Memoirs on Differential Equations and Mathematical Physics

Volume 86, 2022, 85–96

Leila Khitri-Kazi-Tani, Hacen Dib

A NEW h-DISCRETE FRACTIONAL OPERATOR, FRACTIONAL POWER AND FINITE SUMMATION OF HYPERGEOMETRIC POLYNOMIALS

Abstract. In the present paper, we introduce the discrete fractional trapezoidal operators T_h^α for $\alpha \in (0,1)$ as the fractional power of the classical trapezoidal formula. Consequently, we derive the fractional power of a triangular matrix. As applications, we determine the eigenvectors of T_h^α and a finite summation formula of the product of hypergeometric polynomials.

2010 Mathematics Subject Classification. 26A33, 39A12, 33C05, 33C45, 47B12, 15A16.

Key words and phrases. Discrete fractional calculus, trapezoidal operator, hypergeometric polynomials, sectorial operator, fractional power, matrix function, Meixner polynomials.

რეზიუმე. ნაშრომში $\alpha \in (0,1)$-თვის შემოყვანილია დისკრეტული წილადური ტრაპეციული T_h^α ოპერატორები, რომლებიც განსაზღვრულია როგორც კლასიკური ტრაპეციების ფორმულის წილადური ხარისხი. შედეგად გამოყვანილია სამკუთხა მატრიცის წილადური ხარისხი. ამ ოპერატორების გამოყენებით განსაზღვრულია T_h^α-ის საკუთრივი ვექტორები და ჰიპერგეომეტრიული მრავალწევრების ნამრავლების სასრული ჯამის ფორმულა.

1 Introduction

The h-discrete fractional calculus is an active research area (cf. [1, 3, 5, 9–11, 13] and the references therein). In [1], some discrete fractional dynamical systems are presented as a part of h-discrete operators with different kernels.

These developments have encouraged the use of such operators in applied mathematics, especially in modeling of some discrete time dynamical systems. We cite, in particular, the work of Atici et al. [4], where a pharmacokinetic-pharmaco-dynamic model describing tumor growth is represented by fractional h-discrete and h-discrete operators.

This work aims to contribute to the development of the theory of fractional calculus on $h\mathbb{Z}$ by introducing a new discrete formula. To this end, we derive a fractional operator as a fractional power of trapezoidal operator expressed in a matrix form. Motivated by many applications, the concept of matrix function has been the subject of multiple research. We mention the relevant work of Higham [7], in which, in particular, algorithms for computing matrix fractional powers were developed (see [8] and [14]). The matrix fractional power arises in several situations of fractional calculus and algorithms for calculating fractional powers of a matrix are still being investigated.

For our purpose we define $\tau_h = \{t_n = a + nh,\ n = 0, \ldots, N,\ Nh = b - a\}$, a subdivision of the interval $[a, b]$ and X_h, the Banach space of functions defined on τ_h endowed with the norm

$$\|f\|_\infty = \max_{t_n \in \tau_h} \|f(t_n)\|$$

and T_h, the trapezoidal operator acting on X_h defined by

$$T_h(f)(t_n) = \frac{h}{2} \sum_{i=1}^{n} \big[f(t_{i-1}) + f(t_i)\big].$$

By convention, $T_h(f)(a) = 0$.

In matrix notation, T_h can be expressed as a lower triangular matrix

$$T_h = \frac{h}{2} \begin{pmatrix} 0 & 0 & & \cdots & & 0 \\ 1 & 1 & & & & \\ 1 & 2 & 1 & \ddots & 0 & \vdots \\ 1 & 2 & 2 & 1 & & \\ \vdots & & & & \ddots & 0 \\ 1 & 2 & 2 & \cdots & 2 & 1 \end{pmatrix}.$$

Evidently, the operator T_h is a bounded linear operator on X_h and 0, $\frac{h}{2}$ are eigenvalues of T_h. In this study, we derive an explicit formula for the fractional power of this matrix by using the classical Balakrishnan integral representation of fractional power of sectorial operator developed by Haase [6].

The paper is organized as follows. In Section 2, we introduce some definitions and formulas. In Section 3, we derive the fractional power of the operator T_h. In Section 4, we obtain the fractional power of the h-rising factorial function defined in [3]. Then we get the summation formula. We end by a conclusion.

2 Preliminary definitions and results

In this section, we recall some concepts used within the fractional power of sectorial operators (for more details see [6]).

Let A be a linear operator in a Banach space X. For a complex number λ, the resolvent of A is $R(\lambda, A) := (\lambda I - A)^{-1}$ if exists; the resolvent set $\rho(A)$ is the set of all $\lambda \in \mathbb{C}$ for which $R(\lambda, A)$ exists, and $\sigma(A) = \mathbb{C} \setminus \rho(A)$ is the spectrum.

Definition 2.1 (Sectorial operator)**.** Let A be a linear operator in X. Then A is called *sectorial* if

$$\exists\, \omega \in [0, \pi) : \quad \sigma(A) \subseteq S_\omega := \big\{\lambda \in \mathbb{C} \setminus \{0\}, |\arg z| \le \omega\big\} \cup \{0\}$$

and

$$\forall\, \omega' \in (\omega, \pi), \quad M_{\omega'} := \sup_{\lambda \in \mathbb{C} \setminus S_{\omega'}} \|\lambda R(\lambda, A)\| < \infty.$$

The minimum of all angles ω such that A is sectorial is called the *angle of sectoriality.*

Let us now define the fractional powers of A with the help of Balakrishnan representation.

Theorem 2.1 (Balakrishnan representation)**.** *If A is sectorial, then for all $x \in D(A)$ and for all α such that $0 < \operatorname{Re} \alpha < 1$,*

$$A^{\alpha} x = \frac{\sin \alpha\pi}{\pi} \int\limits_{0}^{+\infty} \lambda^{\alpha-1} (\lambda I + A)^{-1} A x \, d\lambda.$$

Now we recall some definitions and miscellaneous formulas.

Definition 2.2. The Gamma function is defined by

$$\Gamma(s) = \int\limits_{0}^{\infty} e^{-t} t^{s-1} \, dt,$$

where s is a complex number such that $\operatorname{Re}(s) > 0$. The Pochhammer symbol is defined by

$$(a)_0 := 1, \quad (a)_k := a(a+1), \ldots, (a+k-1), \quad a \in \mathbb{C}, \quad k \in \mathbb{N}.$$

This can be expressed by

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} \,.$$

The Pochhammer symbol satisfies the useful identities

$$\Gamma(a-k) = (-1)^k \, \frac{\Gamma(a)}{(1-a)_k} \tag{2.1}$$

and for $a = -n$ $(n \in \mathbb{N})$,

$$(-n)_k = \begin{cases} (-1)^k \, \dfrac{n!}{(n-k)!} & \text{for } \ 0 \le k \le n, \\ 0 & \text{for } \ k > n. \end{cases} \tag{2.2}$$

Definition 2.3. Let a, b and c be the complex numbers such that $c \notin \{0, -1, -2, \ldots\}$. The Gaussian hypergeometric function ${}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix} ; y \right)$ is defined by

$${}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix} ; y \right) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \, \frac{y^k}{k!} \,,$$

where y is a complex variable such that $|y| < 1$. If a or b is a non-positive integer, the series terminates and ${}_2F_1 \left(\begin{matrix} -n, b \\ c \end{matrix} ; y \right)$ is a polynomial of degree n in y.

The Gaussian hypergeometric function is related to the Meixner orthogonal polynomials in the following way (see [12, formula (15.9.9)]):

$$M_n(x; \beta, c) = {}_2F_1 \left(\begin{matrix} -n, -x \\ \beta \end{matrix} ; 1 - \frac{1}{c} \right). \tag{2.3}$$

Definition 2.4. Let a, b, c, d and e be the complex numbers such that $d, e \notin \{0, -1, -2, \dots\}$. The generalized hypergeometric function ${}_3F_2\left(\begin{matrix}a,b,c\\ d,e\end{matrix};y\right)$ is defined by

$$ {}_3F_2\left(\begin{matrix}a,b,c\\ d,e\end{matrix};y\right) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k(c)_k}{(d)_k(e)_k}\,\frac{y^k}{k!}\,, $$

where y is a complex variable such that $|y| < 1$. When one or more of the top parameters a, b, c is a non-positive integer, the series terminates and the generalized hypergeometric function is a polynomial in y.

Also, it is easy to verify the following identity:

$$ {}_2F_1\left(\begin{matrix}-n,b\\ c\end{matrix};y\right) = 1 - \frac{nby}{c}\;{}_3F_2\left(\begin{matrix}-n+1,b+1,1\\ c+1,2\end{matrix};y\right). \tag{2.4} $$

Definition 2.5 (see [3])**.** For arbitrary $\beta, t \in \mathbb{R}$ and $h > 0$, the h-rising factorial function is defined by

$$ t_h^{\overline{\beta}} = h^{\beta}\,\frac{\Gamma(\frac{t}{h}+\beta)}{\Gamma(\frac{t}{h})}\,. $$

Definition 2.6. The unilateral $\mathcal{Z}$-transform of a sequence $a[n]$, $n \geq 0$, is defined by

$$ \mathcal{Z}(a[n])(z) = \sum_{n=0}^{\infty} a[n] z^{-n}. $$

The $\mathcal{Z}$-transform is the discrete analogue of the classical Laplace transform.

3 Main results

The purpose of this theorem is to prove the sectoriality of T_h.

Theorem 3.1. *The operator T_h is sectorial of angle $\pi/2$.*

Proof. Let us compute the resolvent of T_h. Consider the discrete problem to find $g \in X_h$ from the equation

$$ \lambda g(t_n) - T_h g(t_n) = f(t_n). \tag{3.1} $$

Notice first that for $n = 0$, $g(t_0) = \frac{1}{\lambda} f(t_0)$. Despite of the triangular form of matrix $\lambda I - T_h$ for which the inverse can be computed directly, we propose another way using unilateral $\mathcal{Z}$-transforms.

Let $G(z)$ and $F(z)$ be the $\mathcal{Z}$-transforms of the sequences $g[n] = g(t_n)$ and $f[n] = f(t_n)$, respectively, completed by zeros, that is, $g[n] = 0$ for $n > N$. Then

$$ \mathcal{Z}\Big(\frac{h}{2}\sum_{i=1}^{n}\big[g(t_{i-1})+g(t_i)\big]\Big)(z) = \sum_{n=0}^{\infty}\frac{h}{2}\sum_{i=1}^{n}\big(g[i-1]+g[i]\big)z^{-n} = \frac{h}{2}\,\frac{1}{1-z^{-1}}\,\big[z^{-1}G(z)+G(z)-g[0]\big]. $$

It follows that for $\lambda \neq 0$, the $\mathcal{Z}$-transform of equation (3.1) becomes

$$ \Big[\lambda + \frac{h}{2}\,\frac{1+z^{-1}}{1-z^{-1}}\Big] G(z) = F(z) + \frac{h}{2\lambda}\,\frac{1}{1-z^{-1}}\,f[0]. $$

Taking $\lambda \in \mathbb{C} \setminus \{0, h/2\}$, we get

$$ G(z) = \Big[\frac{2}{2\lambda-h}\,\frac{1-z^{-1}}{1+\frac{2\lambda+h}{2\lambda-h}\,z^{-1}}\Big] F(z) - \frac{h}{\lambda(2\lambda-h)}\,\frac{1}{1-\frac{2\lambda+h}{2\lambda-h}\,z^{-1}}\,f[0]. $$

This implies that

$$G(z) = \Big(\frac{2}{2\lambda-h} + \frac{4h}{(2\lambda-h)^2}\,\frac{z^{-1}}{1-\frac{2\lambda+h}{2\lambda-h}\,z^{-1}}\Big)F(z) - \frac{hf[0]}{\lambda(2\lambda-h)}\,\frac{1}{1-\frac{2\lambda+h}{2\lambda-h}\,z^{-1}}.$$

Using geometric series and the definition of $F(z)$, we have

$$G(z) = \left(\frac{2}{2\lambda-h} + \frac{4h}{(2\lambda-h)^2}\sum_{k=0}^{\infty}\Big(\frac{2\lambda+h}{2\lambda-h}\Big)^k z^{-k-1}\right)\sum_{i=0}^{\infty}(f[i])z^{-i} - \frac{hf[0]}{\lambda(2\lambda-h)}\sum_{k=0}^{\infty}\Big(\frac{2\lambda+h}{2\lambda-h}\Big)^k z^{-k},$$

which in turn yields

$$g(t_n) = \frac{2}{2\lambda-h}\,f(t_n) + \frac{4h}{(2\lambda-h)^2}\sum_{i=0}^{n-1}\Big(\frac{2\lambda+h}{2\lambda-h}\Big)^{n-i-1} f(t_i) - \frac{h}{\lambda(2\lambda-h)}\,\Big(\frac{2\lambda+h}{2\lambda-h}\Big)^n f(t_0)$$

and

$$g(t_n) = \frac{2}{2\lambda-h}\,f(t_n) + \frac{4h}{(2\lambda-h)^2}\sum_{i=1}^{n-1}\Big(\frac{2\lambda+h}{2\lambda-h}\Big)^{n-i-1} f(t_i) + \frac{h(2\lambda+h)^{n-1}}{\lambda(2\lambda-h)^n}\,f(t_0).$$

In a matrix form, the resolvent is

$$R(\lambda, T_h) = \begin{pmatrix} \frac{1}{\lambda} & 0 & 0 & \cdots & & 0 \\ \frac{h}{\lambda(2\lambda-h)} & \frac{2}{2\lambda-h} & 0 & & & \\ \frac{h(2\lambda+h)}{\lambda(2\lambda-h)^2} & \frac{4h}{(2\lambda-h)^2} & \frac{2}{2\lambda-h} & \ddots & 0 & \vdots \\ \frac{h(2\lambda+h)^2}{\lambda(2\lambda-h)^3} & \frac{4h(2\lambda+h)}{(2\lambda-h)^3} & \frac{4h}{(2\lambda-h)^2} & \frac{2}{2\lambda-h} & & \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ \frac{h(2\lambda+h)^{N-1}}{\lambda(2\lambda-h)^N} & \frac{4h(2\lambda+h)^{N-2}}{(2\lambda-h)^N} & \cdots & \frac{4h(2\lambda+h)}{(2\lambda-h)^3} & \frac{4h}{(2\lambda-h)^2} & \frac{2}{2\lambda-h} \end{pmatrix}.$$

In order to estimate

$$\|\lambda R(\lambda, T_h)\|_\infty = \max_{0\le n\le N}\sum_{j=0}^{N}\big|\lambda(R(\lambda,T_h))_{nj}\big|,$$

let us take $\lambda \in \mathbb{C}\setminus S_{\omega_0}$ with $\frac{\pi}{2} < \omega_0 < \pi$.

Denote

$$|R(\zeta)|_n := \sum_{j=0}^{N}\big|\lambda(R(\lambda,T_h))_{nj}\big|$$

with $\zeta = \frac{2\lambda}{h}$. Then $|R(\zeta)|_0 = 1$ and for $0 < n \le N$,

$$|R(\zeta)|_n = \frac{|\zeta|}{|\zeta-1|} + \frac{2|\zeta|}{|\zeta-1|^2}\sum_{j=0}^{n-2}\Big|\frac{\zeta+1}{\zeta-1}\Big|^j + \frac{1}{|\zeta-1|}\,\Big|\frac{\zeta+1}{\zeta-1}\Big|^n.$$

Clearly,

$$|\zeta-1|^2 = 1+|\zeta|^2-2|\zeta|\cos(\arg\zeta) > 1+|\zeta|^2-2|\zeta|\cos(\omega_0)$$

and

$$\frac{1}{|\zeta-1|} < 1, \quad \frac{|\zeta|}{|\zeta-1|} < 1, \quad \Big|\frac{\zeta+1}{\zeta-1}\Big| < 1$$

which leads to

$$|R(\zeta)|_n < 2 + \frac{2|\zeta|}{|\zeta-1|^2}\,\frac{1}{1-|\frac{\zeta+1}{\zeta-1}|} < 2\Big(1+\frac{|\zeta|}{|\zeta-1|(|\zeta-1|-|\zeta+1|)}\Big).$$

Therefore, it remains to estimate an upper bound for the term $A(\zeta) = \frac{\zeta}{|\zeta-1|(|\zeta-1|-|\zeta+1|)}$. We have

$$A(\zeta) < \frac{|\zeta|}{\sqrt{1-2|\zeta|\cos\omega_0+|\zeta|^2}\,\big(\sqrt{1-2|\zeta|\cos\omega_0+|\zeta|^2}-\sqrt{1+2|\zeta|\cos\omega_0+|\zeta|^2}\big)}\,.$$

By using the conjugate, we get

$$A(\zeta) < -\frac{1}{4\cos\omega_0}\left(1+\sqrt{\frac{1+2|\zeta|\cos\omega_0+|\zeta|^2}{1-2|\zeta|\cos\omega_0+|\zeta|^2}}\right).$$

For every $\tau\in(0,+\infty)$, let φ be a function defined by

$$\varphi(\tau) = -\frac{1}{4\cos\omega_0}\left(1+\sqrt{\frac{1+2\tau\cos\omega_0+\tau^2}{1-2\tau\cos\omega_0+\tau^2}}\right),$$

then

$$\lim_{\tau\to+\infty}\varphi(\tau) = -\frac{1}{2\cos\omega_0} \quad \text{and} \quad \varphi(0) = -\frac{1}{2\cos\omega_0}\,.$$

The derivative of φ is

$$\varphi'(\tau) = \Big(\frac{1+2\tau\cos\omega_0+\tau^2}{1-2\tau\cos\omega_0+\tau^2}\Big)^{-1/2}\,\frac{\tau^2-1}{2(1-2\tau\cos\omega_0+\tau^2)^2}\,.$$

The function φ has a minimum at $\tau=1$ and we can assert that $\forall\,\tau\in(0,+\infty)$, $\varphi(\tau) < -\frac{1}{2\cos\omega_0}$.
Thus

$$A(\zeta) \le -\frac{1}{2\cos\omega_0}\,.$$

Consequently,

$$|R(\zeta)|_n < 2 - \frac{1}{\cos\omega_0}$$

which proves that T_h is sectorial with the angle $\frac{\pi}{2}$. □

We can now construct the fractional power of the trapezoidal operator.

Theorem 3.2. *For every $0<\alpha<1$ and for every $f\in X_h$,*

$$T_h^\alpha f(t_n) = \Big(\frac{h}{2}\Big)^\alpha \sum_{i=1}^{n} {}_2F_1\left(\begin{matrix}-n+i,1-\alpha\\ 1\end{matrix};2\right)\big[f(t_{i-1})+f(t_i)\big].$$

Proof. From Balakrishnan's formula, we have

$$T_h^\alpha f(t_n) = -\frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^\infty \lambda^{\alpha-1}R(-\lambda,T_h)T_hf(t_n)\,d\lambda.$$

From the definition of the operators T_h and $R(\lambda,T_h)$, we get

$$T_h^\alpha f(t_n) = J_1+J_2,$$

where

$$J_1 = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^\infty \lambda^{\alpha-1}\Big(\frac{h}{2\lambda+h}\sum_{j=1}^{n}\big[f(t_{j-1})+f(t_j)\big]\Big)\,d\lambda$$

and

$$J_2 = \frac{-1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^\infty \lambda^{\alpha-1} \Big(\frac{2h^2}{(2\lambda+h)^2} \sum_{j=1}^{n-1} \Big(\frac{2\lambda-h}{2\lambda+h}\Big)^{n-j-1} \sum_{i=1}^{j} \big[f(t_{i-1}) + f(t_i)\big] \Big)\, d\lambda.$$

By interchanging the summations and by expressing partial sums, we have

$$J_2 = \frac{-1}{\Gamma(\alpha)\Gamma(1-\alpha)} \sum_{i=1}^{n-1} \big[f(t_{i-1}) + f(t_i)\big] \Big(\int_0^\infty \lambda^{\alpha-1} \frac{h}{2\lambda+h}\, d\lambda \Big)$$
$$+ \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \sum_{i=1}^{n-1} \big[f(t_{i-1}) + f(t_i)\big] \Big(\int_0^\infty \lambda^{\alpha-1} \frac{h}{2\lambda+h} \Big(\frac{2\lambda-h}{2\lambda+h}\Big)^{n-i} d\lambda \Big).$$

It follows that

$$T_h^\alpha f(t_n) = \sum_{i=1}^{n} \big[f(t_{i-1}) + f(t_i)\big] \Big(\frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^\infty \lambda^{\alpha-1} \frac{h}{2\lambda+h} \Big(\frac{2\lambda-h}{2\lambda+h}\Big)^{n-i} d\lambda \Big).$$

It all comes down to calculating the integral

$$J = \frac{h}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^\infty \lambda^{\alpha-1} (2\lambda-h)^{n-i} (2\lambda+h)^{-n+i-1}\, d\lambda.$$

By using again the variable $\zeta = \frac{2\lambda}{h}$, the integral J becomes

$$J = \Big(\frac{h}{2}\Big)^\alpha \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^\infty \zeta^{\alpha-1} (\zeta-1)^{n-i} (\zeta+1)^{-n+i-1}\, d\zeta.$$

As

$$\int_0^\infty \zeta^{\alpha-1} (\zeta-1)^{n-i} (\zeta+1)^{-n+i-1}\, d\zeta = \int_0^\infty \frac{\zeta^{\alpha-1}}{\zeta+1} \Big(1 - \frac{2}{\zeta+1}\Big)^{n-i} d\zeta,$$

the binomial formula implies

$$\int_0^\infty \zeta^{\alpha-1} (\zeta-1)^{n-i} (\zeta+1)^{-n+i-1}\, d\zeta$$
$$= \int_0^\infty \frac{\zeta^{\alpha-1}}{\zeta+1} \sum_{k=0}^{n-i} \binom{n-i}{k} \Big(-\frac{2}{\zeta+1}\Big)^k d\zeta = \sum_{k=0}^{n-i} \binom{n-i}{k} (-2)^k \int_0^\infty \zeta^{\alpha-1} \Big(\frac{1}{\zeta+1}\Big)^{k+1} d\zeta.$$

Using Euler's beta integral [12, formula (5.12.1)], we have

$$\int_0^\infty \zeta^{\alpha-1} \Big(\frac{1}{\zeta+1}\Big)^{k+1} d\zeta = B(\alpha, k-\alpha+1) = \frac{\Gamma(\alpha)\Gamma(k-\alpha+1)}{k!}$$

and then

$$J = \Big(\frac{h}{2}\Big)^\alpha \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n-i} (-1)^k \frac{(n-i)!}{k!(n-i)!} \frac{\Gamma(k-\alpha+1)}{k!} 2^k.$$

Now, according to (2.2), we can write

$$J = \Big(\frac{h}{2}\Big)^\alpha \sum_{k=0}^{n-i} \frac{(-n+i)_k (1-\alpha)_k}{(1)_k} \frac{(2)^k}{k!} = \Big(\frac{h}{2}\Big)^\alpha {}_2F_1 \left(\begin{matrix} -n+i, 1-\alpha \\ 1 \end{matrix} ; 2 \right).$$

We can therefore formulate

$$T_h^\alpha f(t_n) = \left(\frac{h}{2}\right)^\alpha \sum_{i=1}^{n} {}_2F_1\left(\begin{matrix}-n+i, 1-\alpha\\ 1\end{matrix}; 2\right)\left[f(t_{i-1}) + f(t_i)\right].$$

Thanks to the formula of contiguous functions (see [12, formula 15.5.16]), the operator T_h^α is written in its expanded form:

$$\begin{aligned} T_h^\alpha f(t_n) &= \left(\frac{h}{2}\right)^\alpha {}_2F_1\left(\begin{matrix}-n+1, 1-\alpha\\ 1\end{matrix}; 2\right) f(t_0) \\ &\quad + 2\alpha\left(\frac{h}{2}\right)^\alpha \sum_{i=1}^{n-1} {}_2F_1\left(\begin{matrix}-n+i+1, 1-\alpha\\ 2\end{matrix}; 2\right) f(t_i) + \left(\frac{h}{2}\right)^\alpha f(t_n). \end{aligned} \tag{3.2}$$

The theorem is proved. $\square$

Remark 3.1. In the matrix form, the fractional power of the matrix T_h is $T_h^\alpha = ((T_h^\alpha)_{n,j})_{0\le n,j\le N}$, where

$$(T_h^\alpha)_{i,j} = \left(\frac{h}{2}\right)^\alpha \times \begin{cases} 1 & \text{if } j = n \text{ and } n \neq 0, \\ {}_2F_1\left(\begin{matrix}-n+1, 1-\alpha\\ 1\end{matrix}; 2\right) & \text{if } j = 0 \text{ and } 1 \le n \le N, \\ 2\alpha\ {}_2F_1\left(\begin{matrix}-n+1+j, 1-\alpha\\ 2\end{matrix}; 2\right) & \text{if } 1 \le j < n \le N, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 3.2. Obviously, we obtain the spectral mapping theorem

$$\sigma(T_h^\alpha) = \left\{\lambda^\alpha,\ \lambda \in \sigma(T_h)\right\}.$$

Unlike the classic discrete fractional calculus where the laws of exponents must be proved, we get the first and second laws of the exponent as an immediate consequence of the functional calculus.

Theorem 3.3. *Let $\alpha, \beta \in]0,1[$. Then*

1. $$T_h^{\alpha+\beta} = T_h^\alpha T_h^\beta. \tag{3.3}$$

2. $$(T_h^\alpha)^\beta = T_h^{\alpha\beta}. \tag{3.4}$$

Result (3.3) is an immediate consequence of [6, Proposition 3.1.1 c)] and result (3.4) arises from [6, Corollary 3.1.5].

4 Applications

For the first application of fractional power T_h^α, we prove the following result.

Theorem 4.1. *For $0 < \alpha < 1$ and $\beta > 0$, we have*

$$T_h^\alpha((t_n)_h^{\overline{\beta}}) = \left(\frac{h}{2}\right)^\alpha {}_2F_1\left(\begin{matrix}-n+1, -\alpha\\ 1+\beta\end{matrix}; 2\right)(t_n)_h^{\overline{\beta}}.$$

Proof. By formula (3.2), we get

$$T_h^\alpha((t_n)_h^{\overline{\beta}}) = 2\alpha\left(\frac{h}{2}\right)^\alpha \sum_{i=1}^{n-1} {}_2F_1\left(\begin{matrix}-n+i+1, 1-\alpha\\ 2\end{matrix}; 2\right)((t_i)_h^{\overline{\beta}}) + \left(\frac{h}{2}\right)^\alpha ((t_n)_h^{\overline{\beta}}).$$

Since

$$\sum_{i=1}^{n-1} {}_2F_1\left({-n+i+1, 1-\alpha \atop 2}; 2\right)((t_i)_h^{\overline{\beta}}) = h^\beta \sum_{i=1}^{n-1}\sum_{k=0}^{n-i-1}(-1)^k \frac{\Gamma(n-i)}{\Gamma(n-i-k)} \frac{(1-\alpha)_k}{(2)_k} \frac{(2)^k}{k!} \frac{\Gamma(i+\beta)}{\Gamma(i)},$$

by interchanging order of summation, we get

$$\begin{aligned}\sum_{i=1}^{n-1} {}_2F_1\left({-n+i+1, 1-\alpha \atop 2}; 2\right)((t_i)_h^{\overline{\beta}}) &= h^\beta \sum_{k=0}^{n-2}(-1)^k \frac{(1-\alpha)_k}{(2)_k} \frac{(2)^k}{k!} \sum_{i=1}^{n-k-1} \frac{\Gamma(n-i)}{\Gamma(n-i-k)} \frac{\Gamma(i+\beta)}{\Gamma(i)} \\ &= h^\beta \sum_{k=0}^{n-2}(-1)^k \frac{(1-\alpha)_k}{(2)_k} \frac{(2)^k}{k!} \Gamma(k+1)\Gamma(\beta+1) \sum_{i=0}^{n-k-2} \frac{(k+1)_{n-k-2-i}}{\Gamma(n-k-2-i)!} \frac{(\beta+1)_i}{(i)!}.\end{aligned}$$

Using the Chu–Vandermonde identity [2, Remark 2.2.1]

$$\sum_{k=0}^{n} \frac{(a)_k (b)_{n-k}}{k!(n-k)!} = \frac{(a+b)_n}{n!}$$

and (2.1), we get

$$\begin{aligned}\sum_{i=1}^{n-1} {}_2F_1\left({-n+i+1, 1-\alpha \atop 2}; 2\right)((t_n)_h^{\overline{\beta}}) &= \Gamma(\beta+1) h^\beta \sum_{k=0}^{n-2}(-1)^k \frac{(1-\alpha)_k}{(2)_k} \frac{(2)^k}{k!} \Gamma(k+1) \frac{(\beta+k+2)_{n-k-2}}{(n-k-2)!} \\ &= \frac{h^\beta}{\beta+1} \frac{\Gamma(\beta+n)}{\Gamma(n-1)} \sum_{k=0}^{n-2} \frac{(-n+2)_k (1-\alpha)_k (1)_k}{(2)_k (\beta+2)_k} \frac{(2)^k}{k!},\end{aligned}$$

which leads to

$$T_h^\alpha((t_n)_h^{\overline{\beta}}) = h^{\beta+\alpha} \frac{\Gamma(n+\beta)}{\Gamma(n)} \left(\frac{1}{2^\alpha} + \frac{2\alpha}{2^\alpha} \frac{n-1}{\beta+1} \sum_{k=0}^{n-2} \frac{(-n+2)_k (1-\alpha)_k (1)_k}{(2)_k (\beta+2)_k} \frac{(2)^k}{k!}\right).$$

Depending on the function ${}_3F_2$, $T_h^\alpha((t_n)_h^{\overline{\beta}})$ is written as

$$T_h^\alpha((t_n)_h^{\overline{\beta}}) = h^{\beta+\alpha} \frac{\Gamma(n+\beta)}{\Gamma(n)} \left(\frac{1}{2^\alpha} + \frac{2\alpha}{2^\alpha} \frac{n-1}{\beta+1}\, {}_3F_2\left({-n+2, 1-\alpha, 1 \atop 2, 2+\beta}; 2\right)\right).$$

From the identity (2.4), we obtain

$${}_3F_2\left({-n+2, 1-\alpha, 1 \atop 2, 2+\beta}; 2\right) = \frac{\beta+1}{2\alpha(n-1)} \left({}_2F_1\left({-n+1, -\alpha \atop 1+\beta}; 2\right) - 1\right).$$

Consequently, we have

$$T_h^\alpha((t_n)_h^{\overline{\beta}}) = h^{\beta+\alpha} \frac{\Gamma(n+\beta)}{\Gamma(n)} \frac{1}{2^\alpha}\, {}_2F_1\left({-n+1, -\alpha \atop 1+\beta}; 2\right),$$

whence the theorem follows. □

An interesting consequence may be extracted from the above theorem.

For $\beta = 1$, we have

$$T_h^{\alpha}((t_n)_h^{\overline{1}}) = h^{1+\alpha}\,\frac{n}{2^{\alpha}}\;{}_2F_1\left(\begin{matrix}-n+1,-\alpha\\ 2\end{matrix};2\right),$$

which implies

$$T_h^{\mu}T_h^{\alpha}\big((t_n)_h^{\overline{1}}\big) = T^{\mu}\left(h^{1+\alpha}\,\frac{n}{2^{\alpha}}\;{}_2F_1\left(\begin{matrix}-n+1,-\alpha\\ 2\end{matrix};2\right)\right).$$

Using formula (3.2), we get

$$\begin{aligned}
T^{\mu}&\left(h^{1+\alpha}\frac{n}{2^{\alpha}}\;{}_2F_1\left(\begin{matrix}-n+1,-\alpha\\ 2\end{matrix};2\right)\right)\\
&= 2\mu\Big(\frac{h}{2}\Big)^{\mu}\,\frac{h^{1+\alpha}}{2^{\alpha}}\sum_{i=1}^{n-1} i\;{}_2F_1\left(\begin{matrix}-n+i+1,1-\mu\\ 2\end{matrix};2\right)\;{}_2F_1\left(\begin{matrix}-i+1,-\alpha\\ 2\end{matrix};2\right)\\
&\qquad + \Big(\frac{h}{2}\Big)^{\mu}\,h^{1+\alpha}\frac{n}{2^{\alpha}}\;{}_2F_1\left(\begin{matrix}-n+1,-\alpha\\ 2\end{matrix};2\right)\\
&= h\Big(\frac{h}{2}\Big)^{\mu+\alpha}\left(2\mu\sum_{i=1}^{n-1} i\;{}_2F_1\left(\begin{matrix}-n+i+1,1-\mu\\ 2\end{matrix};2\right)\;{}_2F_1\left(\begin{matrix}-i+1,-\alpha\\ 2\end{matrix};2\right) + n\;{}_2F_1\left(;2\begin{matrix}-n+1,-\alpha\\ 2\end{matrix};2\right)\right).
\end{aligned}$$

On the other hand,

$$T_h^{\alpha+\mu}((t_n)_h^{\overline{1}}) = h^{1+\alpha+\mu}\,\frac{n}{2^{\alpha+\mu}}\;{}_2F_1\left(\begin{matrix}-n+1,-\alpha-\mu\\ 2\end{matrix};2\right).$$

Putting the equalities together, we get

$$\begin{aligned}
\sum_{i=1}^{n-1} i\;{}_2F_1&\left(\begin{matrix}-n+i+1,1-\mu\\ 2\end{matrix};2\right)\;{}_2F_1\left(\begin{matrix}-i+1,-\alpha\\ 2\end{matrix};2\right)\\
&= \frac{n}{2\mu}\left({}_2F_1\left(\begin{matrix}-n+1,-\alpha-\mu\\ 2\end{matrix};2\right) - {}_2F_1\left(\begin{matrix}-n+1,-\alpha\\ 2\end{matrix};2\right)\right).
\end{aligned}$$

We have then proved the following finite summation formula of the product of Gauss hypergeometric functions. For every $0 < a, b < 1$,

$$\begin{aligned}
\sum_{i=1}^{n} i\;{}_2F_1&\left(\begin{matrix}-n+i,a\\ 2\end{matrix};2\right)\;{}_2F_1\left(\begin{matrix}-i+1,-b\\ 2\end{matrix};2\right)\\
&= \frac{n+1}{2(1-a)}\left({}_2F_1\left(\begin{matrix}-n,a-b-1\\ 2\end{matrix};2\right) - {}_2F_1\left(;2\begin{matrix}-n,-b\\ 2\end{matrix};2\right)\right),
\end{aligned}$$

which, to our knowledge, seems to be unknown. Expressed with the Meixner polynomials, the above formula becomes

$$\sum_{i=1}^{n} iM_{n-i}(-a;2,-1)M_{i-1}(b;2,-1) = \frac{n+1}{2(1-a)}\big(M_n(-a+b+1;2,1) - M_n(b;2,-1)\big). \tag{4.1}$$

5 Conclusion

In this paper, we proposed a new fractional h-discrete sum operator. On the one hand, we obtained the fractional power of a particular lower triangular matrix. On the other hand, we found a finite summation formula of a product of two hypergeometric polynomials. Two open problems arise naturally. First: can T_h^{α} be used as an approximation formula for Riemann-Liouville operator? This will be the fractional version of the well known result about the approximation of $\int_a^b f(x)\,dx$ by T_h. The second problem concerns the meaning of formula (4.1). Is there any relation with the general theory of orthogonal polynomials?

Acknowledgments

The authors are grateful to the reviewer for his valuable comments and remarks which contributed to improve the quality of the paper.

References

[1] T. Abdeljawad, Different type kernel h-fractional differences and their fractional h-sums. *Chaos Solitons Fractals* **116** (2018), 146–156.

[2] G. E. Andrews, R. Askey and R. Roy, *Special Functions*. Encyclopedia of Mathematics and its Applications, 71. Cambridge University Press, Cambridge, 1999.

[3] F. M. Atıcı, K. Dadashova and J. M. Jonnalagadda, Linear fractional-order h-difference equations. *Int. J. Difference Equ.* **15** (2020), no. 2, 281–300.

[4] F. M. Atıcı, N. Nguyen, K. Dadashova, S. E. Pedersen and G. Koch, Pharmacokinetics and pharmacodynamics models of tumor growth and anticancer effects in discrete time. *Comput. Math. Biophys.* **8** (2020), 114–125.

[5] R. A. C. Ferreira and D. F. M. Torres, Fractional h-difference equations arising from the calculus of variations. *Appl. Anal. Discrete Math.* **5** (2011), no. 1, 110–121.

[6] M. Haase, *The Functional Calculus for Sectorial Operators*. Operator Theory: Advances and Applications, 169. Birkhäuser Verlag, Basel, 2006.

[7] N. J. Higham, *Functions of Matrices. Theory and Computation*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2008.

[8] N. J. Higham and L. Lin,A Schur–Padé algorithm for fractional powers of a matrix. *SIAM J. Matrix Anal. Appl.* **32** (2011), no. 3, 1056–1078.

[9] L. Khitri-Kazi-Tani and H. Dib, On the approximation of Riemann-Liouville integral by fractional nabla h-sum and applications. *Mediterr. J. Math.* **14** (2017), no. 2, Paper No. 86, 21 pp.

[10] X. Liu, F. Du, D. Anderson and B. Jia, Monotonicity results for nabla fractional h-difference operators. *Math. Methods Appl. Sci.* **44** (2021), no. 2, 1207–1218.

[11] D. Mozyrska, E. Girejko and M. Wyrwas, Comparison of h-difference fractional operators. *Advances in the theory and applications of non-integer order systems*, 191–197, Lect. Notes Electr. Eng., 257, *Springer, Cham*, 2013.

[12] F. W. J. Olver D. W. Lozier, R. F. Boisvert, Ch. W. Clark, (Eds.), NIST handbook of mathematical functions. Cambridge University Press, Cambridge, 2010.

[13] I. Suwan, S. Owies and T. Abdeljawad, Monotonicity results for h-discrete fractional operators and application. *Adv. Difference Equ.* **2018**, Paper No. 207, 17 pp.

[14] F. Tatsuoka, T. Sogabe, Y. Miyatake and S.-L. Zhang, A note on computing the matrix fractional power using the double exponential formula. *Transactions of the Japan Society for Industrial and Applied Mathematics (Web)* **28** (2018), no. 3, 142–161.

(Received 05.06.2021; revised 14.07.2021; accepted 03.08.2021)

Authors' addresses:

Leila Khitri-Kazi-Tani

Laboratoire de Statistiques et modélisation aléatoire, Department of Mathematics, Abou Bekr Belkaid University, Tlemcen, Algeria.

E-mail: `kazitani.leila13@gmail.com`

Hacen Dib

Laboratoire de Statistiques et modélisation aléatoire, Department of Mathematics, Abou Bekr Belkaid University, Tlemcen, Algeria.

E-mail: `hacen.dib13@gmail.com`