S)]^2 \tag{5.8}$$

is Fredholm with index zero.

Let us show that for $\omega \notin J_D(\Omega^+)$ the operator $Q_{\omega,D}$ is injective. Indeed, let $g \in [H^{-1/2}(S)]^5$ and $h \in H^{1/2}(S)$ be solutions of the homogeneous system

$$Q_{\omega,D}(g,h)^\top = 0 \ \text{ on } S.$$

Construct a vector-function $U = \mathbf{V}_\omega g$ and a scalar function $\mathrm{w} = (W_\omega + \mu V_\omega)h$ with $\mu \in \mathbb{C}$, $\operatorname{Im}\mu \neq 0$; Clearly, the pair (U, w) solves the homogeneous problem (D_ω). Since $\omega \notin J_D(\Omega^+)$, from Theorem 2.1 we have that

$$U = \mathbf{V}_\omega g = 0 \ \text{ in } \Omega^+, \quad \mathrm{w} = (W_\omega + \mu V_\omega)h = 0 \ \text{ in } \Omega^-.$$

In view of the equation $\{\mathrm{w}\}^- = \mathcal{N}(h) = 0$ on S and the invertibility of the operator $\mathcal{N}$ we deduce that $h = 0$ on S. From continuity of a single layer potential we have $\{U\}^+ = \{U\}^- = 0$ on S.

Thus $U=\mathbf{V}_\omega g$ solves the exterior homogeneous Dirichlet problem

$$A(\partial,\omega)U=0 \ \text{ on } \Omega^-, \quad \{U\}^-=0 \ \text{ on } S. \tag{5.9}$$

$U=\mathbf{V}_\omega g\in M_{m_1,m_2,m_3}(\mathbf{P})$ and, by Theorem 3.4, $U=\mathbf{V}_\omega g\equiv 0$ in Ω^-. Using the jump formula $\{TU\}^- - \{TU\}^+ = g$ on S, we get $g=0$ on S. Thus the null space of the Fredholm operator (5.8) is trivial and since the index equals to zero we conclude that (5.8) is invertible.

These results imply the following assertion.

Theorem 5.1. *If $\omega\notin J_D(\Omega^+)$, then problem (D_ω) is uniquely solvable.*

Now let us consider the case where ω is Jones's frequency, $\omega\in J_D(\Omega^+)$.

The operator adjoint to $Q_{\omega,D}$ has the following form:

$$Q^*_{\omega,D}=\begin{pmatrix} [\mathbf{H}^{*kl}_{\ \omega} n_l]_{5\times 1} & \big[(-2^{-1}I_4+\mathbf{K}^*_\omega)^{kj}\big]_{5\times 3} & [\mathbf{H}^{*k4}_{\ \omega}]_{5\times 1} & [\mathbf{H}^{*k5}_{\ \omega}]_{5\times 1} \\ -\overline{b}_1\mathcal{M}^* & [-\overline{b}_2\mathcal{N}^* n_j]_{1\times 3} & 0 & 0\end{pmatrix}_{6\times 6}, \quad j=\overline{1,3}, \quad k=\overline{1,5},$$

where

$$\mathbf{H}^*_\omega(g)(z)=\int_S [\overline{\Gamma(y-z,\omega)}]^\top g(y)\,d_yS, \quad z\in S,$$

$$\mathbf{K}^*_\omega(g)(z)=\int_S \big[T(\partial_y,n(y)\,\overline{\Gamma(y-z,\omega)})\big]^\top g(y)d_yS, \quad z\in S,$$

$$\mathcal{N}^*(h)(z)=\big(-2^{-1}I_1+\overline{\mathcal{K}_\omega}\big)(h)(z)+\overline{\mu}\mathcal{H}^*_\omega(h)(z), \quad z\in S,$$

$$\mathcal{M}^*(h)(z)=\mathcal{L}^*_\omega(h)(z)+\overline{\mu}\big(2^{-1}I_1+\overline{\mathcal{K}}^*_\omega\big)(h)(z), \quad z\in S,$$

while

$$\overline{\mathcal{K}}_\omega(h)(z)=\int_S \partial_{n(z)}\overline{\gamma(z-y,\omega)}h(y)\,d_yS, \quad z\in S,$$

$$\overline{\mathcal{K}}^*_\omega(h)(z)=\int_S \partial_{n(y)}\overline{\gamma(z-y,\omega)}h(y)\,d_yS, \quad z\in S,$$

$$\mathcal{H}^*_\omega(h)(z)=\int_S \overline{\gamma(z-y,\omega)}h(y)\,d_yS, \quad z\in S,$$

$$\mathcal{L}^*_\omega(h)(z)=\big\{\partial_{n(z)}\widetilde{W}_\omega(h)(z)\big\}^\pm, \quad z\in S,$$

$$\widetilde{W}_\omega(h)(x)=\int_S \partial_{n(y)}\overline{\gamma(x-y,\omega)}h(y)\,d_yS, \quad x\notin S,$$

$$\widetilde{V}_\omega(h)(x)=\int_S \overline{\gamma(x-y,\omega)}h(y)\,d_yS, \quad x\notin S.$$

The adjoint operator possesses the following mapping property:

$$Q^*_{\omega,D}:[H^{1/2}(S)]^4\times[H^{-1/2}(S)]^2\to[H^{1/2}(S)]^5\times H^{-1/2}(S).$$

Let $\Psi:=(\psi_1,\psi_2,\psi_3,\psi_4,\psi_5,\psi_6)^\top\in[H^{1/2}(S)]^4\times[H^{-1/2}(S)]^2$ be a solution of the homogeneous adjoint system

$$Q^*_{\omega,D}\Psi=0. \tag{5.10}$$

Construct the potentials

$$\widetilde{U}=\widetilde{\mathbf{V}}_\omega\Psi^{(1)}+\widetilde{\mathbf{W}}_\omega\Psi^{(2)}+\widetilde{\mathbf{V}}_\omega\Psi^{(3)} \ \text{ in } \Omega^-, \tag{5.11}$$

$$\widetilde{\mathrm{w}}=-\overline{b}_1\widetilde{W}_\omega\psi_1-\overline{b}_2\widetilde{V}_\omega[\Psi'\cdot n] \ \text{ in } \Omega^+, \tag{5.12}$$

where

$$\Psi^{(1)} := (n\psi_1, 0)^\top, \quad \Psi^{(2)} := (\Psi', 0)^\top, \quad \Psi^{(3)} := (0,0,0,\psi_5,\psi_6)^\top, \quad \Psi' = (\psi_2,\psi_3,\psi_4)^\top,$$
$$\widetilde{\mathbf{V}}_\omega(g)(x) := \int\limits_S [\overline{\Gamma(y-x,\omega)}]^\top g(y)\, d_yS, \;\; x \in \Omega^+,$$
$$\widetilde{\mathbf{W}}_\omega(g)(x) := \int\limits_S \big[T(\partial_y, n(y))\overline{\Gamma(y-x,\omega)}\,\big]^\top g(y)\, d_yS, \;\; x \in \Omega^+.$$

The vectors $\widetilde{\mathbf{V}}_\omega(g)$ and $\widetilde{\mathbf{W}}_\omega(g)$ are the single and double layer potentials associated with the operator $A^*(\partial,\omega)$.

From (5.10) it follows that

$$\{\widetilde{U}\}^- = 0 \;\text{ and }\; \{\partial_n\widetilde{\mathrm{w}} + \overline{\mu}\widetilde{\mathrm{w}}\}^+ = 0 \;\text{ on } S,$$

where $\mu = \mu_1 + i\mu_2$, $\mu_2 \neq 0$.

Since the vector $\widetilde{U} \in [H^1_{loc}(\Omega^-)]^5 \cap M_{m_1,m_2,m_3}(\mathbf{P}^*)$ and solves the homogeneous Dirichlet problem

$$A^*(\partial,\omega)\widetilde{U} = 0 \;\text{ in } \Omega^-, \quad \{\widetilde{U}\}^- = 0 \;\text{ on } S,$$

the uniqueness Theorem 3.5 implies that $\widetilde{U} = 0$ in Ω^-.

On the other hand, the function $\widetilde{\mathrm{w}} \in H^1(\Omega^+)$ solves the homogeneous Robin type problem

$$(\Delta + \rho_2\omega^2)\widetilde{\mathrm{w}} = 0 \;\text{ in } \Omega^+, \tag{5.13}$$
$$\{\partial_n\widetilde{\mathrm{w}} + \overline{\mu}\widetilde{\mathrm{w}}\}^+ = 0 \;\text{ on } S. \tag{5.14}$$

This problem possesses only the trivial solution. Indeed, the following Green's first formula holds:

$$\int\limits_{\Omega^+} (\Delta + \rho_2\omega^2)\widetilde{\mathrm{w}}\overline{\widetilde{\mathrm{w}}}\, dx + \int\limits_{\Omega^+} |\nabla\widetilde{\mathrm{w}}|\, dx - \rho_2\omega^2 \int\limits_{\Omega^+} |\widetilde{\mathrm{w}}|\, dx = \big\langle \{\partial_n\widetilde{\mathrm{w}}\}^+, \{\widetilde{\mathrm{w}}\}^+ \big\rangle_S, \tag{5.15}$$

Taking into account equation (5.13) and the boundary condition (5.14), from (5.15) we get

$$\int\limits_{\Omega^+} |\nabla\widetilde{\mathrm{w}}|\, dx - \rho_2\omega^2 \int\limits_{\Omega^+} |\widetilde{\mathrm{w}}|\, dx = -\mu_1 \int\limits_S \big|\{\widetilde{\mathrm{w}}\}^+\big|^2 dS + i\mu_2 \int\limits_S \big|\{\widetilde{\mathrm{w}}\}^+\big|^2 dS.$$

Therefore, $\{\widetilde{\mathrm{w}}\}^+ = 0$. For a solution $\widetilde{\mathrm{w}} \in H^1(\Omega^+)$ to the homogeneous equation (5.13) we have the following integral representation:

$$\widetilde{\mathrm{w}} = W_\omega\big(\{\widetilde{\mathrm{w}}\}^+\big) - V_\omega\big(\{\partial_n\widetilde{\mathrm{w}}\}^+\big) \;\text{ in } \Omega^+. \tag{5.16}$$

Since $\{\widetilde{\mathrm{w}}\}^+ = 0$ and $\{\partial_n\widetilde{\mathrm{w}}\}^+ = 0$, from the representation formula (5.16) we find that $\widetilde{\mathrm{w}} = 0$ in Ω^+.

Using the jump formulae for potentials (5.11) and (5.12), we derive that on the surface S the following relations hold:

$$\begin{aligned}
\{\widetilde{\mathrm{w}}\}^- &= \overline{b}_1\psi_1,\\
\{\partial_n\widetilde{\mathrm{w}}\}^- &= -\overline{b}_2\, \Psi' \cdot n,\\
\{[\widetilde{T}\widetilde{U}]_j\}^+ &= -n_j\psi_1, \;\; j = 1,2,3,\\
\{[\widetilde{T}\widetilde{U}]_4\}^+ &= -\psi_5,\\
\{[\widetilde{T}\widetilde{U}]_5\}^+ &= -\psi_6,\\
\{\widetilde{U}\}^+ &= (\Psi', 0)^\top,\\
\{\widetilde{U}_4\}^+ &= 0,\\
\{\widetilde{U}_5\}^+ &= 0.
\end{aligned}$$

Hence we deduce that $\widetilde{U}=(\widetilde{U}_1,\widetilde{U}_2,\widetilde{U}_3,\widetilde{U}_4,\widetilde{U}_5)^\top=(\widetilde{U}',\widetilde{U}_4,\widetilde{U}_5)^\top$ with $\widetilde{U}'=(\widetilde{U}_1,\widetilde{U}_2,\widetilde{U}_3,)^\top$ and $\widetilde{\mathrm{w}}$ solve the following homogeneous transmission problem:

$$
\begin{aligned}
A^*(\partial,\omega)\widetilde{U}&=0 \ \text{ in } \Omega^+,\\
(\Delta+\rho_2\omega^2)\widetilde{\mathrm{w}}&=0 \ \text{ in } \Omega^-,\\
\{\widetilde{U}'\cdot n\}^+ +\overline{b}_2^{\,-1}\{\partial_n\widetilde{\mathrm{w}}\}^-&=0 \ \text{ on } S,\\
\big\{[\widetilde{T}(\partial,n)\widetilde{U}]_j\big\}^+ +\overline{b}_1^{\,-1}\{\widetilde{\mathrm{w}}\}^- n_j&=0 \ \text{ on } S, \quad j=1,2,3,\\
\{\widetilde{U}_4\}^+&=0 \ \text{ on } S,\\
\{\widetilde{U}_5\}^+&=0 \ \text{ on } S,
\end{aligned}
$$

From the uniqueness result (see Remark 2.3) it follows that $\widetilde{\mathrm{w}}=0$ in Ω^- and $\widetilde{U}\in X^*_{D,\omega}(\Omega^+)$, i.e., $\widetilde{U}$ belongs to the space of Jones modes $X^*_{D,\omega}(\Omega^+)$. Then we obtain

$$\psi_1=0,\quad \psi_{j+1}=\{\widetilde{U}_j\}^+ \ \ j=1,2,3,\quad \psi_5=-\big\{[\widetilde{T}\widetilde{U}]_4\big\}^+,\quad \psi_6=-\big\{[\widetilde{T}\widetilde{U}]_5\big\}^+.$$

Vice versa, if $\widetilde{U}\in X^*_{D,\omega}(\Omega^+)$, then from the representation formula

$$\widetilde{U}=\widetilde{\mathbf{W}}_\omega\{\widetilde{U}\}^+ -\widetilde{\mathbf{V}}_\omega\{\widetilde{T}\widetilde{U}\}^+ \ \text{ in } \Omega^+ \tag{5.17}$$

it is easy to show that the vector-function $\widetilde{\Psi}:=(0,\{\widetilde{U}_1\}^+,\{\widetilde{U}_2\}^+,\{\widetilde{U}_3\}^+,-\{[\widetilde{T}\widetilde{U}]_4\}^+,-\{[\widetilde{T}\widetilde{U}]_5\}^+)^\top$ is a solution of the adjoint homogeneous system (5.10). Indeed, let us substitute $\widetilde{\Psi}$ in system (5.10). Therefore, we obtain the equalities

$$\big[(-2^{-1}I_4+\mathbf{K}^*_\omega)^{kj}\big]_{5\times3}\{\widetilde{U}'\}^+ -[\mathbf{H}^{*k4}_{\ \omega}]_{5\times1}\big\{[\widetilde{T}\widetilde{U}]_4\big\}^+ -[\mathbf{H}^{*k5}_{\ \omega}]_{5\times1}\big\{[\widetilde{T}\widetilde{U}]_5\big\}^+=0, \qquad j=\overline{1,3},\ \ k=\overline{1,5}, \tag{5.18}$$

$$-\overline{b}_2\mathcal{N}^*\big(\{\widetilde{U}'\}^+\cdot n\big)=0, \tag{5.19}$$

where $\widetilde{U}'=(\widetilde{U}_1,\widetilde{U}_2,\widetilde{U}_3)^\top$.

By taking a trace of the representation formula (5.17), we get

$$\{\widetilde{U}\}^+=2^{-1}\{\widetilde{U}\}^+ +\mathbf{K}^*_\omega\{\widetilde{U}\}^+ -\mathbf{H}^*_\omega\{\widetilde{T}\widetilde{U}\}^+ \ \text{ on } S,$$

i.e., we have

$$(-2^{-1}I+\mathbf{K}^*_\omega)\{\widetilde{U}\}^+ -\mathbf{H}^*_\omega\{\widetilde{T}\widetilde{U}\}^+=0 \ \text{ on } S. \tag{5.20}$$

Since $\widetilde{U}\in X^*_{D,\omega}(\Omega^+)$, we have

$$\{\widetilde{U}_4\}^+=0,\quad \{\widetilde{U}_5\}^+=0,\quad \big\{[\widetilde{T}\widetilde{U}]_j\big\}^+=0,\ \ j=1,2,3, \tag{5.21}$$

$$\{\widetilde{U}'\}^+\cdot n=0. \tag{5.22}$$

Therefore, taking into account (5.21) in equality (5.20), we find that (5.18) is true, and it follows from (5.22) that (5.19) is true.

Therefore,

$$\dim\ker Q_{\omega,D}=\dim\ker Q^*_{\omega,D}=\dim X^*_{D,\omega}(\Omega^+).$$

Thus the orthogonality condition

$$\sum_{j=1}^{3}\big\langle f_j,\{\widetilde{U}_j\}^+\big\rangle_S-\Big\langle\big\{[\overline{\widetilde{T}\widetilde{U}}]_4\big\}^+,\overline{f}_1^{(D)}\Big\rangle_S-\Big\langle\big\{[\overline{\widetilde{T}\widetilde{U}}]_5\big\}^+,\overline{f}_2^{(D)}\Big\rangle_S=0 \ \ \forall\,\widetilde{U}\in X^*_{D,\omega}(\Omega^+), \tag{5.23}$$

is necessary and sufficient for the system of pseudodifferential equations (5.1)–(5.4) to be solvable.

We can now formulate the following existence theorem.

Theorem 5.2. *If $\omega \in J_D(\Omega^+)$, then the Dirichlet type problem (D_ω) is solvable if and only if the orthogonality condition* (5.23) *holds, and a solution is defined modulo Jones modes $X_{D,\omega}(\Omega^+)$.*

Remark 5.3. Let $(f_1, f_2, f_3) = n\psi$, where ψ is a scalar function and n is the unit normal vector to S (see (1.18)). Then the necessary and sufficient condition (5.23) reads as

$$\left\langle \left\{ [\widetilde{T}\widetilde{U}]_4 \right\}^+, f_1^{(D)} \right\rangle_S + \left\langle \left\{ [\widetilde{T}\widetilde{U}]_5 \right\}^+, f_2^{(D)} \right\rangle_S = 0 \;\; \forall \widetilde{U} \in X^*_{D,\omega}(\Omega^+).$$

Clearly, if the Dirichlet datum for the electric potential and magnetic potential are constant, or $\omega \notin J^*_D(\Omega^+)$, then problem (D_ω) is always solvable.

5.2 Existence of solution to the Neumann type problem (N_ω)

We look for a solution of the Neumann type problem (N_ω) in the form of the following potentials:

$$U = \mathbf{V}_\omega g \;\text{ in } \Omega^+, \quad \mathrm{w} = (W_\omega + \mu V_\omega)h \;\text{ in } \Omega^-,$$

where $g \in [H^{-1/2}(S)]^5$ and $h \in H^{1/2}(S)$ are unknown densities. From Theorems 6.1 and 6.3 of Appendix it follows that $U \in [H^1(\Omega^+)]^5$ and $\mathrm{w} \in H^1_{loc}(\Omega^-)$.

Transmission conditions (1.11), (1.12) and the Neumann boundary conditions (1.16), (1.17) lead to the following system of pseudodifferential equations with respect to the unknowns g and h:

$$[\mathbf{H}_\omega g]_l n_l - b_1 \mathcal{M}(h) = f_0 \;\text{ on } S, \tag{5.24}$$

$$\left[(-2^{-1}I_5 + \mathbf{K}_\omega)g\right]_j - b_2 n_j \mathcal{N}(h) = f_j \;\text{ on } S, \;\; j = 1,2,3, \tag{5.25}$$

$$\left[(-2^{-1}I_5 + \mathbf{K}_\omega)g\right]_4 = f_1^{(N)} \;\text{ on } S, \tag{5.26}$$

$$\left[\big(-2^{-1}I_5 + \mathbf{K}_\omega\big)g\right]_5 = f_2^{(N)} \;\text{ on } S. \tag{5.27}$$

The operator generated by the left-hand side of system (5.24)–(5.27) reads as

$$Q_{\omega,N} = \begin{pmatrix} [n_l \mathbf{H}^{lk}_\omega]_{1\times5} & -b_1\mathcal{M} \\ \left[(-2^{-1}I_5 + \mathbf{K}_\omega)^{jk}\right]_{3\times5} & [-b_2 n_j \mathcal{N}]_{3\times1} \\ \left[(-2^{-1}I_5 + \mathbf{K}_\omega)^{4k}\right]_{1\times5} & 0 \\ \left[(-2^{-1}I_5 + \mathbf{K}_\omega)^{5k}\right]_{1\times5} & 0 \end{pmatrix}_{6\times6}, \quad j = \overline{1,3}, \;\; k = \overline{1,5}.$$

Due to Theorem 6.5 (see Appendix), it is evident that the operator

$$Q_{\omega,N} : [H^{-1/2}(S)]^5 \times H^{1/2}(S) \to [H^{-1/2}(S)]^6$$

is bounded.

It follows from (5.5) and (5.6) that the operator

$$Q_{\omega,N} - Q_{\tau,N} : [H^{-1/2}(S)]^5 \times H^{1/2}(S) \to [H^{-1/2}(S)]^6$$

is compact, where $Q_{\tau,N} := \mathcal{P}_{\tau,N}\mathcal{T}_\tau$ with the operator $\mathcal{T}_\tau$ defined in (5.7). Since the operator $Q_{\tau,N}$ is Fredholm with index zero (see Section 4), we have that the operator

$$Q_{\omega,N} : [H^{-1/2}(S)]^5 \times H^{1/2}(S) \to [H^{-1/2}(S)]^6$$

is Fredholm with index zero.

Recall that $J_N(\Omega^+) = \mathbb{R}$, due to Theorem 2.2 (see the end of Subsection 2.1).

The operator adjoint to $Q_{\omega,N}$ has the form

$$Q^*_{\omega,N} = \begin{pmatrix} [\mathbf{H}^{*kl}_{\;\omega} n_l]_{5\times1} & \left[(-2^{-1}I_5 + \mathbf{K}^*_\omega)^{kj}\right]_{5\times3} & [(-2^{-1}I_5 + \mathbf{K}^*_\omega)^{k4}]_{5\times1} & \left[(-2^{-1}I_5 + \mathbf{K}^*_\omega)^{k5}\right]_{5\times1} \\ -\bar{b}_1\mathcal{M}^* & \left[-\bar{b}_2\mathcal{N}^* n_j\right]_{1\times3} & 0 & 0 \end{pmatrix}_{6\times6},$$

$$j = \overline{1,3}, \;\; k = \overline{1,5},$$

and

$$Q^*_{\omega,N} : [H^{1/2}(S)]^6 \to [H^{1/2}(S)]^5 \times H^{-1/2}(S)$$

is bounded.

Let $\Phi := (\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6)^\top \in [H^{1/2}(S)]^6$ be a solution of the homogeneous adjoint system

$$Q^*_{\omega,N}\Phi = 0. \tag{5.28}$$

Construct the potentials

$$\widetilde{U} = \widetilde{\mathbf{V}}_\omega \Phi^{(1)} + \widetilde{\mathbf{W}}_\omega \Phi^{(2)} \ \text{ in } \Omega^-, \tag{5.29}$$

$$\widetilde{\mathrm{w}} = -\overline{b}_1 \widetilde{W}_\omega \varphi_1 - \overline{b}_2 \widetilde{V}_\omega [\Phi' \cdot n] \ \text{ in } \Omega^+, \tag{5.30}$$

where $\Phi^{(1)} := (n\varphi_1, 0)^\top$, $\Phi^{(2)} := (\Phi', \varphi_5, \varphi_6)^\top$, $\Phi' := (\varphi_2, \varphi_3, \varphi_4)^\top$.

From (5.28) we have

$$\{\widetilde{U}\}^- = 0 \ \text{ on } S,$$

$$\big\{\partial_n \widetilde{\mathrm{w}} + \overline{\mu}\widetilde{\mathrm{w}}\big\}^+ = 0 \ \text{ on } S,$$

where $\widetilde{U} \in [H^1_{loc}(\Omega^-)]^5 \cap M_{m_1,m_2,m_3}(\mathbf{P}^*)$ and $\widetilde{\mathrm{w}} \in H^1(\Omega^+)$.

Therefore, from the uniqueness results for the exterior Dirichlet problem (see Theorem 3.5) and interior Robin type problem, we conclude that $\widetilde{U} = 0$ in Ω^- and $\widetilde{\mathrm{w}} = 0$ in Ω^+.

From jump formulae for potentials (5.29) and (5.30) we find that on the surface S the following relations hold:

$$\{\widetilde{\mathrm{w}}\}^- = \overline{b}_1 \varphi_1, \tag{5.31}$$

$$\{\partial_n \widetilde{\mathrm{w}}\}^- = -\overline{b}_2 \Phi' \cdot n, \tag{5.32}$$

$$\{\widetilde{U}\}^+ = (\Phi', \varphi_5, \varphi_6)^\top, \tag{5.33}$$

$$\big\{[\widetilde{T}\widetilde{U}]_j\big\}^+ = -n_j \varphi_1, \ \ j = 1, 2, 3, \tag{5.34}$$

$$\big\{[\widetilde{T}\widetilde{U}]_4\big\}^+ = 0, \tag{5.35}$$

$$\big\{[\widetilde{T}\widetilde{U}]_5\big\}^+ = 0. \tag{5.36}$$

Hence we obtain that $\widetilde{U} = (\widetilde{U}_1, \widetilde{U}_2, \widetilde{U}_3, \widetilde{U}_4, \widetilde{U}_5)^\top = (\widetilde{U}', \widetilde{U}_4, \widetilde{U}_5)^\top$ with $\widetilde{U}' = (\widetilde{U}_1, \widetilde{U}_2, \widetilde{U}_3)^\top$ and $\widetilde{\mathrm{w}}$ solve the following homogeneous problem:

$$A^*(\partial, \omega)\widetilde{U} = 0 \ \text{ in } \Omega^+,$$

$$(\Delta + \rho_2\omega^2)\widetilde{\mathrm{w}} = 0 \ \text{ in } \Omega^-,$$

$$\{\widetilde{U}' \cdot n\}^+ + \overline{b}_2^{\,-1}\{\partial_n \widetilde{\mathrm{w}}\}^- = 0 \ \text{ on } S,$$

$$\big\{[\widetilde{T}(\partial, n)\widetilde{U}]_j\big\}^+ + \overline{b}_1^{\,-1}\{\widetilde{\mathrm{w}}\}^- n_j = 0 \ \text{ on } S, \ \ j = 1, 2, 3,$$

$$\big\{[\widetilde{T}\widetilde{U}]_4\big\}^+ = 0 \ \text{ on } S,$$

$$\big\{[\widetilde{T}\widetilde{U}]_5\big\}^+ = 0 \ \text{ on } S.$$

From uniqueness result (see Remark 2.4) we have $\widetilde{\mathrm{w}} = 0$ in Ω^- and $\widetilde{U} \in X^*_{N,\omega}(\Omega^+)$, i.e., $\widetilde{U}$ belongs to the space of Jones modes $X^*_{N,\omega}(\Omega^+)$.

From (5.31) and (5.33) we get

$$\varphi_1 = 0, \ \ \varphi_{j+1} = \{\widetilde{U}_j\}^+, \ \ j = \overline{1,5}.$$

On the other hand, if $\widetilde{U} \in X^*_{N,\omega}(\Omega^+)$, then using the representation formula (5.17) it is easy to show that the vector-function $\widetilde{\Phi} := (0, \{\widetilde{U}_1\}^+, \{\widetilde{U}_2\}^+, \{\widetilde{U}_3\}^+, \{\widetilde{U}_4\}^+, \{\widetilde{U}_5\}^+)^\top$ is a solution of the homogeneous adjoint system (5.28). Indeed, let us substitute $\widetilde{\Phi}$ in system (5.28). Therefore, we obtain the equalities

$$\big[(-2^{-1}I_5 + \mathbf{K}^*_\omega)\big]\{\widetilde{U}\}^+ = 0, \tag{5.37}$$

$$-\bar{b}_2\mathcal{N}^*\big(\{\widetilde{U}'\}^+ \cdot n\big) = 0. \tag{5.38}$$

Taking the trace of the representation formula (5.17), we get

$$(-2^{-1}I + \mathbf{K}^*_\omega)\{\widetilde{U}\}^+ - \mathbf{H}^*_\omega\{\widetilde{T}\widetilde{U}\}^+ = 0 \ \text{ on } S. \tag{5.39}$$

Since $\widetilde{U} \in X^*_{N,\omega}(\Omega^+)$, we have

$$\{\widetilde{T}\widetilde{U}\}^+ = 0, \tag{5.40}$$

$$\{\widetilde{U}'\}^+ \cdot n = 0. \tag{5.41}$$

Therefore, taking into account (5.40) in equality (5.39), we obtain that (5.37) is true, and it follows from (5.41) that (5.38) is true.

Therefore,

$$\dim\ker Q_{\omega,N} = \dim\ker Q^*_{\omega,N} = \dim X^*_{N,\omega}(\Omega^+).$$

Thus the orthogonality condition

$$\sum_{j=1}^{3}\big\langle f_j, \{\widetilde{U}_j\}^+\big\rangle_S + \big\langle f_1^{(N)}, \{\widetilde{U}_4\}^+\big\rangle_S + \big\langle f_2^{(N)}, \{\widetilde{U}_5\}^+\big\rangle_S = 0 \ \ \forall\, \widetilde{U} \in X^*_{N,\omega}(\Omega^+) \tag{5.42}$$

is necessary and sufficient for the system of pseudodifferential equations (5.24)-(5.27) to be solvable.

The following existence theorem follows directly.

Theorem 5.4. *The Neumann type problem (N_ω) is solvable if and only if the orthogonality condition* (5.42) *holds, and a solution is defined modulo Jones modes $X_{N,\omega}(\Omega^+)$.*

Remark 5.5. If $(f_1, f_2, f_3) = n\psi$, where ψ is a scalar function and n is the unit normal vector to S (see (1.18)), then the necessary and sufficient condition (5.42) can be written in the form

$$\big\langle f_1^{(N)}, \{\widetilde{U}_4\}^+\big\rangle_S + \big\langle f_2^{(N)}, \{\widetilde{U}_5\}^+\big\rangle_S = 0 \ \ \forall\, \widetilde{U} \in X^*_{N,\omega}(\Omega^+).$$

Clearly, if $f_1^{(N)} = f_2^{(N)} = 0$, then problem (N_ω) is always solvable.

6 Appendix

For the readers convenience, we collect here some results describing properties of the layer potentials. Here, we preserve the notation from the main text of the paper. For the potentials associated with the Helmholtz equation, the following theorems hold (see [13, 20, 32, 37]).

Theorem 6.1. *Let $s \in \mathbb{R}$, $1 < p < \infty$, $S \in C^\infty$. Then the single and double layer scalar potentials can be extended to the following continuous operators:*

$$V_\omega : H^s(S) \to H^{s+3/2}(\Omega^+), \quad V_\omega : Hs(S) \to H^{s+3/2}_{loc}(\Omega^-),$$
$$W_\omega : H^s(S) \to H^{s+1/2}(\Omega^+), \quad W_\omega : H^s(S) \to H^{s+1/2}_{loc}(\Omega^-).$$

Theorem 6.2. *Let $s \in \mathbb{R}$, $1 < p < \infty$, $S \in C^\infty$. Then the operators*

$$\mathcal{H}_\omega : H^s(S) \to H^{s+1}(S),$$
$$\mathcal{K}_\omega, \mathcal{K}^*_\omega : H^s(S) \to H^{s+1}(S),$$
$$\mathcal{L}_\omega : H^s(S) \to H^{s-1}(S)$$

are continuous.

For the potentials of steady state oscillation and pseudo-oscillation equations, the following theorems hold (see [5–8, 12]).

Theorem 6.3. *Let $s \in \mathbb{R}$, $1 < p < \infty$, $S \in C^\infty$. Then the vector potentials $\mathbf{V}_\omega$, $\mathbf{W}_\omega$, $\mathbf{V}_\tau$ and $\mathbf{W}_\tau$ are continuous in the following spaces:*

$$\mathbf{V}_\omega, \mathbf{V}_\tau : [H^s(S)]^5 \to [H^{s+3/2}(\Omega^+)]^5 \quad \Big([H^s(S)]^5 \to [H^{s+3/2}_{loc}(\Omega^-)]^5 \Big),$$
$$\mathbf{W}_\omega, \mathbf{W}_\tau : [H^s(S)]^5 \to [H^{s+1/2}_p(\Omega^+)]^5 \quad \Big([H^s(S)]^5 \to [H^{s+1/2}_{loc}(\Omega^-)]^5 \Big).$$

Theorem 6.4. *Let $s \in \mathbb{R}$, $1 < p < \infty$, $S \in C^\infty$. Then the operators*

$$\mathbf{H}_\tau : [H^s(S)]^5 \to [H^{s+1}(S)]^5,$$
$$\mathbf{K}_\tau, \widetilde{\mathbf{K}}_\tau : [H^s(S)]^5 \to [H^s(S)]^5,$$
$$\mathbf{L}_\tau : [H^s(S)]^5 \to [H^{s-1}(S)]^5$$

are bounded.

The operators $\mathbf{H}_\tau$ and $\boldsymbol{L}_\tau$ are strongly elliptic pseudodifferential operators of order -1, and 1 respectively, while the operators $\pm 2^{-1} I_5 + \mathbf{K}_\tau$ and $\pm 2^{-1} I_5 + \widetilde{\mathbf{K}}_\tau$ are elliptic pseudodifferential operators of order 0.

Moreover, the operators $\mathbf{H}_\tau$, $2^{-1} I_5 + \widetilde{\mathbf{K}}_\tau$ and $2^{-1} I_5 + \mathbf{K}_\tau$ are invertible, whereas the operators $\mathbf{L}_\tau$, $-2^{-1} I_5 + \widetilde{\mathbf{K}}_\tau$ and $-2^{-1} I_5 + \mathbf{K}_\tau$ are Fredholm operators with index zero.

Theorem 6.5. *Let $s \in \mathbb{R}$, $1 < p < \infty$, $S \in C^\infty$. Then the operators*

$$\mathbf{H}_\omega : [H^s(S)]^5 \to [H^{s+1}(S)]^5,$$
$$\pm 2^{-1} I_5 + \mathbf{K}_\omega : [H^s(S)]^5 \to [H^s(S)]^5,$$
$$\pm 2^{-1} I_5 + \widetilde{\mathbf{K}}_\omega : [H^s(S)]^5 \to [H^s(S)]^5,$$
$$\mathbf{L}_\omega : [H^s(S)]^5 \to [H^{s-1}(S)]^5$$

are bounded Fredholm operators with index zero.

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