Memoirs on Differential Equations and Mathematical Physics

Volume 87, 2022, 111–118

Mikhail V. Mulyukov

NECESSARY CONDITIONS OF THE STABILITY OF ONE HYBRID SYSTEM

Dedicated to the blessed memory of Professor N. V. Azbelev

Abstract. One linear autonomous discrete-continuous system (also known as hybrid system) with uncertain coefficients is considered. The continuous component is described by a delay differential equation. The necessary conditions of asymptotic stability for this system are obtained.

2020 Mathematics Subject Classification. 34K06, 34K20, 34K25, 34K34.

Key words and phrases. Hybrid systems, discrete-continuous systems, asymptotic stability, delay differential equations.

რეზიუმე. განხილულია ერთი წრფივი ავტონომიური დისკრეტული უწყვეტი სისტემა (ასევე ცნობილი როგორც ჰიბრიდული სისტემა) განუსაზღვრელი კოეფიციენტებით. უწყვეტი კომპონენტი აღწერილია დაგვიანებული დიფერენციალური განტოლებით. ამ სისტემისთვის მიღებულია ასიმპტოტური სტაბილურობის აუცილებელი პირობები.

1 Introduction

A continuous-discrete system of functional-differential equations is a system that contains both continuous and discrete components. Such systems are also called *hybrid*.

Various works are devoted to the stability of hybrid systems. The Lyapunov-based approach is applied in [4–6]. Application of the fixed point principle is considered in [3]. Azbelev's W-method is applied in [2, 9].

Hybrid systems can be divided into two classes. The first class includes systems whose asymptotic properties are determined by some auxiliary difference (finite-dimensional) system. In particular, this class includes such hybrid systems, whose continuous part satisfies ordinary differential equations. As a rule, effective coefficient criteria are obtained for asymptotic stability of such hybrid systems (see [7, 8, 10]).

The second class includes hybrid systems, whose continuous part satisfies a delay differential equation. The coefficient conditions of stability for such systems are investigated not enough. At the same time, one can expect to obtain the necessary and sufficient stability conditions for linear autonomous hybrid systems from this class.

Consider the differential equation

$$\dot{x}(t) + \alpha x(t-h) + \beta x\Big(\Big[\frac{t}{h}\Big]\Big) = 0, \quad t \in \mathbb{R}_+,$$

where $[\,\cdot\,]$ is the integer part of the number, $\mathbb{R}_+ = [0; +\infty)$, $h > 0$, $\alpha, \beta \in \mathbb{R}$.

Without loss of generality, we can assume that $h = 1$;

$$\dot{x}(t) + ax(t-1) + bx\big([t]\big) = 0, \quad t \in \mathbb{R}_+, \tag{1.1}$$

where $a = \alpha h$, $b = \beta h$.

Equation (1.1) can be rewritten in the equivalent form

$$\begin{cases} \dot{x}(t) + ax(t-1) = -by(n), & t \in [n, n+1), \\ y(n+1) = x(n), \end{cases} \qquad n \in \mathbb{N}_0.$$

One can see that the first equation is a delay differential equation. The stability criterion for this system is unknown. In this paper, we consider the necessary conditions of asymptotic stability for equation (1.1).

2 Main result

We say that the Cauchy problem for equation (1.1) is posed if the equation is considered with the initial data $x(0) \in \mathbb{R}$ and the initial function ψ such that

$$x(t) = \psi(t) \ \text{ if } \ t \in [-1, 0).$$

It is assumed that the function ψ is summable.

The solution of the Cauchy problem exists and is unique in the space of locally absolutely continuous functions [1].

Definition 2.1. Equation (1.1) is called *asymptotically stable* if $\lim\limits_{t\to\infty} \|x(t)\| = 0$ for any initial data $x(0) \in \mathbb{R}$ and any summable function ψ.

Denote by a^* the unique positive root of the equation $e^a = 2a + 1$.

Theorem 2.1. *Equation* (1.1) *is asymptotically stable only if $a + b > 0$ and one of the following conditions holds:*

- $b < a \coth \frac{a}{2}$ *and* $a \le a^*$,
- $a < e^{b-1} - b$ *and* $a > a^*$.

Proof. Denote by x_n the solution of equation (1.1) in the interval $[n, n+1)$. We have

$$\dot{x}_n(t) + ax_{n-1}(t-1) + bx(n) = 0, \quad t \in [n, n+1). \tag{2.1}$$

Suppose $a = 0$. Then

$$x_n(t) = x(n)(1 - b(t-n)).$$

Hence

$$x(n+1) = x(n)(1-b) \ \text{ and } \ x(n) = x(0)(1-b)^n.$$

Thus, equation (1.1) is asymptotically stable if and only if $b \in (0, 2)$.
Suppose $a \neq 0$.
Suppose that $\psi(t) = e^{\lambda t}$ and

$$x_n(t) = \frac{e^{\lambda(t-n)} - 1}{e^{\lambda} - 1}\, x(n+1) + \frac{1 - e^{\lambda(t-n-1)}}{1 - e^{-\lambda}}\, x(n). \tag{2.2}$$

One can see that

$$x_n(n) = x(n) \ \text{ and } \ x_n(n+1) = x(n+1).$$

Hence

$$\dot{x}_n(t) = \frac{\lambda e^{\lambda(t-n)}}{e^{\lambda} - 1}\, x(n+1) - \frac{\lambda e^{\lambda(t-n-1)}}{1 - e^{-\lambda}}\, x(n) \tag{2.3}$$

and

$$x_{n-1}(t-1) = \frac{e^{\lambda(t-n)} - 1}{e^{\lambda} - 1}\, x(n) + \frac{1 - e^{\lambda(t-n-1)}}{1 - e^{-\lambda}}\, x(n-1). \tag{2.4}$$

To find a value of λ, we substitude (2.3) and (2.4) into (2.1):

$$\frac{e^{\lambda(t-n)}}{e^{\lambda} - 1}\big(\lambda x(n+1) - \lambda x(n) - ax(n-1) + ax(n)\big) + \frac{ae^{\lambda}x(n-1) - ax(n) + bx(n)(e^{\lambda} - 1)}{e^{\lambda} - 1} = 0.$$

This equality holds for any t if and only if λ satisfies the following system:

$$\begin{cases} \lambda x(n+1) - \lambda x(n) - ax(n-1) + ax(n) = 0, \\ ae^{\lambda}x(n-1) - ax(n) + bx(n)(e^{\lambda} - 1) = 0. \end{cases} \tag{2.5}$$

Denote

$$\omega = \frac{ae^{\lambda}}{a + b(1 - e^{\lambda})}\,. \tag{2.6}$$

It follows from the second equation of system (2.5) that

$$x(n+1) = \omega x(n).$$

Hence from the first equation of system (2.5) it follows that

$$\lambda(\omega x(n) - x(n)) = a\Big(x(n) - \frac{x(n)}{\omega}\Big).$$

This equality holds if $\omega\lambda + a = 0$. We substitude (2.6) into the latter equation:

$$\frac{ae^{\lambda}}{a + b(1 - e^{\lambda})}\,\lambda + a = 0.$$

We transform the last equation as

$$(b - \lambda)e^{\lambda} = a + b. \tag{2.7}$$

So, the solution of equation (1.1) has form (2.2) if λ satisfies equation (2.7).

It follows from equation (2.6) that if $|\omega| \geq 1$, then

$$|x(n+1)| \geq |x(n)|.$$

The inequality $|\omega| \geq 1$ is equivalent to the inequality $|\lambda| \leq |a|$. Thus equation (1.1) is asymptotically stable only if $|\lambda| > |a|$ for any root of equation (2.7).

The study of the case $\lambda \in \mathbb{C}$ is a difficult problem. In this paper, we consider only the case $\lambda \in \mathbb{R}$.

Consider the function

$$f(\lambda) = (b-\lambda)e^{\lambda}.$$

This function has one extremum at the point $\lambda = b-1$. The maximum value is

$$f(b-1) = e^{b-1}.$$

We have

$$\lim_{\lambda\to-\infty} f(\lambda) = 0, \qquad \lim_{\lambda\to+\infty} f(\lambda) = -\infty.$$

Thus, equation (1.1) has

- no real solutions if $a > e^{b-1} - b$,
- one real solution $\lambda = b-1$ if
$$a = e^{b-1} - b, \tag{2.8}$$
- two real solutions λ_1, λ_2 such that $\lambda_1 \in (-\infty, b-1)$, $\lambda_2 \in (b-1, b)$ if $a \in (-b, e^{b-1} - b)$,
- one real solution $\lambda = b$ if $a = -b$,
- one real solution $\lambda > b$ if $a < -b$.

Equation (2.7) has the root $\lambda = -a$ if and only if

$$a + b = 0 \tag{2.9}$$

or

$$a = 0. \tag{2.10}$$

Equation (2.7) has the root $\lambda = a$ if and only if

$$b = a \coth\frac{a}{2}. \tag{2.11}$$

This function is defined by the continuity at zero.

Consider the point $P^*(a^*, a^*+1)$. If we substitute $a = a^*$ and $b = a^*+1$ into (2.8) or (2.11), we obtain

$$a = e^{b-1} - b = b - 1,$$
$$b = a\coth\frac{a}{2} = a + 1.$$

If we substitute $a = a^*$ and $b = a^*+1$ into the derivatives of functions (2.8) or (2.11), we respectively obtain

$$\left.\frac{\partial a}{\partial b}\right|_{\substack{a=a^*\\ b=b^*}} = e^{a^*} - 1 = 2a^*,$$
$$\left.\frac{\partial b}{\partial a}\right|_{\substack{a=a^*\\ b=b^*}} = \frac{e^{2a^*} - 1 - 2a^*e^{a^*}}{(e^{a^*}-1)^2} = \frac{1}{2a^*}\,.$$

The functions (2.8), (2.11) are convex. Hence the graphs of these functions are touching at the point P^*.

Obviously, the roots of equation (2.7) depend on a, b continuously (except the line $a+b=0$). Lines (2.8)–(2.11) divide the plane Oab into 9 domains (see Fig. 1). If two points belong to one domain,

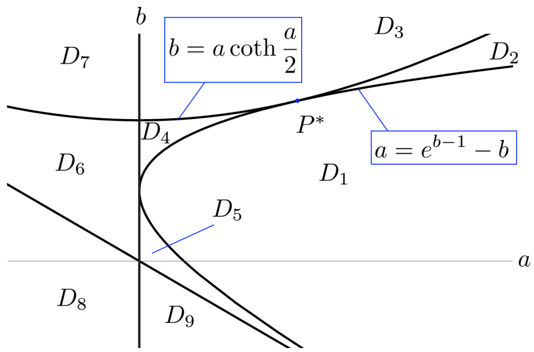

Figure 1. Domains D_1–D_9.

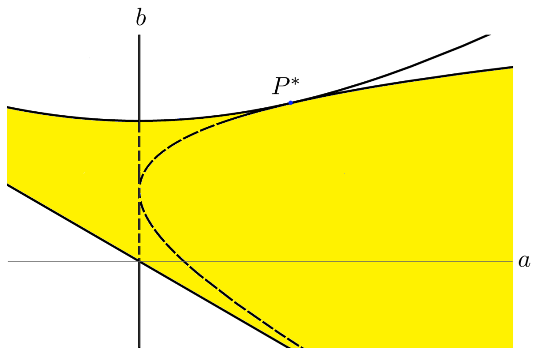

Figure 2. Necessary conditions of asymptotic stability for (1.1).

then the corresponding equations have the same number of real roots in the interval $(-|a|, |a|)$. We call this number the *index* of a domain.

The index of the domain D_1 is equal to zero because equation (2.7) has no real roots at all.

The index of the domain D_2 is equal to 2 because this domain contains the point $a = b$, where a is positive and sufficiently large. Equation (2.7) can be written in the form $2ae^{\lambda} = a - \lambda$. One can see that this equation has two real roots in the interval $(0, a)$.

On the curve line (2.11), we have

$$\frac{\partial \lambda}{\partial b} = \frac{1 - e^{-\lambda}}{1 + \lambda - b}\bigg|_{\substack{\lambda = a \\ b = a \coth \frac{a}{2}}} = e^{-a}\,\frac{(e^a - 1)^2}{e^a - 2a - 1}\,.$$

Thus,

$$\operatorname{sgn} \frac{\partial \lambda}{\partial b} = \operatorname{sgn}(a - a^*).$$

Hence if the point (a, b) crosses line (2.11) along the axis b to the right of the point P^*, then one real positive root of equation (2.7) leaves interval $(-a, a)$ throw the point $\lambda = a$. And if the point (a, b) crosses line (2.11) along the axis b to the left of the point P^*, then one real positive root of equation (2.7) moves inward to the interval $(-a, a)$ throw the point $\lambda = a$. Hence the index of the domain D_3 is equal to 1 and the index of the domain D_4 is equal to 0.

The domain D_5 contains the point $(a, 0)$ such that a is positive and sufficiently small. Equation (2.7) can be written in the form $e^{-\lambda} = -\lambda/a$. One can see that this equation has two real roots in the half-interval $(-\infty, -a)$. Thus, the index of this domain is equal to 0.

The domain D_6 contains the point $(-1, 2)$. Equation (2.7) can be written in the form $e^{-\lambda} = 2 - \lambda$. One can see that there are no roots of this equation in the interval $(-1, 1)$. Hence the index of this domain is equal to 0 and the index of the domain D_7 is equal to 1.

The points $(-1, 0)$ and $(1, -2)$ belong to the domains D_8 and D_9. Equation (2.7) can be written in the form $e^{-\lambda} = \lambda$ and $e^{-\lambda} = \lambda + 2$, respectively. The unique real root of each equation belongs to the interval $(0, 1)$. Thus, the indices of these domains are equal to 1.

Above, we have showed that on the line (2.8) there is a unique root $\lambda = b - 1$. This root belongs to the interval $(-a, a)$ if and only if

$$b - 1 < e^{b-1} - b.$$

This inequality holds if and only if $b < b^*$.

Finally, equation (1.1) is asymptotically stable only if the point (a, b) belongs to the domains D_1, D_4, D_5, D_6 or border between them (yellow domain on Fig. 2). On dotted lines, the asymptotic stability is possible. On solid lines, the asymptotic stability is impossible. □

Conclusion

We have obtained the necessary conditions of asymptotic stability for (1.1) by investigating the location of roots of quasipolynomial (2.7). In this paper, we have considered only the real roots. To improve the results, the complex roots of this quasipolynomial should be considered.

Acknowledgments

This work is supported by the Russian Science Foundation, Project # 22–21–00517.

References

[1] N. Azbelev, V. Maksimov and L. Rakhmatullina, *Introduction to the Theory of Linear Functional-Differential Equations.* Advanced Series in Mathematical Science and Engineering, 3. World Federation Publishers Company, Atlanta, GA, 1995.

[2] E. Bravyi, V. Maksimov and P. Simonov, Some economic dynamics problems for hybrid models with aftereffect. *Mathematics* **2020**, no. 8(10), 1832.

[3] M. De la Sen, Total stability properties based on fixed point theory for a class of hybrid dynamic systems. *Fixed Point Theory Appl.* **2009**, Art. ID 826438, 19 pp.

[4] E. Fridman, Stability of linear descriptor systems with delay: a Lyapunov-based approach. *J. Math. Anal. Appl.* **273** (2002), no. 1, 24–44.

[5] R. I. Kozlov and O. R. Kozlova, Investigation of the stability of nonlinear continuous-discrete models of economic dynamics by the method of vector Lyapunov functions. I. (Russian) *Izv. Ross. Akad. Nauk Teor. Sist. Upr.* **2009**, no. 2, 104–113; translation in *J. Comput. Syst. Sci. Int.* **48** (2009), no. 2, 262–271.

[6] R. I. Kozlov and O. R. Kozlova, Investigation of the stability of nonlinear continuous-discrete models of economic dynamics by the method of vector Lyapunov functions. II. (Russian) *Izv. Ross. Akad. Nauk Teor. Sist. Upr.* **2009**, no. 3, 41–50; translation in *J. Comput. Syst. Sci. Int.* **48** (2009), no. 3, 370–378.

[7] V. M. Marchenko and Zh. Zh. Luazo, On the stability of hybrid differential-difference systems. (Russian) *Differ. Uravn.* **45** (2009), no. 5, 728–740; translation in *Differ. Equ.* **45** (2009), no. 5, 743–756.

[8] B. G. Marchenko and L. N. Shcherbak, Continuous and impulse two-dimensional linear random fields. (Russian) *Otbor i Obrabotka Informatsii*, No. 1(77) (1988), 7–12.

[9] I. V. Simonov, On the stability of motion of an elongated body of revolution in an elastoplastic medium with flow separation. (Russian) *Prikl. Mat. Mekh.* **64** (2000), no. 2, 311–320; translation in *J. Appl. Math. Mech.* **64** (2000), no. 2, 299–307.

[10] H. Ye, A. N. Michel and L. Hou, Stability theory for hybrid dynamical systems. Hybrid control systems. *IEEE Trans. Automat. Control* **43** (1998), no. 4, 461–474.

(Received 15.08.2022; accepted 01.09.2022)

Author's addresses:

Mikhail V. Mulyukov

1. Department of Information System and Mathematical Methods in Economics, Perm State National Research University, 15 Bukirev Str., Perm 614990, Russia.

2. Department of Computational Mathematics, Mechanics and Biomechanics, Perm National Research Polytechnic University, 29 Komsomolskiy Ave., Perm 614990, Russia.

E-mail: mulykoff@gmail.com