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ON THE EXISTENCE AND STABILITY OF SOLUTIONS OF STOCHASTIC DIFFERENTIAL SYSTEMS DRIVEN BY THE G-BROWNIAN MOTION

Abstract. In this paper, we study the Carathéodory approximate solution for a class of stochastic differential systems driven by G-Brownian motion. Based on the Carathéodory approximation scheme, we prove under some suitable conditions that our system has a unique solution and show that the Carathéodory approximate solutions converge to the solution of the system. Moreover, we prove a stability theorem for our system.

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Key words and phrases. G-expectation, G-brownian motion, G-stochastic differential equations, Carathéodory approximation scheme.

რეზიუმე. სტატიაში შესწავლილია კარათეოდორის მიახლოებითი ამონახსნი სტოქასტურ დიფერენციალურ სისტემათა კლასისთვის, რომელიც განპირობებულია G-ბრაუნის მოძრაობით. კარათეოდორის მიახლოების სქემაზე დაყრდნობით დამტკიცებულია, რომ შესაფერის პირობებში სისტემას აქვს ერთადერთი ამონახსნი და ნაჩვენებია, რომ კარათეოდორის მიახლოებითი ამონახსნები კრებადია სისტემის ამ ამონახსნისკენ. მოცემული სისტემისთვის აგრეთვე დამტკიცებულია მდგრადობის თეორემა.

1 Introduction

This paper is intended to study stochastic differential equations (SDE, for short) which have been the object of sustained attention in recent years because of their interesting structure and usefulness in various applied fields. The motivation for studying SDEs comes originally from the stochastic optimal control theory, that is, the adjoint equation in the Pontryagin type maximum principle. After this, extensive study of SDEs was initiated, and potential for its application was found in applied and theoretical areas such as stochastic control, mathematical finance, differential geometry, et al. It is worth pointing out that the SDEs have also been successfully applied to model and to resolve some interesting problems in mathematical finance, such as problems involving term structure of interest rates and hedging contingent claims for large investors, etc. See, e.g., [1, 3, 11, 13, 15, 16, 18, 20, 21] and [24–27, 29].

Recently, the theory of G-Brownian motion was introduced by S. Peng. The existence and uniqueness of solutions for some stochastic differential equations under G-Brownian motion (G-SDEs) with Lipschitz continuous coefficients were developed by Peng and Gao. In 2006, Peng in [24] (for more details see [10] and [19, 24–29]) introduced the theory of nonlinear expectation, the G-Brownian motion and defined the related stochastic calculus, especially, stochastic integrals of Itô's type with respect to the G-Brownian motion, and derived the related Itô's formula. In addition, the notion of G-normal distribution plays the same important role in the theory of nonlinear expectation as that of the normal distribution with the classical probability. In 2009, Gao in [10] studied pathwise properties and homeomorphic property with respect to the initial values for stochastic differential equations driven by the G-Brownian motion. Later, Faizullah et *al.* extended this theory (see, e.g., [4–9]).

In general, one cannot obtain the explicit solutions of SDEs. The fact that these systems model phenomena of the real world, the important mathematical questions that concern them are: the existence and uniqueness of a solution, stability, asymptotic behavior of a solution, etc.

There are many theoretical, analytical and numerical methods and techniques for processing and studying SDEs. We find this in the references mentioned and others. In this work, we will focus on the Carathéodory approximation scheme that has been used by many mathematicians to prove the existence theorem of solutions of ordinary differential equations under weak regularity conditions (see, e.g., [2, 5, 14, 18, 22, 23]).

Furthermore, in [5], Faizullah introduced the Carathéodory approximation scheme for vector-valued stochastic differential equations under the G-Brownian motion. It is shown that the Carathéodory approximate solutions converge to the unique solution of the equation. The existence and uniqueness theorem for G-SDEs is established by using the stated Lipschitz method and the linear growth conditions

$$X(t) = X(0) + \int_0^t f(s, X(s))\, ds + \int_0^t g(s, X(s))\, d\langle B\rangle(s) + \int_0^t h(s, X(s))\, dB(s), \quad t \in [0, T]. \tag{1.1}$$

The existence and the uniqueness of the solution $X(t)$ for G-SDEs (1.1) under different conditions were proved in [1, 4–10, 15, 17] and [19, 24–29].

In this paper, we study the existence, uniqueness and stability of the solution for the following stochastic differential system driven by the G-Brownian motion (SG-DEs):

$$
\begin{cases}
X_1(t) = X_1(0) + \int\limits_0^t f_{1,1}(s, X_1(s), \dots, X_n(s))\, ds \\
\qquad\qquad + \int\limits_0^t f_{2,1}(s, X_1(s), \dots, X_n(s))\, d\langle B\rangle(s) + \int\limits_0^t f_{3,1}(s, X_1(s), \dots, X_n(s))\, dB(s), \\
\quad \vdots \\
X_n(t) = X_n(0) + \int\limits_0^t f_{1,n}(s, X_1(s), \dots, X_n(s))\, ds \\
\qquad\qquad + \int\limits_0^t f_{2,n}(s, X_1(s), \dots, X_n(s))\, d\langle B\rangle(s) + \int\limits_0^t f_{3,n}(s, X_1(s), \dots, X_n(s))\, dB(s),
\end{cases}
\tag{1.2}
$$

where $(X_1(0), \dots, X_n(0))$ is the given initial condition, $(\langle B(t)\rangle)_{t\geq 0}$ is the quadratic variation process of the G-Brownian motion $(B(t))_{t\geq 0}$, and all the coefficients $f_{i,j}(t, x_1, \dots, x_n)$ for $1 \leq i \leq 3$ and $1 \leq j \leq n$ satisfy the Lipschitz and the linear growth conditions with respect to $(x_1, \dots, x_n)$. These results are obtained by using the technics adopted by F. Faizullah [5] in the case where the Lipschitz and the linear growth constants are time dependant.

The article is organized as follows. In Section 2, we provide some results and definitions necessary to understand the content of this work. Section 3 is devoted to the existence and uniqueness of the solution of system (1.2) using the Carathéodory approximation scheme. In the last Section 4 we give a result of the stability.

2 Preliminaries

In this section, we recall some basic notions, definitions and theorems necessary to understand the content of this work. For more details concerning this section see, e.g., [5, 10–12, 15, 26–28] and [24].

Let Ω be a given non-empty set and let $\mathcal{H}$ be a linear space of real valued functions defined on Ω such that any arbitrary constant $c \in \mathcal{H}$ and if $X \in \mathcal{H}$, then $|X| \in \mathcal{H}$. We consider that $\mathcal{H}$ is the space of random variables.

Definition 2.1. A functional $\mathbb{E} : \mathcal{H} \to \mathbb{R}$ is called sublinear expectation, if for all X, Y in $\mathcal{H}$, c in $\mathbb{R}$ and $\lambda \geq 0$, the following properties are satisfied:

- (i) (Monotonicity): if $X \geq Y$, then $\mathbb{E}[X] \geq \mathbb{E}[Y]$;
- (ii) (Constant preserving): $\mathbb{E}[c] = c$;
- (iii) (Sub-additivity): $\mathbb{E}[X + Y] \leq \mathbb{E}[X] + \mathbb{E}[Y]$;
- (iv) (Positive homogeneity): $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$.

The triple $(\Omega, \mathcal{H}, \mathbb{E})$ is called a sublinear expectation space.

We assume that if $X_1, X_2, \dots, X_n \in \mathcal{H}$, then $\varphi(X_1, X_2, \dots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{\ell,\mathrm{Lip}}(\mathbb{R}^n)$, the set of functions $\varphi : \mathbb{R}^n \to \mathbb{R}$ satisfying the condition:

$$
|\varphi(x) - \varphi(y)| \leq C(1 + |x|^m + |y|^m)|x - y| \quad \text{for all} \quad x, y \in \mathbb{R}^n,
$$

where C is a positive constant and $m \in \mathbb{N}^*$ depending only on φ.

Definition 2.2. Let X, Y be two n-dimensional random vectors defined on nonlinear expectation spaces $(\Omega_1, \mathcal{H}_1, \mathbb{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \mathbb{E}_2)$, respectively. They are called identically distributed, denoted by $X \overset{d}{=} Y$, if

$$
\mathbb{E}_2[\varphi(Y)] = \mathbb{E}_1[\varphi(X)] \quad \text{for each} \quad \varphi \in C_{\ell,\mathrm{Lip}}(\mathbb{R}^n).
$$

Definition 2.3. In a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$, a random vector $Y \in \mathcal{H}^n$ is said to be independent of another random vector $X \in \mathcal{H}^m$ if

$$\mathbb{E}[\varphi(X,Y)] = \mathbb{E}\big[\mathbb{E}[\varphi(x,Y)]_{x=X}\big] \ \ \forall\, \varphi \in C_{\ell,\mathrm{Lip}}(\mathbb{R}^m \times \mathbb{R}^n).$$

$\widetilde{X}$ is called an independent copy of X if $\widetilde{X} \overset{d}{=} X$ and $\widetilde{X}$ is independent of X.

Let Γ be a closed bounded and convex subset of $\mathbb{S}_+(d)$, the set of positive and symmetric d-dimensional matrices. Let

$$\Sigma = \big\{\gamma\gamma^{\mathrm{Tr}} : \ \gamma \in \Gamma\big\}$$

and let $G : \mathbb{S}_+(d) \to \mathbb{R}$ is defined by

$$G(A) = \frac{1}{2} \sup_{\gamma \in \Gamma} \mathrm{Tr}(\gamma\gamma^{\mathrm{Tr}} A)$$

Definition 2.4. In a sublinear expectation space $(\Omega, \mathcal{H}, E)$, a d-dimensional vector of random variables $X \in \mathcal{H}^d$ is G-normal distributed if for each $\varphi \in C_{\ell,\mathrm{Lip}}(\mathbb{R}^d)$, the function $u(t,x) = \mathbb{E}(\varphi(x+\sqrt{t}X))$ is the unique viscosity solution of the following parabolic equation called the G-heat equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} = G(D^2u), & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d, \\ u(0,x) = \varphi(x), & \end{cases}$$

where $D^2u = (\partial^2_{x_ix_j} u)^d_{i,j}$ is the Hessian matrix of u.

Remark 2.5. In fact, if $d = 1$, we have $G(\alpha) = \frac{1}{2}(\overline{\sigma}^2\alpha^+ - \underline{\sigma}^2\alpha^-)$, where $\overline{\sigma}^2 = \mathbb{E}[X^2]$, $\underline{\sigma}^2 = -\mathbb{E}[-X^2]$, $\alpha^+ = \max(\alpha, 0)$ and $\alpha^- = \max\{-\alpha, 0\}$ (for more details see [24]). We write $X \sim \mathcal{N}(0; [\underline{\sigma}^2, \overline{\sigma}^2])$.

Definition 2.6. A process $(B(t))_{t\geq 0}$ in a sublinear expectation space (Ω, H, E) is called a G-Brownian motion if the following properties are satisfied:

(i) $B(0) = 0$;

(ii) for each t, $s \geq 0$, the increment $B(t+s) - B(t)$ is $N(0; [\underline{\sigma}^2 s, \overline{\sigma}^2 s]$-distributed and is independent of $(B(t_1), \dots, B(t_n))$ for each $n \in \mathbb{N}$ and $0 \leq t_1 \leq \dots \leq t_n \leq t$.

We denote by $\Omega = C_0(\mathbb{R})$ the space of all $\mathbb{R}$-valued continuous functions ω defined on $\mathbb{R}_+$ such that $\omega(0) = 0$, equipped with the distance

$$\rho(\omega_1, \omega_2) = \sum_{i=1}^{\infty} 2^{-i} \max_{t \in [0,i]} \big[\big|(\omega_1(t) - \omega_2(t)) \wedge 1\big|\big].$$

For each fixed $T > 0$, let

$$\Omega_T = \big\{\omega(\,._{\wedge T}), \ \ \omega \in \Omega\big\},$$

$$\mathrm{Lip}(\Omega_T) = \Big\{\varphi(B(t_1), \dots, B(t_m)), \ \ m \geq 1, \ \ t_1, \dots, t_m \in [0,T], \ \ \varphi \in C_{\ell,\mathrm{Lip}}(\mathbb{R}^m)\Big\},$$

where

$$\mathrm{Lip}(\Omega) = \bigcup_{n=1}^{\infty} \mathrm{Lip}(\Omega_n).$$

In [24], Peng constructs a sublinear expectation $\mathbb{E}$ on $(\Omega, \mathrm{Lip}(\Omega))$ under which the canonical process $(B(t))_{t\geq 0}$ (i.e., $B(t,\omega) = \omega(t)$) is a G-Brownian motion. In what follows, we consider this G-Brownian motion.

We denote by $L^p_G(\Omega_T)$, $p \geq 1$, the completion of $\mathrm{Lip}(\Omega_T)$ under the norm $||X||_p = (\mathbb{E}[|X|^p])^{\frac{1}{p}}$. Similarly, we denote by $L^p_G(\Omega)$ the completion space of $\mathrm{Lip}(\Omega)$. It was shown in [28] and [24] that there exists a family of probability measures $\mathcal{P}$ on Ω such that

$$\mathbb{E}[X] = \sup_{P \in \mathcal{P}} E^P[X] \ \text{ for } \ X \in L^1_G(\Omega),$$

where E^P stands for the linear expectation under the probability P. We say that a property holds quasi surely (q.s.) if it holds for each $P \in \mathcal{P}$.

For a finite partition of $[0,T]$, $\pi_T = \{t_0, t_1, \dots, t_N\}$, we set

$$\mu(\pi_T) = \max\left\{|t_{i+1} - t_i|, \ \ 0 \le i \le N-1\right\}.$$

Consider the collection $M_G^{p,0}(0,T)$ of simple processes defined by

$$\eta(t,\omega) = \sum_{i=0}^{N-1} \xi_i(\omega) I_{[t_i;t_{i+1}[}(t),$$

where

$$\xi_i \in L_G^p(\Omega_{t_i}), \ \ 0 \le i \le N-1 \ \text{ and } \ p \ge 1.$$

The completion of $M_G^{p,0}(0,T)$ under the norm

$$\|\eta\| = \left\{ \frac{1}{T} \int_0^T \mathbb{E}\big[|\eta(t)|^p\big]\, dt \right\}^{\frac{1}{p}}$$

is denoted by $M_G^p(0,T)$. Note that

$$M_G^q(0,T) \subset M_G^p(0,T) \ \text{ for } \ 1 \le p \le q.$$

Definition 2.7. For each $\eta \in M_G^{2,0}(0,T)$, the G-Itô integral is defined by

$$I(\eta) = \int_0^T \eta(v)\, dB(v) = \sum_{i=0}^{N-1} \xi_i (B(t_{i+1}) - B(t_i)).$$

The mapping $\eta \longmapsto I(\eta)$ can be extended continuously to $M_G^2(0,T)$.

Definition 2.8. The increasing continuous process $(\langle B\rangle(t))_{t\ge 0}$ with $\langle B\rangle(0) = 0$ defined by

$$\langle B\rangle(t) = B^2(t) - 2\int_0^t B(v)\, dB(v)$$

is called the quadratic variation process of $(B(t))_{t\ge 0}$. Note that $\langle B\rangle(t)$ can be regarded as the limit in $L_G^2(\Omega_t)$ of $\sum\limits_{j=1}^{N} (B(t_{i+1}^N) - B(t_i^N))^2$, where $\pi_T^N = \{t_0^N, t_1^N, \dots, t_k^N\}$ is a sequence of partitions of $[0,T]$ such that $\mu(\pi_T^N)$ tends to 0 when N goes to infinity.

The following Burkholder–Davis–Gundy inequalities play an important role in the study of our system (see [10] and [29]).

Lemma 2.9. *Let $p \ge 1$, $\eta \in M_G^p(0,T)$ and $0 \le s \le t \le T$. Then*

$$\mathbb{E}\left[\sup_{s\le u\le t} \left| \int_s^u \eta(r)\, d\langle B\rangle(r) \right|^p \right] \le C_1 (t-s)^{p-1} \int_s^t \mathbb{E}\big[|\eta(u)|^p\big]\, du,$$

where $C_1 > 0$ is a constant independent of η.

Lemma 2.10. *Let $p \ge 2$, $\eta \in M_G^p(0,T)$ and $0 \le s \le t \le T$. Then*

$$\mathbb{E}\left[\sup_{s\le u\le t} \left| \int_s^u \eta(r)\, dB(r) \right|^p \right] \le C_2 |t-s|^{\frac{p}{2}-1} \int_s^t \mathbb{E}\big[|\eta(u)|^p\big]\, du,$$

where $C_2 > 0$ is a constant independent of η.

3 Existence and uniqueness results

In this section, we are interested in the study of the existence and uniqueness of the solution to the SG-SDE (1.2), where the initial condition $(X_1(0),\dots,X_n(0))\in(\mathbb{R}^d)^n$ is a given constant and $f_{i,j}(t,x_1,\dots,x_n)\in M_G^2(0,T;(\mathbb{R}^d)^n)$ for $0\le i\le 3$ and $1\le j\le n$.

For system (1.2), the Carathéodory approximation scheme is given as follows. For any integer $k\ge 1$, we define

$$(X_1^k(t),\dots,X_n^k(t))=(X_1(0),\dots,X_n(0)),\ \text{ if } t\in]-1,0],$$

and for $t\in]0,T]$, we have

$$\begin{cases} X_1^k(t)=X_1(0)+\int\limits_0^t f_{1,1}\Big(s,X_1^k\Big(s-\frac{1}{k}\Big),\dots,X_n^k\Big(s-\frac{1}{k}\Big)\Big)\,ds \\ \qquad\qquad +\int\limits_0^t f_{2,1}\Big(s,X_1^k\Big(s-\frac{1}{k}\Big),\dots,X_n^k\Big(s-\frac{1}{k}\Big)\Big)\,d\langle B\rangle(s) \\ \qquad\qquad +\int\limits_0^t f_{3,1}\Big(s,X_1^k\Big(s-\frac{1}{k}\Big),\dots,X_n^k\Big(s-\frac{1}{k}\Big)\Big)\,dB(s), \\ \qquad\vdots \\ X_n^k(t)=X_n(0)+\int\limits_0^t f_{1,n}\Big(s,X_1^k\Big(s-\frac{1}{k}\Big),\dots,X_n^k\Big(s-\frac{1}{k}\Big)\Big)\,ds \\ \qquad\qquad +\int\limits_0^t f_{2,n}\Big(s,X_1^k\Big(s-\frac{1}{k}\Big),\dots,X_n^k\Big(s-\frac{1}{k}\Big)\Big)\,d\langle B\rangle(s) \\ \qquad\qquad +\int\limits_0^t f_{3,n}\Big(s,X_1^k\Big(s-\frac{1}{k}\Big),\dots,X_n^k\Big(s-\frac{1}{k}\Big)\Big)\,dB(s). \end{cases} \tag{3.1}$$

We assume the following assumptions (A1) and (A2) for $f_{i,j}$, $0\le i\le 3$ and $1\le j\le n$:

(A1)

$$\big|f_{i,j}(t,x_1,x_2,\dots,x_n)\big|^2\le g(t)\Big(1+\sum_{j=1}^n|x_j|^2\Big)$$

for each $x_1,\dots,x_n\in\mathbb{R}^d$ and $t\in[0,T]$, where g is a positive and continuous function on $[0,T]$.

(A2)

$$\big|f_{i,j}(t,x_1,\dots,x_n)-f_{i,j}(t,y_1,\dots,y_n)\big|^2\le h(t)\Big(\sum_{j=1}^n|y_j-x_j|^2\Big)$$

for each $x_1,y_1,\dots,x_n,y_n\in\mathbb{R}^d$ and $t\in[0,T]$, where h is a positive and continuous function on $[0,T]$.

In the sequel, the space of processes in $(M_G^2(0,T;\mathbb{R}^d))^n$ will be equipped with the norm

$$\|(X_1,\dots,X_n)\|=\mathbb{E}^{\frac12}\Big[\sup_{0\le t\le T}\Big(\sum_{j=1}^n|X_{j(t)}|^2\Big)\Big].$$

We note that this is a Banach space.

Now, we give first main result of this work.

Theorem 3.1. *Under the assumptions* (A1) *and* (A2)*, system* (1.2) *has a unique solution q.s.,*

$$(X_1(t),\dots,X_n(t))\in(M_G^2(0,T;\mathbb{R}^d))^n.$$

In order to prove this theorem, we need some important lemmas.

Lemma 3.2. *For all integers n, $k \geq 1$ and $0 \leq s < t \leq T$, we have*

$$\sup_{0\leq t\leq T} \mathbb{E}\Big[\sum_{j=1}^{n} |X_j^k(t)|^2\Big] \leq K_n \exp\Big(C_n \int_0^T g(t)\,dt\Big),$$

where

$$K_n = 1 + 4\sum_{j=1}^{n} \mathbb{E}\big[|X_j(0)|^2\big], \quad C_n = 4n(T + C_1T + C_2).$$

Proof. By using (3.1) and the fact that $\big(\sum_{j=1}^{n} a_j\big)^2 \leq n \sum_{j=1}^{n} a_j^2$ for each positive constants a_j, $1 \leq j \leq n$, for all $t \in [0,T]$, we have

$$|X_j^k(t)|^2 \leq 4|X_j(0)|^2 + 4\Big|\int_0^t f_{1,j}\Big(s, X_1^k\Big(s-\frac{1}{k}\Big),\dots,X_n^k\Big(s-\frac{1}{k}\Big)\Big)\,ds\Big|^2$$
$$+4\Big|\int_0^t f_{2,j}\Big(s, X_1^k\Big(s-\frac{1}{k}\Big),\dots,X_n^k\Big(s-\frac{1}{k}\Big)\Big)\,d\langle B\rangle(s)\Big|^2 + 4\Big|\int_0^t f_{3,j}\Big(s, X_1^k\Big(s-\frac{1}{k}\Big),\dots,X_n^k\Big(s-\frac{1}{k}\Big)\Big)\,dB(s)\Big|^2,$$

which, due to Lemmas 2.9 and 2.10, the G-Hölder inequality and the assumption (A1), implies that

$$\begin{aligned}\sup_{0\leq v\leq t} \mathbb{E}\big[|X_j^k(v)|^2\big] &\leq 4\mathbb{E}\big[|X_j(0)|^2\big] + 4(T + C_1T + C_2)\int_0^t g(s)\Big(1 + \mathbb{E}\Big[\sum_{j=1}^{n}\Big|X_j^k\Big(s-\frac{1}{k}\Big)\Big|^2\Big]\Big)\,ds\\ &\leq 4\mathbb{E}\big[|X_j(0)|^2\big] + 4(T + C_1T + C_2)\int_0^t g(s)\Big(1 + \sup_{0\leq v\leq s}\mathbb{E}\Big[\sum_{j=1}^{n}|X_j^k(v)|^2\Big]\Big)\,ds.\end{aligned}$$

Thus

$$1 + \sup_{0\leq v\leq t}\mathbb{E}\Big[\sum_{j=1}^{n}|X_j^k(t|^2\Big] \leq 1 + 4\sum_{j=1}^{n}\mathbb{E}\big[|X_j(0)|^2\big] + C_n\int_0^t g(s)\Big(1 + \sup_{0\leq v\leq s}\mathbb{E}\Big[\sum_{j=1}^{n}|X_j^k(v)|^2\Big]\Big)\,ds,$$

where $C_n = 4n(T + C_1T + C_2)$. Applying Gronwall's lemma, we conclude that

$$1 + \sup_{0\leq v\leq t}\mathbb{E}\Big[\sum_{j=1}^{n}|X_j^k(v)|^2\Big] \leq K_n \exp\Big(C_n\int_0^t g(s)\,ds\Big)$$

and, consequently,

$$\sup_{0\leq t\leq T}\mathbb{E}\Big[\sum_{j=1}^{n}|X_j^k(t)|^2\Big] \leq K_n \exp\Big(C_n\int_0^T g(t)\,dt\Big).$$ □

Lemma 3.3. *For all integers n, $k \geq 1$ and $0 \leq s < t \leq T$, we have*

$$\mathbb{E}\Big[\sum_{j=1}^{n}|X_j^k(t) - X_j^k(s)|^2\Big] \leq L_n[G(t) - G(s)],$$

where

$$G(t) = \int_0^t g(s)\,ds \ \text{ and } \ L_n = \frac{3}{4}C_n\Big[1 + K_n\exp\Big(C_n\int_0^T g(t)\,dt\Big)\Big].$$

Proof. We have

$$X_j^k(t) - X_j^k(s) = \int_s^t f_{1,j}\Big(w, X_1^k\Big(w - \frac{1}{k}\Big), \dots, X_n^k\Big(w - \frac{1}{k}\Big)\Big)\, dw$$

$$+\int_s^t f_{2,j}\Big(w, X_1^k\Big(w-\frac{1}{k}\Big), \dots, X_n^k\Big(w-\frac{1}{k}\Big)\Big)\, d\langle B\rangle(w)+\int_s^t f_{3,j}\Big(w, X_1^k\Big(w-\frac{1}{k}\Big), \dots, X_n^k\Big(w-\frac{1}{k}\Big)\Big)\, dB(w)$$

and so, for each $0 \le s \le v \le u \le t \le T$, we have

$$\mathbb{E}\Big[\sup_{s\le v\le u\le t} |X_j^k(u) - X_j^k(v)|^2\Big] \le 3\mathbb{E}\Big[\sup_{s\le v\le u\le t}\Big|\int_v^u f_{1,j}\Big(w, X_1^k\Big(w - \frac{1}{k}\Big), \dots, X_n^k\Big(w - \frac{1}{k}\Big)\Big)\, dw\Big|^2\Big]$$

$$+ 3\mathbb{E}\Big[\sup_{s\le v\le u\le t}\Big|\int_v^u f_{2,j}\Big(w, X_1^k\Big(w - \frac{1}{k}\Big), \dots, X_n^k\Big(w - \frac{1}{k}\Big)\Big)\, d\langle B\rangle(w)\Big|^2\Big]$$

$$+ 3\mathbb{E}\Big[\sup_{s\le v\le u\le t}\Big|\int_v^u f_{3,j}\Big(w, X_1^k\Big(w - \frac{1}{k}\Big), \dots, X_n^k\Big(w - \frac{1}{k}\Big)\Big)\, dB(w)\Big|^2\Big].$$

Owing to Lemmas 2.9, 2.10 and the assumption (A1), we obtain

$$\mathbb{E}\Big[\sup_{s\le v\le u\le t} |X_j^k(u) - X_j^k(v)|^2\Big] \le 3T\int_s^t \mathbb{E}\Big[\Big|f_{1,j}(w, X_j^k\Big(w - \frac{1}{k}\Big), \dots, X_n^k\Big(w - \frac{1}{k}\Big)\Big)\Big|^2\Big]\, dw$$

$$+ 3C_1T\int_s^t \mathbb{E}\Big[\Big|f_{2,j}(w, X_j^k\Big(w - \frac{1}{k}\Big), \dots, X_n^k\Big(w - \frac{1}{k}\Big)\Big)\Big|^2\Big]\, dw$$

$$+ 3C_2\int_s^t \mathbb{E}\Big[\Big|f_{3,j}(w, X_j^k\Big(w - \frac{1}{k}\Big), \dots, X_n^k\Big(w - \frac{1}{k}\Big)\Big)\Big|^2\Big]\, dw$$

$$\le 3(T + C_1T + C_2)\int_s^t g(w)\Big(1 + \mathbb{E}\Big[\sum_{j=1}^n \Big|X_j^k\Big(w - \frac{1}{k}\Big)\Big|^2\Big]\Big)\, dw$$

$$\le 3(T + C_1T + C_2)[G(t) - G(s)] + 3(T + C_1T + C_2)\int_s^t g(w)\mathbb{E}\Big[\sum_{j=1}^n \Big|X_j^k\Big(w - \frac{1}{k}\Big)\Big|^2\Big]\, dw.$$

Using Lemma 3.2, we get

$$\mathbb{E}\Big[\sup_{s\le v\le u\le t} |X_j^k(u) - X_j^k(v)|^2\Big] \le 3(T + C_1T + C_2)\Big[1 + K_n \exp\Big(C_n\int_0^T g(t)\, dt\Big)\Big]\, [G(t) - G(s)].$$

Thus

$$\sum_{j=1}^n \mathbb{E}\Big[\sup_{s\le v\le u\le t} |X_j^k(u) - X_j^k(v)|^2\Big] \le 3n(T + C_1T + C_2)\Big[1 + K_n \exp\Big(C_n\int_0^T g(t)\, dt\Big)\Big]\, [G(t) - G(s)].$$

Then

$$\sum_{j=1}^n \mathbb{E}\big[|X_j^k(t) - X_j^k(s)|^2\big] \le L_n[G(t) - G(s)],$$

where

$$L_n = \frac{3}{4} C_n \Big[1 + K_n \exp\Big(C_n \int_0^T g(t)\,dt \Big) \Big],$$

which proves the desired result. □

Proof of Theorem 3.1. We will prove the theorem in three steps.

Step 1: Suppose that $(X_1(t), \ldots, X_n(t))$ and $(Y_1(t), \ldots, Y_n(t))$ are two solutions of system (1.2) with the initial conditions $(X_1(0), \ldots, X_n(0))$ and $(Y_1(0), \ldots, Y_n(0))$, respectively. Then we for $1 \leq j \leq n$, we have

$$\begin{aligned} |Y_j(t) - X_j(t)|^2 \leq 4|X_j(0) - Y_j(0)|^2 & \\ + 4\Big| \int_0^t f_{1,j}(s, X_1(s), \ldots, X_n(s)) - f_{1,j}(s, Y_1(s), \ldots, Y_n(s))\,ds \Big|^2 & \\ + 4\Big| \int_0^t f_{2,j}(s, X_1(s), \ldots, X_n(s)) - f_{2,j}(s, Y_1(s), \ldots, Y_n(s))\,d\langle B\rangle(s) \Big|^2 & \\ + 4\Big| \int_0^t f_{3,j}(s, X_1(s), \ldots, X_n(s)) - f_{3,j}(s, Y_1(s), \ldots, Y_n(s))\,dB(s) \Big|^2 &. \end{aligned}$$

Now, by using Lemmas 2.9, 2.10 and the assumption (A2), for $0 \leq r \leq t \leq T$, we have

$$\begin{aligned} &\mathbb{E}\Big[\Big| \int_0^r \big(f_{1,j}(s, X_1(s), \ldots, X_n(s)) - f_{1,j}(s, Y_1(s), \ldots, Y_n(s))\big)\,ds \Big|^2 \Big] \\ &\qquad \leq T \int_0^t \mathbb{E}\Big[\big| f_{1,j}(s, X_1(s), \ldots, X_n(s)) - f_{1,j}(s, Y_1(s), \ldots, Y_n(s)) \big|^2 \Big]\,ds \\ &\qquad \leq T \int_0^t h(s)\mathbb{E}\Big[\Big(\sum_{j=1}^n |Y_j(s) - X_j(s)|^2 \Big) \Big]\,ds, \end{aligned}$$

$$\begin{aligned} &\mathbb{E}\Big[\sup_{0\leq r\leq t} \Big| \int_0^r f_{2,j}(s, X_1(s), \ldots, X_n(s)) - f_{2,j}(s, Y_1(s), \ldots, Y_n(s))\,d\langle B\rangle(s) \Big|^2 \Big] \\ &\qquad \leq C_1 T \int_0^t \mathbb{E}\Big[\big| f_{2,j}(s, X_1(s), \ldots, X_n(s)) - f_{2,j}(s, Y_1(s), \ldots, Y_n(s)) \big|^2 \Big]\,ds \\ &\qquad \leq C_1 T \int_0^t h(s)\mathbb{E}\Big[\Big(\sum_{j=1}^n |Y_j(s) - X_j(s)|^2 \Big) \Big]\,ds, \end{aligned}$$

and

$$\begin{aligned} &\mathbb{E}\Big[\sup_{0\leq r\leq t} \Big| \int_0^r f_{3,j}(s, X_1(s), \ldots, X_n(s)) - f_{3,j}(s, Y_1(s), \ldots, Y_n(s))\,dB(s) \Big|^2 \Big] \\ &\qquad \leq C_2 \int_0^t \mathbb{E}\Big[\big| f_{3,j}(s, X_1(s), \ldots, X_n(s)) - f_{3,j}(s, Y_1(s), \ldots, Y_n(s)) \big|^2 \Big]\,ds \end{aligned}$$

$$\leq C_2 \int_0^t h(s)\mathbb{E}\Big[\Big(\sum_{j=1}^n |Y_j(s) - X_j(s)|^2\Big)\Big]\, ds.$$

Therefore,

$$\mathbb{E}\Big[\sup_{0\leq r\leq t} |Y_j(r) - X_j(r)|^2\Big]$$
$$\leq 4|Y_j(0) - X_j(0)|^2 + 4(T + C_1T + C_2) \int_0^t h(s)\mathbb{E}\Big[\sum_{j=1}^n |Y_j(s) - X_j(s)|^2\Big]\, ds.$$

We obtain

$$\mathbb{E}\Big[\sup_{0\leq r\leq t} \Big(\sum_{j=1}^n |Y_j(r) - X_j(r)|^2\Big)\Big] \leq 4\sum_{j=1}^n |Y_j(0) - X_j(0)|^2 + C_n \int_0^t h(s)\mathbb{E}\Big[\sum_{j=1}^n |Y_j(s) - X_j(s)|^2\Big]\, ds.$$

Using Gronwall's lemma, we get

$$\mathbb{E}[\sup_{0\leq r\leq t} \Big(\sum_{j=1}^n |Y_j(r) - X_j(r)|^2\Big)\Big] \leq 4\sum_{j=1}^n |Y_j(0) - X_j(0)|^2 \exp\Big(C_n \int_0^t h(s)\, ds\Big).$$

Now, taking

$$(X_1(0), \ldots, X_n(0)) = (Y_1(0), \ldots, Y_n(0)),$$

we can see that for $t = T$,

$$\mathbb{E}\Big[\sup_{0\leq r\leq T} \Big(\sum_{j=1}^n |Y_j(r) - X_j(r)|^2\Big)\Big] = 0,$$

which implies

$$(X_1(t), \ldots, X_n(t)) = (Y_1(t), \ldots, Y_n(t)) \text{ q.s. for each } t \in [0, T].$$

Step 2: We now prove that $(X_1^k(t), \ldots, X_n^k(t))_{k\geq 1}$ in $(M_G^2(0, T; \mathbb{R}^d))^n$ is a Cauchy sequence for each $t \in [0, T]$. By the same arguments as those used in the previous step, for each $\ell > k$, we have

$$\mathbb{E}\Big[\sup_{0\leq t\leq T} \Big(\sum_{j=1}^n |X_j^\ell(t) - X_j^k(t)|^2\Big)\Big] \leq \frac{3}{4}\, C_n \int_0^T h(s)\mathbb{E}\Big[\Big(\sum_{j=1}^n \Big|X_j^\ell\Big(s - \frac{1}{\ell}\Big) - X_j^k\Big(s - \frac{1}{k}\Big)\Big|^2\Big)\Big]\, ds.$$

Since

$$\mathbb{E}\Big[\sum_{j=1}^n \Big|X_j^\ell\Big(s - \frac{1}{\ell}\Big) - X_j^k\Big(s - \frac{1}{k}\Big)\Big|^2\Big]$$
$$\leq 2\mathbb{E}\Big[\sum_{j=1}^n \Big|X_j^\ell\Big(s - \frac{1}{\ell}\Big) - X_j^k\Big(s - \frac{1}{\ell}\Big)\Big|^2\Big] + 2\mathbb{E}\Big[\sum_{j=1}^n \Big|X_j^k\Big(s - \frac{1}{\ell}\Big) - X_j^k\Big(s - \frac{1}{k}\Big)\Big|^2\Big]$$
$$\leq 2\mathbb{E}\Big[\sup_{0\leq u\leq s} \Big(\sum_{j=1}^n |X_j^\ell(u) - X_j^k(u)|^2\Big)\Big] + 2\mathbb{E}\Big[\Big(\sum_{j=1}^n \Big|X_j^k\Big(s - \frac{1}{\ell}\Big) - X_j^k\Big(s - \frac{1}{k}\Big)\Big|^2\Big)\Big],$$

using Lemma 3.3 we get

$$\mathbb{E}\Big[\sup_{0\leq t\leq T} \Big(\sum_{j=1}^n |X_j^\ell(t) - X_j^k(t)|^2\Big)\Big]$$
$$\leq \frac{3}{2}\, C_n \int_0^T h(r)\mathbb{E}\Big[\sup_{0\leq u\leq r} \Big(\sum_{j=1}^n |X_j^\ell(u) - X_j^k(u)|^2\Big)\Big]\, dr + \frac{3}{2}\, C_n L_n\Big[G\Big(s - \frac{1}{\ell}\Big) - G\Big(s - \frac{1}{k}\Big)\Big] \int_0^T h(r)\, dr.$$

Thus, by Gronwall's lemma,

$$\mathbb{E}\Big[\sup_{0\le t\le T}\Big(\sum_{j=1}^{n}|X_j^{\ell}(t)-X_j^{k}(t)|^2\Big)\Big]\le M_n\Big(\frac{1}{k}-\frac{1}{\ell}\Big)\exp\Big(\frac{3}{2}\,C_n\int_0^T h(s)\,ds\Big),$$

where

$$M_n=\frac{3}{2}\,TC_nL_n\sup_{0\le t\le T}[g(t)]\sup_{0\le t\le T}[h(t)],$$

which means that $(X_1^k(t),\dots,X_n^k(t))_{k\ge1}$ is a Cauchy sequence.

Step 3: Here we prove that the limit $(X_1(t),\dots,X_n(t))$ in $(M_G^2(0,T;\mathbb{R}^d))^n$ of $(X_1^k(t),\dots,X_n^k(t))$ is the solution of system (1.2). For the existence, let the initial condition $(X_1(0),\dots,X_n(0))\in(\mathbb{R}^d)^n$ be a given constant.

This results in

$$\begin{aligned}|X_j(u)-X_j^k(u)|^2\le 3\Big|\int_0^u f_{1,j}\Big(s,X_1^k\Big(s-\frac{1}{k}\Big),\dots,X_n^k\Big(s-\frac{1}{k}\Big)\Big)-f_{1,j}(s,X_1(s),\dots,X_n(s))\,ds\Big|^2\\
+3\Big|\int_0^u f_{2,j}\Big(s,X_1^k\Big(s-\frac{1}{k}\Big),\dots,X_n^k\Big(s-\frac{1}{k}\Big)\Big)-f_{2,j}(s,X_1(s),\dots,X_n(s))\,d\langle B\rangle(s)\Big|^2\\
+3\Big|\int_0^u f_{3,j}\Big(s,X_1^k\Big(s-\frac{1}{k}\Big),\dots,X_n^k\Big(s-\frac{1}{k}\Big)\Big)-f_{3,j}(s,X_1(s),\dots,X_n(s))\,dB(s)\Big|^2.\end{aligned}$$

Using Lemmas 2.9, 2.10 and the assumption (A2), we have

$$\begin{aligned}\mathbb{E}\Big[\sup_{0\le u\le T}\big(|X_j^k(u)-X_j(u)|^2\big)\Big]&\le 3(T+C_1T+C_2)\int_0^T h(s)\mathbb{E}\Big[\sum_{j=1}^{n}\Big|X_j^k\Big(s-\frac{1}{k}\Big)-X_j(s)\Big|^2\Big]\,ds\\
&\le 6(T+C_1T+C_2)\int_0^T h(s)\mathbb{E}\Big[\sum_{j=1}^{n}\Big|X_j^k\Big(s-\frac{1}{k}\Big)-X_j^k(s)\Big|^2\Big]\,ds\\
&\quad+6(T+C_1T+C_2)\int_0^T h(s)\mathbb{E}\Big[\sum_{j=1}^{n}|X_j^k(s)-X_j(s)|^2\Big]\,ds.\end{aligned}$$

Thus, using Lemma 3.3,

$$\mathbb{E}\Big[\sup_{0\le u\le T}\big(|X_j^k(u)-X_j(u)|^2\big)\Big]\le\frac{M_n}{nk}+\frac{3C_n}{2n}\int_0^T\mathbb{E}\Big[\sup_{0\le u\le s}\Big(\sum_{j=1}^{n}|X_j^k(u)-X_j(u)|^2\Big)\Big]\,ds,$$

which implies that

$$\mathbb{E}\Big[\sup_{0\le u\le T}\Big(\sum_{j=1}^{n}\big(|X_j^k(u)-X_j(u)|^2\big)\Big)\Big]\le\frac{M_n}{k}+\frac{3}{2}\,C_n\int_0^T h(s)\mathbb{E}\Big[\sup_{0\le u\le s}\sum_{j=1}^{n}\big(|X_j^k(u)-X_j(u)|^2\big)\Big]\,ds.$$

Applying Gronwall's lemma again, we get directly

$$\mathbb{E}\Big[\sup_{0\le u\le T}\Big(\sum_{j=1}^{n}|X_j^k(u)-X_j(u)|^2\Big)\Big]\le\frac{M_n}{k}\exp\Big(\frac{3}{2}\,C_n\int_0^T h(s)\,ds\Big),$$

which shows our result. □

4 Stability result

In this section, we prove another important result on the stability of the following G-SDEs depending on a parameter ε ($\varepsilon \geq 0$) (for more information, see, e.g., [13, 15, 27, 28, 30]):

$$\begin{cases} X_1^\varepsilon(t) = X_1^\varepsilon(0) + \int\limits_0^t f_{1,1}^\varepsilon(s, X_1^\varepsilon(s), \dots, X_n^\varepsilon(s))\, ds \\ \qquad\qquad + \int\limits_0^t f_{2,1}^\varepsilon(s, X_1^\varepsilon(s), \dots, X_n^\varepsilon(s))\, d\langle B\rangle(s) + \int\limits_0^t f_{3,1}^\varepsilon(s, X_1^\varepsilon(s), \dots, X_n^\varepsilon(s))\, dB(s), \\ \quad \vdots \\ X_n^\varepsilon(t) = X_n^\varepsilon(0) + \int\limits_0^t f_{1,n}^\varepsilon(s, X_1^\varepsilon(s), \dots, X_n^\varepsilon(s))\, ds \\ \qquad\qquad + \int\limits_0^t f_{2,n}^\varepsilon(s, X_1^\varepsilon(s), \dots, X_n^\varepsilon(s))\, d\langle B\rangle(s) + \int\limits_0^t f_{3,n}^\varepsilon(s, X_1^\varepsilon(s), \dots, X_n^\varepsilon(s))\, dB(s). \end{cases}$$

We assume the following assumptions (B1), (B2) and (B3) for $f_{i,j}^\varepsilon$, $0 \leq i \leq 3$ and $1 \leq j \leq n$:

(B1)
$$|f_{i,j}^\varepsilon(t, x_1, \dots, x_n)|^2 \leq g(t)\Big(1 + \sum_{j=1}^n |x_j|^2\Big)$$
for each $x_1,\ x_2, \dots, x_n \in \mathbb{R}^d$ and $t \in [0,T]$, where g is a positive and continuous function on $[0,T]$.

(B2)
$$\big|f_{i,j}^\varepsilon(t, x_1, \dots, x_n) - f_{i,j}^\varepsilon(t, y_1, \dots, y_n)\big|^2 \leq h(t)\Big(\sum_{j=1}^n |y_j - x_j|^2\Big)$$
for each $x_1, y_1, \dots, x_n, y_n \in \mathbb{R}^d$ and $t \in [0,T]$, where h is a positive and continuous function on $[0,T]$.

(B3) (i) $\forall\, t \in [0,T]$,
$$\lim_{\varepsilon \to 0} \int\limits_0^t \mathbb{E}\Big[\big|f_{i,j}^\varepsilon(s, X_1^0(s), \dots, X_n^0(s)) - f_{i,j}^0(s, X_1^0(s), \dots, X_n^0(s))\big|\Big]\, ds = 0;$$

(ii)
$$\lim_{\varepsilon \to 0}(X_1^\varepsilon(0), \dots, X_n^\varepsilon(0)) = (X_1^0(0), \dots, X_n^0(0)).$$

Remark 4.1. The assumptions (B1) and (B2) guarantee, for any $\varepsilon \geq 0$, the existence of a unique solution
$$(X_1^\varepsilon(t), \dots, X_n^\varepsilon(t)) \in (M_G^2(0,T;\mathbb{R}^d))^n$$
of our system, while the assumption (B3) allows us to deduce the stability theorem for the system.

The following lemmas are very important, they will be used in the upcoming result. For the proofs see [15].

Lemma 4.2. *For every $p \geq 1$ and for any $T > 0$ and $\eta \in M_G^p(0,T)$, we have*

$$\mathbb{E}\Big[\Big|\int_0^T \eta(t)\,dt\Big|^p\Big] \leq T^{p-1}\int_0^T \mathbb{E}\big[|\eta(t)|^p\big]\,dt,$$

$$\mathbb{E}\Big[\Big|\int_0^T \eta(t)\,d\langle B\rangle(t)\Big|^p\Big] \leq T^{p-1}\int_0^T \mathbb{E}\big[|\eta(t)|^p\big]\,dt.$$

Lemma 4.3. *For every $p \geq 2$, there exists a positive constant C_p such that, for any $T > 0$ and $\eta \in M_G^p(0,T)$,*

$$\mathbb{E}\Big[\Big|\int_0^T \eta(t)\,dB(t)\Big|^p\Big] \leq C_p T^{\frac{p}{2}-1}\int_0^T \mathbb{E}\big[|\eta(t)|^p\big]\,dt.$$

Now, we present our second main result of this work.

Theorem 4.4. *Under the assumptions* (B1), (B2) *and* (B3)*, we have*

$$\forall\, t \in [0,T],\ \ \lim_{\varepsilon\to 0}\mathbb{E}\Big[\sum_{j=1}^n |X_j^\varepsilon(t) - X_j^0(t)|^2\Big] = 0.$$

Proof. For all $1 \leq j \leq n$, we have

$$\begin{aligned}
X_j^\varepsilon(t) &= X_j^\varepsilon(0) + \int_0^t f_{1,j}^\varepsilon(s, X_1^\varepsilon(s),\dots,X_n^\varepsilon(s))\,ds\\
&\quad + \int_0^t f_{2,j}^\varepsilon(s, X_1^\varepsilon(s),\dots,X_n^\varepsilon(s))\,d\langle B\rangle_s + \int_0^t f_{3,j}^\varepsilon(s, X_1^\varepsilon(s),\dots,X_n^\varepsilon(s))\,dB(s),\\
X_j^0(t) &= X_j^0(0) + \int_0^t f_{1,j}^0(s, X_1^0(s),\dots,X_n^0(s))\,ds\\
&\quad + \int_0^t f_{2,j}^0(s, X_1^0(s),\dots,X_n^0(s))\,d\langle B\rangle(s) + \int_0^t f_{3,j}^0(s, X_1^0(s),\dots,X_n^0(s))\,dB(s).
\end{aligned}$$

Then

$$\begin{aligned}
X_j^\varepsilon(t) - X_j^0(t) &= X_j^\varepsilon(0) - X_j^0(0)\\
&\quad + \int_0^t \Big[f_{1,j}^\varepsilon(s, X_1^\varepsilon(s),\dots,X_n^\varepsilon(s)) - f_{1,j}^0(s, X_1^0(s),\dots,X_n^0(s))\Big]\,ds\\
&\quad + \int_0^t \Big[f_{2,j}^\varepsilon(s, X_1^\varepsilon(s),\dots,X_n^\varepsilon(s)) - f_{2,j}^0(s, X_1^0(s),\dots,X_n^0(s))\Big]\,d\langle B\rangle(s)\\
&\quad + \int_0^t \Big[f_{3,j}^\varepsilon(s, X_1^\varepsilon(s),\dots,X_n^\varepsilon(s)) - f_{3,j}^0(s, X_1^0(s),\dots,X_n^0(s))\Big]\,dB(s)
\end{aligned}$$

and

$$\begin{aligned}
X_j^\varepsilon(t) - X_j^0(t) = X_j^\varepsilon(0) - X_j^0(0) + &\\
+ \int_0^t \Big(& f_{1,j}^\varepsilon(s, X_1^\varepsilon(s), \ldots, X_n^\varepsilon(s)) - f_{1,j}^0(s, X_1^0(s), \ldots, X_n^0(s)) \\
& + f_{1,j}^\varepsilon(s, X_1^\varepsilon(s), \ldots, X_n^\varepsilon(s)) - f_{1,j}^\varepsilon(s, X_1^\varepsilon(s), \ldots, X_n^\varepsilon(s)) \Big) \, ds \\
+ \int_0^t \Big(& f_{2,j}^\varepsilon(s, X_1^\varepsilon(s), \ldots, X_n^\varepsilon(s)) - f_{2,j}^0(s, X_1^0(s), \ldots, X_n^0(s)) \\
& + f_{2,j}^\varepsilon(s, X_1^\varepsilon(s), \ldots, X_n^\varepsilon(s)) - f_{2,j}^\varepsilon(s, X_1^\varepsilon(s), \ldots, X_n^\varepsilon(s)) \Big) \, d\langle B\rangle(s) \\
+ \int_0^t \Big(& f_{3,j}^\varepsilon(s, X_1^\varepsilon(s), \ldots, X_n^\varepsilon(s)) - f_{3,j}^\varepsilon(s, X_1^0(s), \ldots, X_n^0(s)) \\
& + f_{3,j}^\varepsilon(s, X_1^\varepsilon(s), \ldots, X_n^\varepsilon(s)) - f_{3,j}^\varepsilon(s, X_1^\varepsilon(s), \ldots, X_n^\varepsilon(s)) \Big) \, dB(s).
\end{aligned}$$

We have

$$\begin{aligned}
|X_j^\varepsilon(t) - X_j^0(t)|^2 \le\; & 7|X_j^\varepsilon(0) - X_j^0(0)|^2 \\
& + 7\Big| \int_0^t \Big[f_{1,j}^\varepsilon(s, X_1^\varepsilon(s), \ldots, X_n^\varepsilon(s)) - f_{1,j}^\varepsilon(s, X_1^0(s), \ldots, X_n^0(s)) \Big] \, ds \Big|^2 \\
& + 7\Big| \int_0^t \Big[f_{1,j}^\varepsilon(s, X_1^0(s), \ldots, X_n^0(s)) - f_{1,j}^0(s, X_1^0(s), \ldots, X_n^0(s)) \Big] \, ds \Big|^2 \\
& + 7\Big| \int_0^t \Big[f_{2,j}^\varepsilon(s, X_1^\varepsilon(s), \ldots, X_n^\varepsilon(s)) - f_{2,j}^\varepsilon(s, X_s^0, y_s^0) \Big] \, d\langle B\rangle(s) \Big|^2 \\
& + 7\Big| \int_0^t \Big[f_{2,j}^\varepsilon(s, X_1^0(s), \ldots, X_n^0(s)) - f_{2,j}^0(s, X_1^0(s), \ldots, X_n^0(s)) \Big] \, d\langle B\rangle(s) \Big|^2 \\
& + 7\Big| \int_0^t \Big[f_{3,j}^\varepsilon(s, X_1^\varepsilon(s), \ldots, X_n^\varepsilon(s)) - f_{3,j}^\varepsilon(s, X_1^0(s), \ldots, X_n^0(s)) \Big] \, dB(s) \Big|^2 \\
& + 7\Big| \int_0^t \Big[f_{3,j}^\varepsilon(s, X_1^0(s), \ldots, X_n^0(s)) - f_{3,j}^0(s, X_1^0(s), \ldots, X_n^0(s)) \Big] \, dB(s) \Big|^2.
\end{aligned}$$

Taking the G-expectation on both sides of the above relation, from Lemmas 4.2 and 4.3 we get

$$\begin{aligned}
\mathbb{E}|X_j^\varepsilon(t) - X_j^0(0)|^2 \le\; & 7\mathbb{E}\big[|X_j^\varepsilon(0) - X_j^0(0)|^2\big] \\
& + 7T \int_0^t \mathbb{E}\Big[\Big| f_{1,j}^\varepsilon(s, X_1^\varepsilon(s), \ldots, X_n^\varepsilon(s)) - f_{1,j}^\varepsilon(s, X_1^0(s), \ldots, X_n^0(s)) \Big|^2\Big] \, ds \\
& + 7T\Big| \int_0^t \mathbb{E}\Big[\Big| f_{1,j}^\varepsilon(s, X_1^0(s), \ldots, X_n^0(s)) - f_{1,j}^0(s, X_1^0(s), \ldots, X_n^0(s)) \Big|^2\Big] \, ds \Big| \\
& + 7T \int_0^t \mathbb{E}\Big[\Big| f_{2,j}^\varepsilon(s, X_1^\varepsilon(s), \ldots, X_n^\varepsilon(s)) - f_{2,j}^\varepsilon(s, X_1^0(s), \ldots, X_n^0(s)) \Big|^2\Big] \, ds
\end{aligned}$$

$$
\begin{aligned}
&+ 7T \int_0^t \mathbb{E}\Big[\Big| f^{\varepsilon}_{2,j}(s, X^0_1(s), \dots, X^0_n(s)) - f^0_{1,j}(s, X^0_1(s), \dots, X^0_n(s))\Big|^2\Big]\, ds \\
&+ 7C \int_0^t \mathbb{E}\Big[\Big| f^{\varepsilon}_{3,j}(s, X^{\varepsilon}_1(s), \dots, X^{\varepsilon}_n(s)) - f^{\varepsilon}_{3,j}(s, X^0_1(s), \dots, X^0_n(s))\Big|^2\Big]\, ds \\
&+ 7C \int_0^t \mathbb{E}\Big[\Big| f^{\varepsilon}_{3,j}(s, X^0_1(s), \dots, X^0_n(s)) - f^0_{3,j}(s, X^0_1(s), \dots, X^0_n(s))\Big|^2\Big]\, ds.
\end{aligned}
$$

By the assumptions (B1)–(B3), we obtain

$$
\mathbb{E}\big[|X^{\varepsilon}_j(t) - X^0_j(t)|^2\big] \le C^{\varepsilon}(T) + 7(2T + C) \int_0^t \mathbb{E}\Big(h(s) \sum_{j=1}^n |X^{\varepsilon}_j(s) - X^0_j(s)|^2\Big)\, ds,
$$

where

$$
\begin{aligned}
C^{\varepsilon}(t) = {}& 7\mathbb{E}\big[|X^{\varepsilon}_j(0) - X^0_j(0)|^2\big] \\
&+ 7T \int_0^t \mathbb{E}\Big[\Big| f^{\varepsilon}_{1,j}(s, X^0_1(s), \dots, X^0_n(s)) - f^0_{1,j}(s, X^0_1(s), \dots, X^0_n(s))\Big|^2\Big]\, ds \\
&+ 7T \int_0^t \mathbb{E}\Big[\Big| f^{\varepsilon}_{2,j}(s, X^0_1(s), \dots, X^0_n(s)) - f^0_{2,j}(s, X^0_1(s), \dots, X^0_n(s))\Big|^2\Big]\, ds \\
&+ 7C \int_0^t \mathbb{E}\Big[\Big| f^{\varepsilon}_{3,j}(s, X^0_1(s), \dots, X^0_n(s)) - f^0_{3,j}(s, X^0_1(s), \dots, X^0_n(s))\Big|^2\Big]\, ds.
\end{aligned}
$$

Then

$$
\begin{aligned}
\mathbb{E}\Big[\sum_{j=1}^n |X^{\varepsilon}_j(t) - X^0_j(t)|^2\Big] &\le \sum_{j=1}^n \mathbb{E}|X^{\varepsilon}_j(t) - X^0_j(t)|^2 \\
&\le C^{\varepsilon}_n(T) + C_n(T) \int_0^t h(s) \sum_{j=1}^n \mathbb{E}|X^{\varepsilon}_j(s) - X^0_j(s)|^2\, ds,
\end{aligned}
$$

where

$$
C^{\varepsilon}_n(T) = nC^{\varepsilon}(T) \;\text{ and }\; C_n(T) = 7n(2T + C).
$$

Hence, by Gronwall's inequality, we have

$$
\mathbb{E}\Big[\sum_{j=1}^n |X^{\varepsilon}_j(t) - X^0_j(t)|^2\Big] \le C^{\varepsilon}_n(T) \exp\Big(C_n(T) \int_0^T h(t)\, dt\Big).
$$

Since $C^{\varepsilon}_n(T) \to 0$ as $\varepsilon \to 0$, we finally get

$$
\forall\, t \in [0, T], \quad \lim_{\varepsilon \to 0} \mathbb{E}\Big[\sum_{j=1}^n |X^{\varepsilon}_j(t) - X^0_j(t)|^2\Big] = 0,
$$

hence the desired result follows. □

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