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COMPARISON THEOREM AND SOLVABILITY OF THE BOUNDARY VALUE PROBLEM OF A FRACTIONAL DIFFERENTIAL EQUATION

Abstract. When the nonlinearities satisfy the growth conditions on a finite interval, some existence results of solutions to the boundary value problems of fractional differential equations are established via comparison theorem, upper and lower solutions method and fixed point theorems. An example is presented to illustrate the applications of the obtained results.

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რეზიუმე. შედარების თეორემის, ზედა და ქვედა ამონახსნების მეთოდისა და უძრავი წერტილის თეორემების საშუალებით დადგენილია რამდენიმე შედეგი ფრაქციული დიფერენციალური განტოლებისთვის სასაზღვრო ამოცანების ამონახსნის არსებობის შესახებ, როდესაც არაწრფივობა აკმაყოფილებს ზრდის პირობებს სასრულ ინტერვალზე. მიღებული შედეგების გამოყენების საილუსტრაციოდ მოყვანილია მაგალითი.

1 Introduction

Fractional calculus has played a significant role in engineering, science, economy, and other fields. Most of papers and books on fractional calculus are devoted to the solvability of linear initial fractional differential equations in terms of special functions. Recently, there appeared some papers dealing with the existence of solutions (or positive solutions) of nonlinear initial value problems of fractional differential equation using the techniques of nonlinear analysis (see [2,9] and the references therein).

In the literature, ${}^cD^{\alpha}_{0+}u(t) + f(t, u(t)) = 0$ is known as a single-term equation. This kind of fractional differential equation has many applications and has been studied widely. Equations containing more than one fractional differential terms are called multi-term fractional differential equations; they have some concrete applications in many fields. Due to the complexity of such a kind of equations, it seems that there has been no result for a general multi-term fractional differential equation. Only some special cases have been investigated. A classical example is the so-called Bagley–Torvik equation (B-T equation for short) [12],

$$Au''(t) + B^cD^{\frac{3}{2}}_{0+}u(t) + Cu(t) = f(t),$$

where A, B and C are certain constants, ${}^cD^{\alpha}_{0+}$ is the Caputo fractional derivative and f is a given function. This equation arises from the mathematical model of the motion of a thin plate in a Newtonian fluid. The B-T equation, as well as various generalizations, have wide applications in fluid dynamics and hence attracted much attention. The analytic solution and the numerical solution for the B-T equation were studied in [4] and [5], respectively.

J. Cermak et al. [3] investigated the two-term fractional differential equation

$$u''(t) + B^cD^{\beta}_{0+}u(t) + bu(t) = 0$$

with coefficients $a, b \in R$ and positive real orders $0 < \beta < 2$. It contains the important case such as the B-T equation for $\beta = \frac{3}{2}$. Qualitative properties of the true and numerical solutions were described and numerical stability regions for the classical and fractional models were compared.

In [14], S. Zhang discussed the following boundary-value problems for two-point nonlinear fractional differential equation:

$$\begin{cases} D^{\alpha}_{0+}u(t) + q(t)f\big(u(t), u'(t), u''(t), \dots, u^{(n-2)}(t)\big) = 0, \;\; t \in (0,1), \\ u(0) = u'(1) = u''(0) = \cdots = u^{(n-2)}(0) = u^{(n-2)}(1) = 0, \end{cases}$$

where α is a positive number, D^{α}_{0+} is the Riemann–Liouville's fractional derivative, q may be singular at $t = 0$ and $f(x_0, x_1, \dots, x_{n-2})$ may be singular at $x_0 = 0, x_1 = 0, x_2 = 0, \dots, x_{(n-2)} = 0$. The existence of positive solutions to the problem is obtained by the fixed point theorem for the mixed monotone operator.

In [7], the authors have investigated the existence of solutions for two-point boundary value problems

$$\begin{cases} D^{\alpha}_{0+}u(t) + f\big(t, u(t), D^{\alpha-2}_{0+}u(t)\big) = 0, \;\; t \in (0,1), \\ u^{(k)}(0) = 0, \;\; k = 0, 1, \dots, n-3, \;\; n = [\alpha] + 1, \\ D^{\alpha-2}_{0+}u(1) = D^{\alpha-1}_{0+}u(0) = 0, \end{cases}$$

for fractional differential equations of arbitrary order $\alpha > 2$, by applying upper and lower solutions method together with Schauder's fixed point theorem. First, they transformed the posed problem to an ordinary first order initial value problem that they modified to prove the existence of solutions for the problem. Moreover, they gave the explicit expression of the upper and lower solutions of the problem.

Recently, in [13], the authors considered the existence of solutions of the boundary-value problem for two-term three-point nonlinear fractional differential equation:

$$\begin{cases} \lambda D^{\alpha}_{0+}u(t) + D^{\beta}_{0+}u(t) = f(t, u(t)), \;\; t \in [0,T], \\ u(0) = 0, \;\; \mu D^{\gamma_1}_{0+}u(T) + D^{\gamma_2}_{0+}u(\eta) = \gamma_3, \end{cases}$$

where $1 < \alpha \le 2$, $1 \le \beta < \alpha$, $0 < \lambda \le 1$, $0 \le \mu \le 1$, $0 \le \gamma_1 \le \alpha - \beta$, $\gamma_2 \ge 0$, $0 < \eta < T$ are the constants, D^α_{0+}, D^β_{0+} are the Riemann–Liouville fractional derivative, and $f : [0,T] \times R \to R$ is continuous. By means of the fixed point theorems and Gronwall type inequality, some results on the existence of solutions and the Hyers–Ulam stability are obtained. (For more results see [1, 6, 10, 11] and the references therein.)

Motivated by the above results, in this paper we deal with the boundary value problem of the two-term fractional differential equation:

$$\begin{cases} D^{2+\alpha}_{0+} u(t) + f\big(t, u(t), D^\alpha_{0+} u(t)\big) = 0, \quad t \in (0,1), \\ u(0) = 0, \quad D^\alpha_{0+} u(t)\big|_{t=0} = D^\alpha_{0+} u(t)\big|_{t=1} = 0, \end{cases} \tag{1.1}$$

where $0 < \alpha \le 1$ is a real number and D^α_{0+} is the standard Riemann–Liouville fractional derivative, $f : [0,1] \times R^2 \to R$ is continuous. We prove a new comparison theorem, and then establish the existence of solutions for the above-given problem using the comparison theorem, fixed point theory and the method of upper and lower solutions. By these methods, we can obtain the iterative scheme for this problem, which implies that the solutions are computable.

The paper is organized as follows. In Section 2, a new comparison theorem is proved. The existence results for problem (1.1) are established in Section 3. In the same section, we give the proof of the main result. An example is presented in the last section to illustrate the application of our results.

2 Preliminaries and comparison theorem

In this section, we first recall some standard definitions and notation.

Let $\alpha > 0$ be a constant.

Definition 2.1 ([8])**.** The Riemann–Liouville fractional integral $I^\alpha_{a+} f$ of order α is defined by

$$I^\alpha_{a+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(x)}{(t-x)^{1-\alpha}}\, dx, \quad t > a,$$

provided that the right-hand side is defined point-wisely, where Γ is the Gamma function.

Definition 2.2 ([8])**.** The Riemann–Liouville fractional derivatives $D^\alpha_{a+} f$ of order α are defined by

$$D^\alpha_{a+} f(t) = \Big(\frac{d}{dx}\Big)^n (I^{n-\alpha}_{a+} f)(t) = \Big(\frac{d}{dx}\Big)^n \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}}\, dx, \quad n = [\alpha] + 1, \quad t > a,$$

provided that the right-hand side is defined point-wisely, where $[\alpha]$ denotes the integer part of α.

Lemma 2.3 ([8])**.** *Let $m \in N_+$ and $D = d/dt$. If the fractional derivatives $(D^\alpha_{a+} f)(t)$ and $(D^{\alpha+m}_{a+} f)(t)$ exist, then*

$$(D^m D^\alpha_{a+} f)(t) = (D^{\alpha+m}_{a+} f)(t).$$

Remark 2.4.

(1) The Riemann–Liouville fractional integral satisfies the equality

$$I^\alpha_{0+} t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}\, t^{\beta+\alpha}, \quad \alpha > 0, \quad \beta > -1, \quad t > 0.$$

(2) The equality $D^\alpha_{0+} I^\alpha_{0+} u(t) = u(t)$ holds for $u \in L(0,1)$.

(3) If $\alpha \in (0,1]$, then for $u \in L(0,1)$, $D^\alpha_{a+} u \in L(0,1)$ and arbitrary $c \in R$, the equality

$$I^\alpha_{0+} D^\alpha_{0+} u(t) = u(t) + ct^{\alpha-1}$$

holds.

The following comparison theorem is crucial in this paper.

Lemma 2.5. *Let λ_1, λ_2 be two nonnegative numbers, $r>0$ be a constant. If $m(t)\in C^2[0,1]$ satisfies*

$$m''(t) \geq \frac{\lambda_1}{\Gamma(r)} \int_0^t (t-s)^{r-1} m(s)\,ds + \lambda_2 m(t), \quad 0<t<1, \quad m(0)\leq 0, \quad m(1)\leq 0,$$

then $m(t)\leq 0$, $\forall\, t\in[0,1]$, provided that $0\leq \lambda_1+\lambda_2\Gamma(r+1)\leq 2\Gamma(r+1)$.

Proof. We will verify the assertion in the following cases.

Case 1. If $\lambda_1=\lambda_2=0$, then we have $m''(t)\geq 0$, which implies that $m(t)$ is a convex function on $[0,1]$. Hence, we have $m(t)\leq \min\{m(0),m(1)\}\leq 0$, $t\in[0,1]$.

Case 2. Let $\lambda_1=0, 0<\lambda_2<2$.

Conversely, suppose there exists $t_0\in(0,1)$ such that $m_0=m(t_0)=\max m(t)>0$, then $m'(t_0)=0$, $m''(t_0)\leq 0$. But $m''(t_0)\geq \lambda_2 m(t_0)$ implies $m''(t_0)>0$, which is a contradiction.

Case 3. Let $\lambda_1>0$, $\lambda_2\geq 0$ and $0<\lambda_1+\lambda_2\Gamma(r+1)\leq 2\Gamma(r+1)$.

Assume that there exists $t_0\in(0,1)$ such that $m_0=m(t_0)=\max\limits_{0\leq t\leq 1} m(t)>0$, then $m'(t_0)=0$, $m''(t_0)\leq 0$. Hence, by

$$0\geq m''(t_0)\geq \frac{\lambda_1}{\Gamma(r)} \int_0^{t_0} (t_0-s)^{r-1} m(s)\,ds + \lambda_2 m(t_0),$$

we have $\int\limits_0^{t_0} (t_0-s)^{r-1} m(s)\,ds<0$.

This implies that there is $t_1\in[0,t_0)$ such that $m_1=m(t_1)=\min\limits_{t\in[0,t_0]} m(t)<0$. According to Taylor's formula, there is $\lambda\in(t_1,t_0)$ such that

$$m_1=m(t_1)=m(t_0)+m'(t_0)(t_1-t_0)+\frac{m''(\lambda)}{2}(t_1-t_0)^2.$$

Since $m_1<0$, we have

$$m''(\lambda)=\frac{2(m_1-m_0)}{(t_1-t_0)^2}<\frac{2m_1}{(t_1-t_0)^2}\,.$$

Hence

$$\begin{aligned}2m_1>m''(\lambda)&\geq \frac{\lambda_1}{\Gamma(r)}\int_0^{\lambda}(\lambda-s)^{r-1}m(s)\,ds+\lambda_2 m(\lambda)\geq \frac{\lambda_1}{\Gamma(r)}\int_0^{\lambda}(\lambda-s)^{r-1}m_1\,ds+\lambda_2 m_1\\ &=\frac{\lambda_1}{\Gamma(r+1)}\,\lambda^r m_1+\lambda_2 m_1>\frac{\lambda_1}{\Gamma(r+1)}\,m_1+\lambda_2 m_1.\end{aligned}$$

This implies that $\lambda_1+\lambda_2\Gamma(r+1)>2\Gamma(r+1)$, which contradicts the assumption that $0\leq\lambda_1+\lambda_2\Gamma(r+1)\leq 2\Gamma(r+1)$.

This ends the proof. □

Corollary 2.6. *Let λ_1, λ_2 be two nonnegative numbers, $0<\alpha\leq 1$ be a constant. If $h(t)\in C^3[0,1]$ satisfies*

$$\begin{cases} D_{0+}^{2+\alpha}h(t)\geq \lambda_1 h(t)+\lambda_2 D_{0+}^{\alpha}h(t), & 0<t<1,\\ h(0)=0, \quad D_{0+}^{\alpha}h(t)\big|_{t=0}\leq 0, \quad D_{0+}^{\alpha}h(t)\big|_{t=1}\leq 0, \end{cases}$$

then $h(t)\leq 0$, $\forall\, t\in[0,1]$ provided that $0\leq\lambda_1+\lambda_2\Gamma(\alpha+1)\leq 2\Gamma(\alpha+1)$.

Proof. Let $m(t) = D_{0+}^{\alpha} h(t)$. Since $h(0) = 0$, we have

$$h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} m(s)\, ds, \quad m''(t) = D_{0+}^{2+\alpha} h(t)$$

and

$$m''(t) \geq \frac{\lambda_1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} m(s)\, ds + \lambda_2 m(t), \quad 0 < t < 1, \quad m(0) \leq 0, \quad m(1) \leq 0.$$

Due to Lemma 2.5, we have $m(t) \leq 0$, $\forall\, t \in [0,1]$. Hence

$$h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} m(s)\, ds \leq 0, \quad \forall\, t \in [0,1].$$

This ends the proof. □

3 The existence criteria

Throughout this section, we assume that $f : [0,1] \times R^2 \to R$ is continuous and there exist non-negative numbers λ_1, λ_2 such that

(H_1) for $t \in [0,1]$, $z \in R$, $x_1 \geq x_2$, $y_1 \geq y_2$

$$f(t, x_1, y_1) - f(t, x_2, y_2) \geq -\lambda_1(x_1 - x_2) - \lambda_2(y_1 - y_2).$$

(H_2) $0 \leq \lambda_1 + \lambda_2 \Gamma(\alpha+1) \leq 2\Gamma(\alpha+1)$.

Definition 3.1. A function $u \in C[0,1]$ is called a solution of problem (1.1) if $D_{0+}^{\alpha} u \in C[0,1]$, and u satisfies the equation in (1.1) for $t \in [0,1]$ and the boundary condition in (1.1).

Lemma 3.2. *If $u \in C[0,1]$ is a solution of the following boundary value problem*

$$\begin{cases} (D_{0+}^{\alpha} u(t))'' + f\big(t, u(t), D_{0+}^{\alpha} u(t)\big) = 0, \quad t \in (0,1), \\ u(0) = 0, \quad D_{0+}^{\alpha} u(t)\big|_{t=0} = D_{0+}^{\alpha} u(t)\big|_{t=1} = 0, \end{cases} \tag{3.1}$$

then u is a solution of (1.1).

Proof. According to Lemma 2.3, we have

$$(D^2 D_{a+}^{\alpha} u)(t) = (D_{a+}^{\alpha+2} u)(t),$$

i.e.,

$$(D_{0+}^{\alpha} u)''(t) = (D_{0+}^{\alpha+2} u)(t).$$

So, if $u \in C[0,1]$ is a solution of (3.1), it is a solution of (1.1). □

The main result reads as follows.

Theorem 3.3. *If $\min\limits_{0 \leq t \leq 1} f(t,0,0) \geq 0$ and there exists $c > 0$ such that*

$$\max\left\{ f(t,x,y) \;\middle|\; (t,x,y) \in [0,1] \times \left[0, \frac{c}{\Gamma(3+\alpha)} \Big(\frac{1+\alpha}{2}\Big)^{1+\alpha}\right] \times \left[0, \frac{c}{4}\right] \right\} \leq 2c,$$

then (1.1) *has a solution u^* satisfying*

$$0 \leq u^*(t) \leq c\Big(\frac{t^{\alpha}}{\Gamma(1+\alpha)} - \frac{2t^{2+\alpha}}{\Gamma(3+\alpha)}\Big).$$

Proof. Let $X = C[0,1]$, the norm on X be $\|\cdot\| : \|x\| = \max\limits_{0\le t\le 1} |x(t)|$ for $x \in X$. Let $K = \{x \in X \mid \ x(t) \ge 0,\ 0 \le t \le 1\}$ and the partial order "$\le$" on X be induced by K: for $x, y \in X$, $y \le x \Longleftrightarrow x - y \in K$, then (X, K) is an ordered Banach space.

Having in mind (3.1) (with $D^{\alpha}_{0+}u$ replaced by h), we discuss the problem

$$\begin{cases} -h''(t) = f\big(t, I^{\alpha}_{0+}h(t), h(t)\big), \\ h(0) = h(1) = 0, \end{cases} \tag{3.2}$$

Let $D = \{h \in X \mid \ h'' \in X,\ h(0) = h(1) = 0\}$. Define $L : D \subset X \to X$ and $N : X \to X$ as follows:

$$Lh = -h''(t) + \lambda_1 I^{\alpha}_{0+}h(t) + \lambda_2 h(t),$$
$$Nh = f\big(t, I^{\alpha}_{0+}h(t), h(t)\big) + \lambda_1 I^{\alpha}_{0+}h(t) + \lambda_2 h(t).$$

By the definition of L and N, (3.2) can be rewritten as

$$Lh = Nh. \tag{3.3}$$

Step 1. *$L : D \subset X \to X$ is a reversible mapping.*

Given $\eta \in X$, we consider the following boundary value problem:

$$\begin{cases} -h''(t) + \lambda_1 I^{\alpha}_{0+}h(t) + \lambda_2 h(t) = \eta(t), \\ h(0) = h(1) = 0. \end{cases}$$

It is known that h is the solution of the above problem if and only if h is the fixed point of the operator $A_\eta : X \to X$, where

$$A_\eta h(t) = \int_0^1 G(t,s)\big[\eta(s) - \lambda_1 I^{\alpha}_{0+}h(s) - \lambda_2 h(s)\big]\, ds$$

and

$$G(t,s) = \begin{cases} s(1-t), & 0 \le s \le t \le 1, \\ t(1-s), & 0 \le t \le s \le 1. \end{cases}$$

Since $\max\limits_{t\in[0,1]} \int_0^1 G(t,s)\, ds = \frac{1}{8}$, we have

$$|A_\eta x(t) - A_\eta y(t)| = \int_0^1 G(t,s)\Big[\lambda_1 I^{\alpha}_{0+}(y(s) - x(s)) + \lambda_2 (y(s) - x(s))\Big]\, ds$$
$$\le \int_0^1 G(t,s)\Big[\lambda_1 I^{\alpha}_{0+}\|x-y\| + \lambda_2 \|x-y\|\Big]\, ds \le \frac{1}{8}\Big[\frac{\lambda_1}{\Gamma(\alpha+1)} + \lambda_2\Big]\|x-y\| \le \frac{1}{4}\,\|x-y\|$$

for all $t \in [0,1]$, $x, y \in X$, which implies that $A_\eta : X \to X$ is contractive.

By the completeness of X and an application of the Banach contraction principle, there exists a unique $h \in X$ such that $A_\eta h = h$, i.e., $Lh = \eta$. In fact, $h \in D$. Hence $L : D \subset X \to X$ is reversible.

Step 2. *$L^{-1} : X \to D$ is continuous.*

Let $\eta \in X$, $\{\eta_n\} \subset X$, $\eta_n \to \eta$, $L^{-1}\eta = x$, $L^{-1}\eta_n = x_n$, then

$$x_n(t) = \int_0^1 G(t,s)\big[\eta_n(s) - \lambda_1 I^{\alpha}_{0+}x_n(s) - \lambda_2 x_n(s)\big]\, ds,$$

$$x(t) = \int_0^1 G(t,s)\big[\eta(s) - \lambda_1 I^{\alpha}_{0+}x(s) - \lambda_2 x(s)\big]\, ds.$$

As a result,

$$\begin{aligned}
|x_n(t)-x(t)| &= \Big|\int_0^1 G(t,s)\Big[\eta_n(s)-\eta(s)+\lambda_1 I_{0+}^{\alpha}(x-x_n)(s)+\lambda_2(x(s)-x_n(s))\Big]\,ds\Big| \\
&\le \int_0^1 G(t,s)\Big[|\eta_n(s)-\eta(s)|+\lambda_1 I_{0+}^{\alpha}|x-x_n|(s)+\lambda_2|x(s)-x_n(s)|\Big]\,ds \\
&\le \frac{1}{8}\Big[\|\eta_n-\eta\|+\Big(\frac{\lambda_1}{\Gamma(\alpha+1)}+\lambda_2\Big)\|x-x_n\|\Big] \\
&\le \frac{1}{8}\,\|\eta_n-\eta\|+\frac{1}{4}\,\|x-x_n\|.
\end{aligned}$$

We have

$$\|x_n-x\|\le \frac{1}{6}\,\|\eta_n-\eta\|.$$

Consequently, $x_n\to x$, when $\eta_n\to\eta$. Therefore, $L^{-1}:X\to D$ is continuous.

Step 3. *$L^{-1}:X\to D$ is compact.*

Let $S\subset X$ be a bounded subset, i.e., there exists a constant $M>0$ such that $\|\eta\|\le M$ for any $\eta\in S$.

Let $\eta\in S, L^{-1}\eta=x$, then

$$x(t)=\int_0^1 G(t,s)\big[\eta(s)-\lambda_1 I_{0+}^{\alpha}x(s)-\lambda_2 x(s)\big]\,ds.$$

As a result,

$$\|x\|\le \frac{1}{8}\,\|\eta\|+\frac{1}{8}\Big(\frac{\lambda_1}{\Gamma(\alpha+1)}+\lambda_2\Big)\|x\|\le \frac{1}{8}\,\|\eta\|+\frac{1}{4}\,\|x\|,$$

hence

$$\|x\|\le\frac{1}{6}\,\|\eta\|\le\frac{1}{6}\,M,$$

which implies that $L^{-1}(S)$ is bounded.

Furthermore, let $t_1,t_2\in[0,1]$, $t_1<t_2$, then for any $x\in L^{-1}(S)$, there exists $\eta\in D$ such that $L^{-1}\eta=x$ and

$$\begin{aligned}
|x(t_1)-x(t_2)| &= |A_\eta x(t_1)-A_\eta x(t_2)| \\
&= \Big|\int_0^1 (G(t_1,s)-G(t_2,s))\big[\eta(s)-\lambda_1 I_{0+}^{\alpha}x(s)-\lambda_2 x(s)\big]\,ds\Big| \\
&\le \int_0^1 |G(t_1,s)-G(t_2,s)|\,|\eta(s)-\lambda_1 I_{0+}^{\alpha}x(s)-\lambda_2 x(s)|\,ds \\
&\le \int_0^1 |G(t_1,s)-G(t_2,s)|\,ds\,\Big[\|\eta\|+\Big(\frac{\lambda_1}{\Gamma(\alpha+1)}+\lambda_2\Big)\|x\|\Big] \\
&\le \frac{4M}{3}\int_0^1 |G(t_1,s)-G(t_2,s)|\,ds.
\end{aligned}$$

Due to the uniform continuity of $G(t,s)$ on $[0,1]\times[0,1]$, for $\forall\,\varepsilon>0$, there exists $\sigma>0$ such that $|t_2-t_1|<\sigma$ implies

$$|G(t_1,s)-G(t_2,s)|<\frac{3}{4M}\,\varepsilon.$$

At the same time, we have

$$|x(t_1) - x(t_2)| \le \frac{4M}{3} \int_0^1 |G(t_1,s) - G(t_2,s)|\, ds < \frac{4M}{3}\,\frac{3}{4M}\,\varepsilon = \varepsilon.$$

Hence $L^{-1}(S)$ is equi-continuous.

Since $L^{-1}(S)$ is bounded and equi-continuous, $L^{-1} : X \to D$ is compact.

Step 4. *$L^{-1}N : X \to D$ is continuous and increasing.*

Since f is continuous, by the definition of N and Step 3, $N : X \to X$ and $L^{-1}N : X \to D$ are continuous.

Moreover, for arbitrary $\eta_1, \eta_2 \in X$, $\eta_1 \le \eta_2$, (H_1) implies $N\eta_1 \le N\eta_2$. Let $v_1 = L^{-1}N\eta_1$, $v_2 = L^{-1}N\eta_2$, then $Lv_1 = N\eta_1 \le N\eta_2 = Lv_2$. Hence we have $L(v_1 - v_2) \le 0$, i.e.,

$$-(v_1 - v_2)''(t) + \frac{\lambda_1}{\Gamma(r)} \int_0^t (t-s)^{r-1}(v_1(s) - v_2(s))\, ds + \lambda_2(v_1(t) - v_2(t)), \quad 0 < t < 1,$$

$$(v_1 - v_2)(0) = (v_1 - v_2)(1) = 0.$$

By Lemma 2.5, we obtain $(v_1 - v_2)(t) \le 0$ for $t \in [0,1]$, i.e., $v_1 \le v_2$. Hence $L^{-1}N : X \to D$ is increasing.

Step 5. *There exist $x, y \in D$, $x \le y$ such that $Lx \le Nx$ and $Ly \ge Ny$.*

Let $v(t) = 0$. Since

$$\min_{0 \le t \le 1} f(t,0,0) \ge 0,$$

we have

$$\begin{cases} D_{0+}^{2+\alpha} v(t) + f(t, v(t), D_{0+}^{\alpha} v(t)) \ge 0, \quad t \in (0,1) \\ v(0) = 0, \quad D_{0+}^{\alpha} v(t)\big|_{t=0} \le 0, \quad D_{0+}^{\alpha} v(t)\big|_{t=1} \le 0. \end{cases}$$

Let

$$w(t) = c\Big(\frac{t^{\alpha}}{\Gamma(1+\alpha)} - \frac{2t^{2+\alpha}}{\Gamma(3+\alpha)}\Big).$$

Noting that for $t \in [0,1]$,

$$D_{0+}^{2+\alpha} w(t) = 2c, \quad w(t) \in \Big[0, \frac{c}{\Gamma(3+\alpha)}\Big(\frac{1+\alpha}{2}\Big)^{1+\alpha}\Big], \quad D_{0+}^{\alpha} w(t) \in \Big[0, \frac{c}{4}\Big]$$

and

$$\max\Big\{ f(t,x,y) \mid \ (t,x,y) \in [0,1] \times \Big[0, \frac{c}{\Gamma(3+\alpha)}\Big(\frac{1+\alpha}{2}\Big)^{1+\alpha}\Big] \times \Big[0, \frac{c}{4}\Big] \Big\} \le 2c,$$

we get

$$\begin{cases} D_{0+}^{2+\alpha} w(t) + f\big(t, w(t), D_{0+}^{\alpha} w(t)\big) \le 0, \quad t \in (0,1), \\ w(0) = 0, \quad D_{0+}^{\alpha} w(t)\big|_{t=0} \ge 0, \quad D_{0+}^{\alpha} w(t)\big|_{t=1} \ge 0. \end{cases}$$

By Step 1, there exist $x, y \in D$ such that

$$Lx = N(D_{0+}^{\alpha} v(t)), \quad Ly = N(D_{0+}^{\alpha} w(t)).$$

Next, we assert that

(1) $x \le y$;

(2) $D_{0+}^{\alpha} v(t) \le x$ and $Lx \le Nx$;

(3) $y \le D_{0+}^{\alpha} w(t)$ and $Ly \ge Ny$.

Since N is nondecreasing, we have $N(D^\alpha_{0+}v(t)) \le N(D^\alpha_{0+}w(t))$, hence $Lx \le Ly$. Lemma 2.5 implies $x \le y$. Assertion (1) is verified.

Next, we verify assertion (2).

In fact, by the definition of x, we have

$$\begin{cases} -x''(t) + \lambda_1 I^\alpha_{0+}x(t) + \lambda_2 x(t) = f\big(t, v(t), D^\alpha_{0+}v(t)\big) + \lambda_1 v(t) + \lambda_2 D^\alpha_{0+}v(t), \\ x(0) = x(1) = 0. \end{cases} \tag{3.4}$$

Let $\phi(t) = D^\alpha_{0+}v(t)$. Then

$$\begin{cases} -\phi''(t) + \lambda_1 I^\alpha_{0+}\phi(t) + \lambda_2 \phi(t) \le f\big(t, v(t), D^\alpha_{0+}v(t)\big) + \lambda_1 v(t) + \lambda_2 D^\alpha_{0+}v(t), \\ \phi(0) \le 0, \quad \phi(1) \le 0. \end{cases} \tag{3.5}$$

(3.4), (3.5) together with the assumption (H_2) lead to

$$\begin{cases} -(x(t) - \phi(t))'' + \lambda_1 I^\alpha_{0+}(x-\phi)(t) + \lambda_2(x(t) - \phi(t)) \ge 0, \\ (x(0) - \phi(0)) \ge 0, \quad (x(1) - \phi(1)) \ge 0. \end{cases}$$

By virtue of Lemma 2.5, we have $x(t) - \phi(t) \ge 0$ i.e., $x(t) \ge \phi(t)$. The nondecreasing of N gives $Nx \ge N\phi$, hence $Lx = N\phi \le Nx$.

$y \le D^\alpha_{0+}w(t)$, $Ny \le Ly$ can be verified similarly.

Step 6. *Problem* (1.1) *has a solution* $u^*(t)$ *satisfying* $v(t) \le u^*(t) \le w(t)$.

Step 4 and Step 5 implies that the operator $L^{-1}N$ maps $[x, y] \cap D$ into $[x, y] \cap D$. Since $[x, y] \cap D$ is convex, closed and bounded and $L^{-1}N$ is completely continuous, an application of Schauder's fixed point theorem implies that $Lh = Nh$ has a solution h^* in $[x, y]$. Let

$$u^*(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h^*(s)\, ds,$$

then $u^*(t)$ is a solution of problem (1.1) satisfying $v(t) \le u^*(t) \le w(t)$. □

Theorem 3.4. *If* $\max\limits_{0\le t\le 1} f(t, 0, 0) \le 0$ *and there exists* $c > 0$ *such that*

$$\min\left\{ f(t,x,y) \mid\ (t,u,v) \in [0,1] \times \Big[-\frac{c}{\Gamma(3+\alpha)} \Big(\frac{1+\alpha}{2}\Big)^{1+\alpha}, 0\Big] \times \Big[-\frac{c}{4}, 0\Big] \right\} \ge -2c,$$

then (1.1) *has a solution* u^* *satisfying*

$$0 \ge u^*(t) \ge -c\Big(\frac{t^\alpha}{\Gamma(1+\alpha)} - \frac{2t^{2+\alpha}}{\Gamma(3+\alpha)}\Big).$$

Proof. In Step 5 of the proof of Theorem 3.3, let

$$v(t) = -c\Big(\frac{t^\alpha}{\Gamma(1+\alpha)} - \frac{2t^{2+\alpha}}{\Gamma(3+\alpha)}\Big), \quad w(t) \equiv 0.$$

Then the conclusion of Theorem 3.4 can be verified in a similar way. □

Theorem 3.5. *If there exists* $c > 0$ *such that*

$$\max\left\{ f(t,x,y) \mid\ (t,x,y) \in [0,1] \times \Big[0, \frac{c}{\Gamma(3+\alpha)} \Big(\frac{1+\alpha}{2}\Big)^{1+\alpha}\Big] \times \Big[0, \frac{c}{4}\Big] \right\} \le 2c,$$

$$\min\left\{ f(t,x,y) \mid\ (t,u,v) \in [0,1] \times \Big[-\frac{c}{\Gamma(3+\alpha)} \Big(\frac{1+\alpha}{2}\Big)^{1+\alpha}, 0\Big] \times \Big[-\frac{c}{4}, 0\Big] \right\} \ge -2c,$$

then (1.1) *has a solution* u^* *satisfying*

$$-c\Big(\frac{t^\alpha}{\Gamma(1+\alpha)} - \frac{2t^{2+\alpha}}{\Gamma(3+\alpha)}\Big) \le u^*(t) \le c\Big(\frac{t^\alpha}{\Gamma(1+\alpha)} - \frac{2t^{2+\alpha}}{\Gamma(3+\alpha)}\Big).$$

Proof. In Step 5 of the proof of Theorem 3.3, let

$$v(t) = -c\Big(\frac{t^{\alpha}}{\Gamma(1+\alpha)} - \frac{2t^{2+\alpha}}{\Gamma(3+\alpha)}\Big), \quad w(t) = c\Big(\frac{t^{\alpha}}{\Gamma(1+\alpha)} - \frac{2t^{2+\alpha}}{\Gamma(3+\alpha)}\Big).$$

Then the conclusion of Theorem 3.5 can be verified in a similar way. □

4 Example and remark

Example 4.1. Consider the following boundary value problem for the fractional differential equation:

$$\begin{cases} D_{0+}^{\frac{5}{2}}u(t) + \cos u(t) + \arctan(D_{0+}^{\frac{1}{2}}u(t)) = 0, \\ u(0) = 0, \ \ D_{0+}^{\frac{1}{2}}u(t)\big|_{t=0} = D_{0+}^{\frac{1}{2}}u(t)\big|_{t=1} = 0. \end{cases}$$

Let

$$f(t,x,y) = \cos x + \arctan y.$$

Then $f(t,0,0) > 0$ and f satisfies $(H_1 - H_2)$ with $\lambda_1 = 1$, $\lambda_2 = 0$, $\alpha = \frac{1}{2}$.

Furthermore, let $c = 4$, we have

$$\max\Big\{f(x,y) \mid \ (x,y) \in \Big[0, \frac{c}{\Gamma(3+\alpha)}\Big(\frac{1+\alpha}{2}\Big)^{1+\alpha}\Big] \times \Big[0, \frac{c}{4}\Big]\Big\} = 1 + \frac{\pi}{4} \leq 2c.$$

Then Theorem 3.3 assures the above problem has a solution between 0 and

$$\frac{8t^{\frac{1}{2}}}{\sqrt{\pi}}\Big(1 - \frac{8t^2}{15}\Big).$$

Remark 4.2. By the proof of Theorem 3.3, we know that the solution of problem (3.3) can be obtained by iterative sequence $\{x_n\}$ or $\{y_n\}$, where

$$\begin{aligned} Lx_{n+1} &= N(x_n), \ \ x_0 = x, \ \ n = 0,1,2,\dots; \\ Ly_{n+1} &= N(y_n), \ \ y_0 = y, \ \ n = 0,1,2,\dots. \end{aligned}$$

This implies that the solution of problem (1.1) is computable.

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References

[1] M. Al-Refai and K. Pal, A maximum principle for a fractional boundary value problem with convection term and applications. *Math. Model. Anal.* **24** (2019), no. 1, 62–71.

[2] A. Atangana, Non validity of index law in fractional calculus: a fractional differential operator with Markovian and non-Markovian properties. *Phys. A* **505** (2018), 688–706.

[3] J. Čermák and T. Kisela, Exact and discretized stability of the Bagley–Torvik equation. *J. Comput. Appl. Math.* **269** (2014), 53–67.

[4] S. Das, Analytical solution of a fractional diffusion equation by variational iteration method. *Comput. Math. Appl.* **57** (2009), no. 3, 483–487.

[5] M. El-Gamel and M. A. El-Hady, Numerical solution of the Bagley–Torvik equation by Legendre-collocation method. *SeMA J.* **74** (2017), no. 4, 371–383.

[6] H. Fazli and J. J. Nieto, An investigation of fractional Bagley–Torvik equation. *Open Math.* **17** (2019), no. 1, 499–512.

[7] N. Khaldi and B. Messirdi, Stability of essential spectra of closed operators under T-compact equivalence and applications. *Bull. Transilv. Univ. Braşov Ser. III* **12(61)** (2019), no. 1, 53–64.

[8] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*. North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[9] I. Podlubny, *Fractional Differential Equations. An Introduction to fractional Derivatives, Fractional Differential Equations, to Methods of their Solution and Some of their Applications*. Mathematics in Science and Engineering, 198. Academic Press, Inc., San Diego, CA, 1999.

[10] Q. Song and Zh. Bai, Positive solutions of fractional differential equations involving the Riemann-Stieltjes integral boundary condition. *Adv. Difference Equ.* **2018**, Paper No. 183, 7 pp.

[11] S. Staněk, Boundary value problems for Bagley–Torvik fractional differential equations at resonance. *Miskolc Math. Notes* **19** (2018), no. 1, 611–622.

[12] P. J. Torvik and R. L. Bagley, On the appearance of the fractional derivative in the behavior of real materials. *J. Appl. Mech.* **51** (1984), no. 2, 294–298.

[13] L. Xu, Q. Dong and G. Li, Existence and Hyers-Ulam stability for three-point boundary value problems with Riemann-Liouville fractional derivatives and integrals. *Adv. Difference Equ.* **2018**, Paper No. 458, 17 pp.

[14] S. Zhang, Positive solutions to singular boundary value problem for nonlinear fractional differential equation. *Comput. Math. Appl.* **59** (2010), no. 3, 1300–1309.

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