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ON A CLASS OF DELAY DIFFERENTIAL EQUATIONS WITH COMPUTABLE OPERATORS

Dedicated to the blessed memory of Professor N. V. Azbelev

Abstract. In this paper, in order to expand the possibilities of the constructive approach to the study of functional differential equations, one way of constructing a new kind of so-called computable operators is discussed. An illustrative example is given.

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1 Introduction

In the present paper we propose one way to expand the possibilities of applying the constructive approach to the study of functional differential equations [1, 4, 6]. Our study is based on the main results of the theory of functional differential equations [1]. In order to illustrate the main idea of constructive approach, let us consider one example. Let $\mathcal{L} : \mathbb{AC}^n \to \mathbb{L}^n$ be a bounded linear operator and $\ell = \operatorname{col}\{\ell^1, \dots, \ell^n\} : \mathbb{AC} \to \mathbb{R}^n$ be a bounded linear vector functional. Here, $\mathbb{L}^n$ is the space of summable $z : [0, T] \to \mathbb{R}^n$ with the standard norm

$$\|z\|_{\mathbb{L}^n} = \int\limits_0^T |z(s)|_n \, ds$$

(here, $| \cdot |_n$ denotes a norm in $\mathbb{R}^n$), $\mathbb{AC}^n$ is the space of absolutely continuous $x : [0, T] \to \mathbb{R}^n$ with the norm

$$\|x\|_{\mathbb{AC}^n} = |x(0)|_n + \|\dot{x}\|_{\mathbb{L}^n}.$$

For $X = (x_1, \dots, x_n)$ with components $x_i \in \mathbb{AC}^n$, ℓX denotes the $(n \times n)$-matrix, whose columns are the values of ℓ on the components of X: $\ell X = (\ell^i x_j)$, $i, j = 1, \dots, m$. Consider the boundary value problem

$$(\mathcal{L}x)(t) = f(t), \ \ \ell x = \alpha, \ \ t \in [0, T], \tag{1.1}$$

where $f \in \mathbb{L}^n$, $\alpha \in \mathbb{R}^n$, under the assumption that the homogeneous equation $\mathcal{L}x = 0$ has the fundamental $(n \times n)$-matrix X. As is known, in this case problem (1.1) has a unique solution if and only if the matrix $\ \ = \ell X$ is invertible. The key idea of the constructive study of the solvability of (1.1) is as follows.

- Two $n \times n$-matrices, ${}^a\mathbf{\Gamma}$ and ${}^v\mathbf{\Gamma}$, with rational elements are constructed according to a specially developed procedure based on a computer-assisted proof [6] such that

  $$\boldsymbol{|}\mathbf{\Gamma} -^a \mathbf{\Gamma}\boldsymbol{|} \le \boldsymbol{|}^v\mathbf{\Gamma}\boldsymbol{|}$$

  ($\boldsymbol{|}\mathbf{A}\boldsymbol{|} \overset{\text{def}}{=} \{|a_{ij}|\}_{i,j=1}^n$ for the $(n \times n)$-matrix A);
- the invertibility of the matrix ${}^a\mathbf{\Gamma}$ is verified with the use of the reliable computer experiment;
- if there exists an inverse matrix ${}^a\mathbf{\Gamma}^{-1}$, then due to the theorem on inverse operator [2, p. 207], the inequality

  $$\|{}^v\mathbf{\Gamma}\|_{R^{n \times n}} < \frac{1}{\|{}^a\mathbf{\Gamma}^{-1}\|_{R^{n \times n}}}$$

  guarantees the invertibility of $\mathbf{\Gamma}$ which, in turn, means the solvability of (1.1).

Matrix ${}^a\mathbf{\Gamma}$ is defined by the equality

$${}^a\mathbf{\Gamma} = {}^a\mathcal{L}^a X,$$

where the operator ${}^a\mathcal{L} : \mathbb{AC}^n \to \mathbb{L}^n$ is an approximation of $\mathcal{L}$ within the class of the so-called *computable* operators, the elements of the matrix ${}^aX$ are piecewise polynomials with rational coefficients (the ways of constructing the matrices ${}^aX$ and ${}^v\mathbf{\Gamma}$ are not discussed in this paper, those are described in [6]).

Notation and definitions

Let $\Omega = \{t_q\}_{q=0}^{m+1}$, where t_q are real numbers, be such that

$$0 = t_0 < t_1 < \dots < t_m < t_{m+1} = T. \tag{1.2}$$

On the partition Ω (1.2), we define a set of intervals

$$\mathfrak{J} = \{\mathfrak{J}_q\}_{q=0}^{m+1}, \tag{1.3}$$

where $\mathfrak{I}_0 = (-\infty, 0)$, each interval $\mathfrak{I}_q$, $q = 1, \dots, m+1$, may have one of the following kinds:

$$[t_{q-1}, t_q], \quad [t_{q-1}, t_q), \quad (t_{q-1}, t_q], \quad (t_{q-1}, t_q).$$

The conditions

$$\bigcup_{q=1}^{m+1} \mathfrak{I}_q = [0, T], \quad \mathfrak{I}_{q_1} \cap \mathfrak{I}_{q_2} = \varnothing, \;\; q_1 \neq q_2,$$

are assumed to be fulfilled.

Definition 1.1. We say that a function $h : [0, T] \to \mathbb{R}$ is a-computable over the partition Ω (1.2) and the set $\mathfrak{I}$ (1.3) if for every $j = 1, \dots, m+1$, there exists an integer q_j, $0 \leq q_j \leq m+1$ such that $h(t) \in \mathfrak{I}_{q_j}$ for $t \in \mathfrak{I}_j$.

Denote by $\mathfrak{P}$ the set of all polynomials with rational coefficients.

Definition 1.2. A function $h : [0, T] \to \mathbb{R}$ is called *computable* over the partition Ω (1.2) and the set $\mathfrak{I}$ (1.3) if the following conditions hold:

(i) $h \in \mathfrak{P}$,
(ii) numbers t_q, $q = 1, \dots, m+1$, from (1.2) are rational,
(iii) the function h is a-computable over Ω (1.2) and $\mathfrak{I}$ (1.3).

Let $\mathcal{L} = \operatorname{col}\{\mathcal{L}_1, \dots, \mathcal{L}_n\} : \mathbb{AC}^n \to \mathbb{L}^n$ be the linear operator given by

$$\begin{gathered}(\mathcal{L}_i x)(t) = \dot{x}_i(t) + \sum_{j=1}^{n} \sum_{k=1}^{n_{ij}} p_{ij}^k(t) x_j[h_{ij}^k(t)], \;\; t \in [0, T], \\ x_i(\xi) = 0, \xi < 0,\end{gathered} \tag{1.4}$$

where $p_{ij}^k \in \mathbb{L}^1$, h_{ij}^k is a piecewise continuous function, $i = 1, \dots, n$.

Definition 1.3. An operator $\mathcal{L}$ (1.4) is called *computable* over the sets Ω (1.2) and $\mathfrak{I}$ (1.3) if the following conditions hold:

(i) $p_{ij}^k \in \mathfrak{P}$,
(ii) the functions h_{ij}^k are computable over Ω (1.2) and $\mathfrak{I}$ (1.3),

$i, j = 1, \dots, n$, $k = 1, \dots, n_{ij}$.

Formulation of the problem

Let t_q, $q = 1, \dots, m$, be rational numbers, ${}^a p_{ij}^k \in \mathfrak{P}$ be approximation of p_{ij}^k such that

$$\|p_{ij}^k - {}^a p_{ij}^k\|_{\mathbb{L}^1} \leq {}^v p_{ij}^k,$$

where ${}^v p_{ij}^k$ are rational numbers, $i, j = 1, \dots, n$, $k = 1, \dots, n_{ij}$. Define the sets $\widetilde{\Omega}$ and $\widetilde{\mathfrak{I}}$ as

$$\begin{gathered}\widetilde{\Omega} = \{t_q\}_{q=0}^{m+1}, \quad \widetilde{\mathfrak{I}} = \{\widetilde{\mathfrak{I}}_q\}_{q=0}^{m+1}, \\ \widetilde{\mathfrak{I}}_0 = (-\infty, 0), \quad \widetilde{\mathfrak{I}}_q = [t_{q-1}, t_q), \;\; q = 1, \dots, m, \quad \widetilde{\mathfrak{I}}_{m+1} = [t_m, T].\end{gathered} \tag{1.5}$$

The functions h_{ij}^k are approximated by piecewise constant functions ${}^a h_{ij}^k$ given by

$${}^a h_{ij}^k(t) = \sum_{q=1}^{m+1} \chi_{\widetilde{\mathfrak{I}}_q}(t) {}^a_q h_{ij}^k, \;\; t \in [0, T],$$

where $\chi_{\widetilde{\mathfrak{I}}_q}(\,\cdot\,)$ is the characteristic function of $\widetilde{\mathfrak{I}}_q$, ${}^a_q h^k_{ij}$ is a rational approximation of $h^k_{ij}(t_{q-1})$ with a rational error estimate ${}^v_q h^k_{ij}$ such that

$$|h^k_{ij}(t_q) - {}^a_q h^k_{ij}| \le {}^v_q h^k_{ij},$$

$i, j = 1, \dots, n$, $k = 1, \dots, n_{ij}$. Define the operator ${}^a\mathcal{L} = \operatorname{col}\{{}^a\mathcal{L}_1, \dots, {}^a\mathcal{L}_n\}$ by the equality

$$\begin{gathered}({}^a\mathcal{L}_i x)(t) = \dot{x}_i(t) + \sum_{j=1}^{n}\sum_{k=1}^{n_{ij}} {}^a p^k_{ij}(t)\, x_j[{}^a h^k_{ij}(t)], \;\; t \in [0, T],\\ x_i(\xi) = 0, \;\; \xi < 0,\end{gathered} \tag{1.6}$$

where $i = 1, \dots, n$. It is clear that the operator ${}^a\mathcal{L}$ (1.6) is computable over $\widetilde{\Omega}_m$ and $\widetilde{\mathfrak{I}}$, since ${}^a h^k_{ij}(t) \in \widetilde{\mathfrak{I}}_q$ as $t \in \widetilde{\mathfrak{I}}_q$, $i, j = 1, \dots, n$, $k = 1, \dots, n_{ij}$. This kind of computable operators has been used for constructive research until recently (see [1, 3, 4, 6–10]). It seems interesting to construct other kinds of computable operators. In [11], one class of the so-called *admissible* delay functions and the corresponding computable operators were proposed. Further, some new kinds of such functions will be considered in Section 2. An example of computable operator will be given in Section 3.

2 Admissible delay functions

2.1 Increasing delay functions

Let τ_* be a real number, $0 < \tau_* < T$. Define some classes of functions $h : [0, T] \to \mathbf{R}$.

(i) The function $h : [0, T] \to \mathbb{R}$ is strictly increasing continuous one passing through the points $(\gamma_1, 0)$, (T, γ_2), where γ_1, γ_2 are real numbers such that

$$\tau_* < \gamma_1 < T, \quad 0 < \gamma_2 < T. \tag{2.1}$$

Define the numbers t_q as follows:

$$t_1 = \gamma_1, \;\; t_q = h^{-1}(t_{q-1}), \;\; q = 2, \dots, m,$$

where m is such that both conditions $t_m < T$ and $h^{-1}(t_m) \ge T$ are fulfilled. By construction, we have

$$h(t) \in \begin{cases} (\infty, 0), & t \in [t_0, t_1), \\ [t_{q-1}, t_q), & t \in [t_q, t_{q+1}), \;\; q = 1, \dots, m, \\ [t_{m-1}, t_m), & t \in [t_m, t_{m+1}], \end{cases}$$
$$t_0 = 0, \;\; t_{m+1} = T.$$

Thus the function h is a-computable over the partition Ω and the set $\mathfrak{I}$ given by

$$\begin{gathered}\Omega = \{t_q\}_{q=0}^{m+1}, \quad \mathfrak{I} = \{\mathfrak{I}_q\}_{q=0}^{m+1},\\ \mathfrak{I}_0 = (-\infty, 0), \quad \mathfrak{I}_q = [t_{q-1}, t_q), \;\; q = 1, \dots, m, \quad \mathfrak{I}_{m+1} = [t_m, T].\end{gathered} \tag{2.2}$$

(ii) The function $h : [0, T] \to \mathbb{R}$ is defined as follows:

$$h(t) = \chi_{[0,\gamma_3)}(t)\widetilde{h}(t), \;\; t \in [0, T],$$

where the function $\widetilde{h} : [0, T] \to \mathbb{R}$ is strictly increasing one passing through the points $(\gamma_1, 0)$, (γ_3, γ_4), here, γ_1, γ_3 and γ_4 are real numbers such that

$$\tau_* < \gamma_1 < \gamma_3 < T, \quad 0 < \gamma_4 < \gamma_3. \tag{2.3}$$

Define the numbers t_q:

$$t_1 = \gamma_1, \quad t_q = h^{-1}(t_{q-1}), \ \ q = 2, \dots, m-1,$$

where m is such that both conditions $t_{m-1} < \gamma_3$ and ${}^a h^{-1}(t_{m-1}) \geq \gamma_3$ are fulfilled. From this we have

$$h(t) \in \begin{cases} (\infty, 0), & t \in [t_0, t_1), \\ [t_{q-1}, t_q), & t \in [t_q, t_{q+1}), \ \ q = 1, \dots, m, \\ [t_0, t_1), & t \in [t_m, t_{m+1}], \end{cases}$$
$$t_0 = 0, \ \ t_m = \gamma_3, \ \ t_{m+1} = T.$$

It is clear that the function h is a-computable over the partition Ω and the set $\mathfrak{I}$ given by

$$\begin{gathered} \Omega = \{t_q\}_{q=0}^{m+1}, \quad \mathfrak{I} = \{\mathfrak{I}_q\}_{q=0}^{m+1}, \\ \mathfrak{I}_0 = (-\infty, 0), \quad \mathfrak{I}_q = [t_{q-1}, t_q), \ \ q = 1, \dots, m, \quad \mathfrak{I}_{m+1} = [t_m, T]. \end{gathered} \tag{2.4}$$

(iii) The function $h : [0, T] \to \mathbb{R}$ is of the form

$$h(t) = \chi_{[\gamma_1, \gamma_3)}(t) \widetilde{h}(t), \ \ t \in [0, T],$$

where the function $\widetilde{h} : [0, T] \to \mathbb{R}$ is strictly increasing one passing through the points (γ_1, γ_2), (γ_3, γ_4), here, γ_1, γ_2, γ_3 and γ_4 are real numbers such that

$$\tau_* < \gamma_1 < \gamma_3 < T, \quad 0 < \gamma_2 < \gamma_1, \quad \gamma_2 < \gamma_4 < \gamma_3. \tag{2.5}$$

Define the numbers t_q as

$$t_1 = \gamma_1, \quad t_q = h^{-1}(t_{q-1}), \ \ q = 2, \dots, m-1,$$

where m is such that both conditions $t_{m-1} < \gamma_3$ and ${}^a h^{-1}(t_{m-1}) \geq \gamma_3$ are fulfilled. Thus we have

$$h(t) \in \begin{cases} [t_0, t_1), & t \in [t_0, t_1), \\ [t_{q-1}, t_q), & t \in [t_q, t_{q+1}), \ \ q = 1, \dots, m, \\ [t_0, t_1), & t \in [t_m, t_{m+1}], \end{cases}$$
$$t_0 = 0, \ \ t_m = \gamma_3, \ \ t_{m+1} = T.$$

It is obvious that the function h is a-computable over the partition Ω and the set $\mathfrak{I}$ given by

$$\begin{gathered} \Omega = \{t_q\}_{q=1}^{m+1}, \quad \mathfrak{I} = \{\mathfrak{I}_q\}_{q=0}^{m+1}, \\ \mathfrak{I}_0 = (-\infty, 0), \quad \mathfrak{I}_q = [t_{q-1}, t_q), \ \ q = 1, \dots, m, \quad \mathfrak{I}_{m+1} = [t_m, T]. \end{gathered} \tag{2.6}$$

Definition 2.1. The functions h proposed above will be called *admissible* functions.

Next, we prove that there exist sets Ω (1.2) and $\mathfrak{I}$ (1.3) such that any finite set of admissible functions are a-computable over these sets. As a preliminary, we prove two auxiliary lemmas. Let h be an admissible function and let h be a-computable over Ω_0 and $\mathfrak{I}_0$, where

$$\begin{gathered} \Omega_0 = \big\{t_0 < t_1 < \dots < t_{m_0} < t_{m_0+1}\big\}, \quad t_q \text{ are real numbers,} \\ \mathfrak{I}_0 = \bigcup_{q=1}^{m+1} \mathfrak{I}_q, \quad \mathfrak{I}_q = [t_{q-1}, t_q). \end{gathered} \tag{2.7}$$

Denote by $\Omega, \mathfrak{I}$ the sets

$$\begin{gathered} \Omega = \Omega_0 \cup \{t_{m+1}\}, \ \ m = m_0 + 1, \ \ t_m < t_{m+1}, \\ \mathfrak{I} = \mathfrak{I}_0 \cup \mathfrak{I}_{m+1}, \quad \mathfrak{I}_{m+1} = [t_m, t_{m+1}). \end{gathered} \tag{2.8}$$

Further, we have the following alternatives:

A1. The function h is a-computable over Ω and $\mathfrak{I}$.

A2. The function h is not a-computable over Ω and $\mathfrak{I}$.

Consider the case where alternative A1 is true. Let $\widetilde{m}$ be a positive integer and

$$\widetilde{\Omega} = \{\tau_1, \dots, \tau_{\widetilde{m}}\}, \ \ t_m < \tau_1 < \dots < \tau_{\widetilde{m}} < t_{m+1}, \ \ \tau_r \text{ are real numbers},$$
$$\widetilde{\mathfrak{I}} = \bigcup_{r=1}^{\widetilde{m}+1} \widetilde{\mathfrak{I}}_r, \quad \widetilde{\mathfrak{I}}_r = [\tau_{r-1}, \tau_r), \quad r = 1, \dots, \widetilde{m}+1.$$

Lemma 2.1. *The function h is a-computable over $\overline{\Omega}$ and $\overline{\mathfrak{I}}$, where*

$$\overline{\Omega} = \Omega \cup \widetilde{\Omega}, \quad \overline{\mathfrak{I}} = \bigcup_{q=1}^{m} \mathfrak{I}_q \cup \bigcup_{r=1}^{\widetilde{m}+1} \widetilde{\mathfrak{I}}_r.$$

Proof. Under the conditions, there exists an interval $\mathfrak{I}_{q_0}$ such that $h(t) \in \mathfrak{I}_{q_0}$ as $t \in \mathfrak{I}_{m+1}$. Also, we find that if $t \in \widetilde{\mathfrak{I}}_r$, then $h(t) \in \mathfrak{I}_{q_0}$, since $\widetilde{\mathfrak{I}}_r \subset \mathfrak{I}_{m+1}$, $r = 1, \dots, \widetilde{m}+1$. □

Let alternative A2 be true. In this case, there are $\widetilde{m}$ elements $t_r^0 \in \Omega$, $\widetilde{m} \geq 1$ such that

$$t_{q_0} < t_r^0 < t_{q_1}, \ \ r = 1, \dots, \widetilde{m}, \quad t_{q_0} = h(t_m), \quad t_{q_1} = h(t_{m+1}).$$

Let $\tau_r = h^{-1}(t_r^0)$, $r = 1, \dots, \widetilde{m}$. Note that $t_m < \tau_r < t_{m+1}$, $r = 1, \dots, \widetilde{m}$. Define the set of numbers $\widetilde{\Omega}$ and the set of intervals $\widetilde{\mathfrak{I}}$ as follows:

$$\widetilde{\Omega} = \Omega \cup \{\tau_r\}_{r=1}^{\widetilde{m}}, \quad \widetilde{\mathfrak{I}} = \bigcup_{q=1}^{m} \mathfrak{I}_q \cup \bigcup_{r=1}^{\widetilde{m}+1} \widetilde{\mathfrak{I}}_r,$$
$$\widetilde{\mathfrak{I}}_r = [\tau_{r-1}, \tau_r), \quad \tau_0 = t_m, \tau_{\widetilde{m}+1} = t_{m+1}.$$

By construction,

$$\begin{aligned} h(t) &\in \mathfrak{I}_{q_0+1}, & t &\in \widetilde{\mathfrak{I}}_1, \\ &\vdots & &\vdots \\ h(t) &\in \mathfrak{I}_{q_1}, & t &\in \widetilde{\mathfrak{I}}_{\widetilde{m}+1}. \end{aligned}$$

So, h is a-computable over $\widetilde{\Omega}$ and $\widetilde{\mathfrak{I}}$. Let $\widetilde{m}_r$ be a positive integer and

$$\tau_{r-1} < \sigma_r^1 < \dots < \sigma_r^{\widetilde{m}_r} < \tau_r, \quad \sigma_r^\nu \text{ be real numbers}, \ \ r = 1, \dots, \widetilde{m}+1.$$

Define the set of numbers $\overline{\Omega}$ and the set of intervals $\overline{\mathfrak{I}}$:

$$\begin{gathered} \overline{\Omega} = \Omega \cup \{\tau_r\}_{r=1}^{\widetilde{m}} \cup \bigcup_{r=1}^{\widetilde{m}+1} \{\sigma_r^\nu\}_{\nu=1}^{\widetilde{m}_r}, \quad \overline{\mathfrak{I}} = \bigcup_{q=1}^{m} \mathfrak{I}_q \cup \bigcup_{r=1}^{\widetilde{m}+1} \bigcup_{\nu=1}^{\widetilde{m}_r+1} \widetilde{\mathfrak{I}}_r^\nu, \\ \widetilde{\mathfrak{I}}_r^\nu = [\sigma_r^{\nu-1}, \sigma_r^\nu), \quad \sigma_r^0 = \tau_{r-1}, \quad \sigma_r^{\widetilde{m}_r+1} = \tau_r. \end{gathered} \tag{2.9}$$

Lemma 2.2. *The function h is a-computable over $\overline{\Omega}$ and $\overline{\mathfrak{I}}$ from* (2.9).

Proof. Under the construction $h(t) \in \widetilde{\mathfrak{I}}_r$ for $t \in \widetilde{\mathfrak{I}}_r^\nu$, $\nu = 1, \dots, \widetilde{m}_r+1$, $r = 1, \dots, \widetilde{m}+1$. □

Let h_i be an admissible function and Ω_i be the corresponding set of points of form (2.2), (2.4) or (2.6), $i = 1, \dots, n$. Define the set $\widetilde{\Omega}$ as

$$\widetilde{\Omega} = \bigcup_{i=1}^{n} \Omega_i. \tag{2.10}$$

Let $\widetilde{\Omega} = \{0 = \widetilde{t}_0 < \widetilde{t}_1 < \dots < \widetilde{t}_{\widetilde{m}} < \widetilde{t}_{\widetilde{m}+1} = T\}$. Without loss of generality, we assume that $\Omega_i \cap \Omega_j = \{0, T\}$, $i \neq j$, $i, j = 1, \dots, n$.

Theorem 2.1. *There exist a set of points* Ω (1.2) *and a set of intervals* $\mathfrak{I}$ (1.3) *such that all functions* h_i, $i = 1, \ldots, n$, *are a-computable over* Ω *and* $\mathfrak{I}$.

Proof. The proof follows from the way of constructing the desired sets Ω and $\mathfrak{I}$. Describe this step by step.

Step 1. Let $q = 1$, $m = 1$, $\Omega = \{0 = t_0, t_1 = \widetilde{t}_1\}$, $\mathfrak{I}_0 = (-\infty, 0)$, $\mathfrak{I}_1 = [t_0, t_1)$. It is clear that all functions h_i are a-computable over Ω and $\mathfrak{I} = \mathfrak{I}_0 \cup \mathfrak{I}_q$, since either $h_i(t) \in \mathfrak{I}_0$ or $h_i(t) \in \mathfrak{I}_1$ as $t \in \mathfrak{I}_1$, $i = 1, \ldots, n$.

Step 2. $q = q + 1$, $m = m + 1$. If $q = \widetilde{m} + 1$, we complete the proof, otherwise we continue. Add the point $\widetilde{t}_q$ to the set Ω: $\Omega = \Omega \cup \{\widetilde{t}_q\}$, and add the new interval $\mathfrak{I}_m = [t_{m-1}, t_m)$ to the set $\mathfrak{I}$: $\mathfrak{I} = \mathfrak{I} \cup \mathfrak{I}_m$.

Step 2.1. Consider each function h_i, $i = 1, \ldots, n$. There are two cases:

- there exists j_i such that $h_i(t) \in \mathfrak{I}_{j_i}$ as $t \in \mathfrak{I}_m$, that is, h_i is a-computable over Ω and $\mathfrak{I}$; let $\Omega_i^q = \emptyset$;
- h_i is not a-computable over Ω and $\mathfrak{I}$, that is, there are $\widetilde{m}_i^q$ elements $t_{i_q}^r \in \Omega$, $\widetilde{m}_i^q \geq 1$ such that

$$t_{q_0} < t_{i_q}^r < t_{q_1}, \quad r = 1, \ldots, \widetilde{m}_i^q, \quad t_{q_0} = h_i(t_{m-1}), \quad t_{q_1} = h(t_m).$$

Let $\tau_{i_q}^r = {h_i}^{-1}(t_{i_q}^r)$, $r = 1, \ldots, \widetilde{m}_i^q$. Define the set of numbers $\widetilde{\Omega}_i^q$ and the set of intervals $\widetilde{\mathfrak{I}}_i^q$ as follows:

$$\widetilde{\Omega}_i^q = \Omega \cup \{\tau_{i_q}^r\}_{r=1}^{\widetilde{m}_i^q}, \quad \widetilde{\mathfrak{I}}_i^q = \bigcup_{q=1}^{m-1} \mathfrak{I}_q \cup \bigcup_{r=1}^{\widetilde{m}_i^q+1} \widetilde{\mathfrak{I}}_{i_q}^r,$$

$$\widetilde{\mathfrak{I}}_{i_q}^r = [\tau_{i_q}^{r-1}, \tau_{i_q}^r), \quad \tau_{i_q}^0 = t_{m-1}, \tau_{\widetilde{m}_i^q+1} = t_m.$$

Note that under the construction, h_i is a-computable over $\widetilde{\Omega}_i^q$ and $\widetilde{\mathfrak{I}}_i^q$. Let $\Omega_i^q = \{\tau_{i_q}^r\}_{r=1}^{\widetilde{m}_i^q}$.

Step 2.2. Main conclusion. Let $\Omega^q = \bigcup\limits_{i=1}^{n} \Omega_i^q$. There are two cases:

1. $\Omega^q = \varnothing$. All functions h_i are a-computable over Ω and $\mathfrak{I}$;
2. $\Omega^q \neq \varnothing$, let $\Omega^q = \{\tau_q^1 < \cdots < \tau_q^{\widetilde{m}_q}\}$ and let $\mathfrak{I}_q^r = [\tau_q^{r-1}, \tau_q^r)$, $r = 1, \ldots, \widetilde{m}_q + 1$, $\tau_q^0 = t_{m-1}$, $\tau_q^{\widetilde{m}_q+1} = t_m$; redefine the sets Ω and $\mathfrak{I}$ as follows:

$$\Omega = \Omega \cup \Omega^q, \quad \mathfrak{I} = \bigcup_{j=1}^{m-1} \mathfrak{I}_j \cup \bigcup_{r=1}^{\widetilde{m}_q+1} \mathfrak{I}_q^r; \tag{2.11}$$

next, for each function h_i, $i = 1, \ldots, n$, we have:

- if the set Ω_i^q is empty, then h_i is a-computable over Ω and $\mathfrak{I}$ (2.11) according to Lemma 2.1,
- if the set Ω_i^q is not empty, then h_i is a-computable over Ω and $\mathfrak{I}$ (2.11) according to Lemma 2.2.

Remark 2.1. Obviously, $\widetilde{m}_q \leq m \times n$.

Step 2.3. Get back ***Step 2.***

□

2.2 Decreasing delay functions

In this section, we propose some classes of decreasing delay functions. The presentation of the material is done in the same way as it has been done in the previous section. Consider the following functions $h : [0, T] \to \mathbb{R}$.

(i) The function $h : [0, T] \to \mathbb{R}$ is given by

$$h(t) = \chi_{[\gamma_3, T]}(t)\widetilde{h}(t), \;\; t \in [0, T],$$

where the function $\widetilde{h} : [0, T] \to \mathbb{R}$ is strictly decreasing one passing through the points (γ_3, γ_4), $(\gamma_1, 0)$, where γ_1, γ_3 and γ_4 are real numbers,

$$\tau_* < \gamma_3 < \gamma_1 < T, \quad 0 < \gamma_4 < \gamma_3. \tag{2.12}$$

Let $t_1 = \gamma_3$, $t_2 = \gamma_1$. Under the construction, we have

$$h(t) \in \begin{cases} [t_0, t_1), & t \in [t_0, t_1], \\ [t_0, t_1), & t \in (t_1, t_2], \\ (-\infty, 0), & t \in (t_2, t_3], \end{cases} \quad t_0 = 0, \;\; t_3 = T.$$

Define the partition Ω and intervals $\mathfrak{I}_q$ as follows:

$$\begin{gathered} \Omega = \{0, t_1, t_2, T\}, \\ \mathfrak{I}_0 = (-\infty, 0), \quad \mathfrak{I}_1 = [t_0, t_1], \quad \mathfrak{I}_2 = (t_1, t_2], \quad \mathfrak{I}_3 = (t_2, T]. \end{gathered} \tag{2.13}$$

Obviously, the function h is a-computable over Ω and $\mathfrak{I} = \{\mathfrak{I}_q\}_{q=0}^{3}$.

(ii) The function $h : [0, T] \to \mathbb{R}$ is defined as follows:

$$h(t) = \chi_{[\gamma_3, T]}(t)\widetilde{h}(t), \;\; t \in [0, T],$$

where the function $\widetilde{h} : [0, T] \to \mathbb{R}$ is strictly decreasing one passing through the points (γ_3, γ_4), (T, γ_2), here, γ_2, γ_3 and γ_4 are the real numbers

$$\tau_* < \gamma_3 < T, \quad 0 < \gamma_4 < \gamma_3, \quad 0 < \gamma_2 < \gamma_4. \tag{2.14}$$

Let $t_1 = \gamma_3$. Next, we have

$$h(t) \in \begin{cases} [t_0, t_1), & t \in [t_0, t_1], \\ [t_0, t_1), & t \in (t_1, t_2], \end{cases} \quad t_0 = 0, \;\; t_2 = T.$$

Thus the function h is a-computable over Ω and $\mathfrak{I} = \{\mathfrak{I}_q\}_{q=0}^{2}$, where

$$\Omega = \{0, t_1, T\}, \quad \mathfrak{I}_0 = (-\infty, 0), \quad \mathfrak{I}_1 = [t_0, t_1], \quad \mathfrak{I}_2 = (t_1, T]. \tag{2.15}$$

- The function $h : [0, T] \to \mathbb{R}$ is defined as

$$h(t) = \chi_{[\gamma_3, \gamma_1]}(t)\widetilde{h}(t), \;\; t \in [0, T],$$

here, the function $\widetilde{h} : [0, T] \to \mathbb{R}$ is strictly increasing one passing through the points (γ_3, γ_4), (γ_1, γ_2), γ_1, γ_2, γ_3 and γ_4 are the real numbers such that

$$\tau_* < \gamma_3 < \gamma_1 < T, \quad 0 < \gamma_4 < \gamma_3, \quad 0 < \gamma_2 < \gamma_4. \tag{2.16}$$

Let $t_1 = \gamma_3$, $t_2 = \gamma_1$. We have

$$h(t) \in \begin{cases} [t_0, t_1), & t \in [t_0, t_1], \\ [t_0, t_1), & t \in (t_1, t_2], \\ [t_0, t_1), & t \in (t_2, t_3], \end{cases} \quad t_0 = 0, \;\; t_3 = T.$$

It is clear that the function h is a-computable over Ω and $\mathfrak{I} = \{\mathfrak{I}_q\}_{q=0}^{3}$ given by

$$\Omega = \{0, t_1, t_2, T\}, \quad \mathfrak{I}_0 = (-\infty, 0), \quad \mathfrak{I}_1 = [t_0, t_1], \quad \mathfrak{I}_2 = (t_1, t_2], \quad \mathfrak{I}_3 = (t_2, T]. \tag{2.17}$$

Definition 2.2. The functions h proposed above will be called *admissible* functions.

Next, it is proved that there exist the sets Ω (1.2) and $\Im$ (1.3) such that any finite set of admissible functions h are a-computable over these sets. As a preliminary, we prove two auxiliary statements. Let h be an admissible function and let h be computable over Ω_0 and $\Im_0$ given by

$$\begin{gathered}\Omega_0 = \{t_0 < t_1 < \cdots < t_{m_0} < t_{m_0+1}\}, \quad t_q \text{ are real numbers,} \\ \Im_0 = \bigcup_{q=1}^{m+1} \Im_q, \quad \Im_q = (t_{q-1}, t_q].\end{gathered} \tag{2.18}$$

Let t_{m+1} be a real number, $m = m_0 + 1$. Define the sets Ω, $\Im$ as

$$\Omega = \Omega_0 \cup \{t_{m+1}\}, \ \ m = m_0 + 1, \ \ t_m < t_{m+1}, \quad \Im = \Im_0 \cup \Im_{m+1}, \quad \Im_{m+1} = (t_m, t_{m+1}]. \tag{2.19}$$

There are two alternatives.

A1. The function h is a-computable over Ω and $\Im$,

A2. The function h is not a-computable over Ω and $\Im$.

Let alternative A1 be true. Define

$$\begin{gathered}\widetilde{\Omega} = \{\tau_1 < \cdots < \tau_{\widetilde{m}}\}, \quad t_m < \tau_1, \ \ \tau_{\widetilde{m}} < t_{m+1}, \ \ \tau_r \text{ are real numbers,} \\ \widetilde{\Im} = \bigcup_{r=1}^{\widetilde{m}+1} \widetilde{\Im}_r, \quad \widetilde{\Im}_r = (\tau_{r-1}, \tau_r], \ \ r = 1, \ldots, \widetilde{m}+1\end{gathered}$$

($\widetilde{m}$ is a positive integer).

Lemma 2.3. *The function h is a-computable over $\overline{\Omega}$ and $\overline{\Im}$, where*

$$\overline{\Omega} = \Omega \cup \widetilde{\Omega}, \quad \overline{\Im} = \bigcup_{q=1}^{m} \Im_q \cup \bigcup_{r=1}^{\widetilde{m}+1} \widetilde{\Im}_r.$$

Proof. By construction, there exists an interval $\Im_{q_0}$ such that $h(t) \in \Im_{q_0}$ as $t \in \Im_{m+1}$. Also, we find that if $t \in \widetilde{\Im}_r$, then $h(t) \in \Im_{q_0}$, since $\widetilde{\Im}_r \subset \Im_{m+1}$, $r = 1, \ldots, \widetilde{m}+1$. ☐

Let alternative A2 be true. In this case, there are $\widetilde{m}$ elements $t_r^0 \in \Omega$, $\widetilde{m} \geq 1$ such that

$$t_{q_0} < t_r^0 < t_{q_1}, \ \ r = 1, \ldots, \widetilde{m}, \quad t_{q_0} = h(t_m), \ \ t_{q_1} = h(t_{m+1}).$$

Let $\tau_r = h^{-1}(t_r^0)$, $r = 1, \ldots, \widetilde{m}$, $t_m < \tau_r < t_{m+1}$, $r = 1, \ldots, \widetilde{m}$. Define the sets $\widetilde{\Omega}$ and $\widetilde{\Im}$ as follows:

$$\begin{gathered}\widetilde{\Omega} = \Omega \cup \{\tau_r\}_{r=1}^{\widetilde{m}}, \quad \widetilde{\Im} = \bigcup_{q=1}^{m} \Im_q \cup \bigcup_{r=1}^{\widetilde{m}+1} \widetilde{\Im}_r, \\ \widetilde{\Im}_r = [\tau_{r-1}, \tau_r), \quad \tau_0 = t_m, \quad \tau_{\widetilde{m}+1} = t_{m+1}.\end{gathered}$$

Obviously,

$$\begin{array}{ll} h(t) \in \Im_{q_0+1}, & t \in \widetilde{\Im}_1, \\ \vdots & \vdots \\ h(t) \in \Im_{q_1}, & t \in \widetilde{\Im}_{\widetilde{m}+1}. \end{array}$$

So, h is a-computable over $\widetilde{\Omega}$ and $\widetilde{\Im}$. Let

$$\tau_{r-1} < \sigma_r^1 < \cdots < \sigma_r^{\widetilde{m}_r} < \tau_r, \ \ r = 1, \ldots, \widetilde{m}+1$$

($\widetilde{m}$ is a positive integer). Define the sets $\overline{\Omega}$ and $\overline{\Im}$ as follows:

$$\begin{gathered}\overline{\Omega} = \Omega \cup \{\tau_r\}_{r=1}^{\widetilde{m}} \cup \bigcup_{r=1}^{\widetilde{m}+1} \{\sigma_r^\nu\}_{\nu=1}^{\widetilde{m}_r}, \quad \overline{\Im} = \bigcup_{q=1}^{m} \Im_q \cup \bigcup_{r=1}^{\widetilde{m}+1} \bigcup_{\nu=1}^{\widetilde{m}_r+1} \widetilde{\Im}_r^\nu, \\ \widetilde{\Im}_r^\nu = [\sigma_r^{\nu-1}, \sigma_r^\nu), \quad \sigma_r^0 = \tau_{r-1}, \quad \sigma_r^{\widetilde{m}_r+1} = \tau_r.\end{gathered} \tag{2.20}$$

Lemma 2.4. *The function h is a-computable over $\overline{\Omega}$ and $\overline{\Im}$ from* (2.20).

Proof. By construction, $h(t) \in \widetilde{\Im}_r$ as $t \in \widetilde{\Im}_r^{\nu}$, $\nu = 1, \dots, \widetilde{m}_r + 1$, $r = 1, \dots, \widetilde{m} + 1$. □

Let h_i be an admissible function, Ω_i be the corresponding set of form (2.13), (2.15) or (2.17), $i = 1, \dots, n$. Define the set $\widetilde{\Omega}$ by the equality

$$\widetilde{\Omega} = \bigcup_{i=1}^{n} \Omega_i. \tag{2.21}$$

We assume that $\widetilde{\Omega} = \{0 = \widetilde{t}_0 < \widetilde{t}_1 < \dots < \widetilde{t}_{\widetilde{m}} < \widetilde{t}_{\widetilde{m}+1} = T\}$ and $\Omega_i \cap \Omega_j = \{0, T\}$, $i \neq j$, $i, j = 1, \dots, n$.

Theorem 2.2. *There exist the sets Ω* (1.2) *and $\Im$* (1.3) *such that all functions h_i, $i = 1, \dots, n$, are a-computable over Ω and $\Im$.*

Proof. The proof is carried out by constructing the desired sets Ω and $\Im$ as follows.

Step 1. Let $q = 1$, $m = 1$, $\Omega = \{0 = t_0, t_1 = \widetilde{t}_1\}$, $\Im_0 = (-\infty, 0)$, $\Im_1 = [t_0, t_1)$. Obviously, all functions h_i are a-computable on Ω and $\Im = \Im_0 \cup \Im_1$, since $h_i(t) \in \Im_1$ as $t \in \Im_1$, $i = 1, \dots, n$.

Step 2. $q = q + 1$, $m = m + 1$. If $q = \widetilde{m} + 1$, we complete the proof, otherwise we continue. Add $\widetilde{t}_q$ to $\Omega : \Omega = \Omega \cup \{\widetilde{t}_q\}$, and add the interval $\Im_m = (t_{m-1}, t_m]$ to the set $\Im : \Im = \Im \cup \Im_m$.

Step 2.1. Consider each function h_i, $i = 1, \dots, n$. There are two cases:

- there exists j_i such that $h_i(t) \in \Im_{j_i}$ as $t \in \Im_m$, that is, h_i is a-computable over Ω and $\Im$; let $\Omega_i^q = \emptyset$;
- h_i is not a-computable over Ω and $\Im$, that is, there are $\widetilde{m}_i^q$ elements $t_{i_q}^r \in \Omega$, $\widetilde{m}_i^q \geq 1$ such that

$$t_{q_0} < t_{i_q}^r < t_{q_1}, \quad r = 1, \dots, \widetilde{m}_i^q, \quad t_{q_0} = h_i(t_{m-1}), \quad t_{q_1} = h(t_m).$$

Let $\tau_{i_q}^r = {h_i}^{-1}(t_{i_q}^r)$, $r = 1, \dots, \widetilde{m}_i^q$. Define the sets $\widetilde{\Omega}_i^q$ and $\widetilde{\Im}_i^q$ as

$$\widetilde{\Omega}_i^q = \Omega \cup \{\tau_{i_q}^r\}_{r=1}^{\widetilde{m}_i^q}, \quad \widetilde{\Im}_i^q = \bigcup_{q=1}^{m-1} \Im_q \cup \bigcup_{r=1}^{\widetilde{m}_i^q+1} \widetilde{\Im}_{i_q}^r,$$

$$\widetilde{\Im}_{i_q}^r = [\tau_{i_q}^{r-1}, \tau_{i_q}^r), \quad \tau_{i_q}^0 = t_{m-1}, \tau_{\widetilde{m}_i^q+1} = t_m.$$

It is clear that h_i is a-computable over $\widetilde{\Omega}_i^q$ and $\widetilde{\Im}_i^q$. Let $\Omega_i^q = \{\tau_{i_q}^r\}_{r=1}^{\widetilde{m}_i^q}$.

Step 2.2. Main conclusion. Define $\Omega^q = \bigcup_{i=1}^{n} \Omega_i^q$. There are two cases:

1. $\Omega^q = \varnothing$, all functions h_i are a-computable over Ω and $\Im$;
2. $\Omega^q \neq \varnothing$, let $\Omega^q = \{\tau_q^1 < \dots < \tau_q^{\widetilde{m}_q}\}$ and let $\Im_q^r = [\tau_q^{r-1}, \tau_q^r)$, $r = 1, \dots, \widetilde{m}_q + 1$, $\tau_q^0 = t_{m-1}$, $\tau_q^{\widetilde{m}_q+1} = t_m$; redefine the sets Ω and $\Im$ as follows:

$$\Omega = \Omega \cup \Omega^q, \quad \Im = \bigcup_{j=1}^{m-1} \Im_j \cup \bigcup_{r=1}^{\widetilde{m}_q+1} \Im_q^r; \tag{2.22}$$

thus for each function h_i, $i = 1, \dots, n$, we have

- if the set Ω_i^q is empty, then h_i is a-computable over Ω and $\Im$ (2.11) according to Lemma 2.3,
- if the set Ω_i^q isn't empty, then h_i is a-computable over Ω and $\Im$ (2.11) according to Lemma 2.4.

Remark 2.2. Obviously, $\widetilde{m}_q \leq m \times n$.

Step 2.3. Get back to ***Step 2***.

□

3 Computable operators

In this section, we propose one way to construct computable operators.

3.1 Approximation of delay functions h_{ij}^k

To simplify the text below, we omit the indices in the function notation h_{ij}^k from (1.4). Let h be a linear *admissible* function from Section 2.1. There are the following cases:

(i) The function h is linear one passing through the points $(\gamma_1, 0)$, (T, γ_2),

$$h(t) = c_0 + c_1 t, \quad c_1 = \frac{\gamma_2}{T - \gamma_1}\,, \quad c_0 = \frac{\gamma_1\,\gamma_2}{T - \gamma_1}\,, \ \ t \in [0, T], \tag{3.1}$$

where the real numbers γ_1, γ_2 are from (2.1). We approximate γ_1, γ_2 by rational numbers ${}^a\gamma_1$, ${}^a\gamma_2$, respectively, such that

$$\tau_* < {}^a\gamma_1 < T, \quad {}^a\gamma_1 \le \gamma_1 \le {}^a\gamma_1 + \varepsilon, \quad T > {}^a\gamma_2 > \varepsilon, \quad {}^a\gamma_2 \ge \gamma_2 \ge {}^a\gamma_2 - \varepsilon, \tag{3.2}$$

where $\varepsilon \ge 0$ is a given rational error bound. Denote by ${}^a h$ the linear function passing through the points $({}^a\gamma_1, 0)$, $(T, {}^a\gamma_2)$ given by

$${}^a h(t) = {}^a c_0 + {}^a c_1 t, \quad {}^a c_1 = \frac{{}^a\gamma_2}{T - {}^a\gamma_1}, \quad {}^a c_0 = \frac{{}^a\gamma_1\,{}^a\gamma_2}{T - {}^a\gamma_1}, \ \ t \in [0, T]. \tag{3.3}$$

Note that under the construction, ${}^a h(t) \ge h(t)$, $t \in [0, T]$. Define the intervals $\mathbf{C}_r$ such that

$$c_r \in \mathbf{C}_r, \quad \mathbf{C}_r = [\underline{C}_r, \overline{C}_r], \quad \underline{C}_r = {}^a c_r - {}^v c_r, \quad \overline{C}_r = {}^a c_r + {}^v c_r,$$

where ${}^v c_r \ge |c_r - {}^a c_r|$, $r = 0, 1$. The intervals $\mathbf{C}_r$, $r = 0, 1$, can be found as a solution of the following system of interval equations [5]:

$$\mathbf{C}_0 + \mathbf{C}_1[{}^a\gamma_1, {}^a\gamma_1 + \varepsilon] = 0, \quad \mathbf{C}_0 + \mathbf{C}_1 T = [{}^a\gamma_2 - \varepsilon, {}^a\gamma_2].$$

Thus we obtain the rational estimates

$$|c_r - {}^a c_r| \le \max\left\{|{}^a c_r - \underline{C}_r|, |{}^a c_r - \overline{C}_r|\right\} \overset{\text{def}}{=} {}^v c_r, \ \ r = 0, 1.$$

Finally, we have

$$|h(t) - {}^a h(t)| \le T\,{}^v c_1 + {}^v c_0 \overset{\text{def}}{=} {}^v h, \ \ t \in [0, T]. \tag{3.4}$$

Remark 3.1. It is obvious that the function ${}^a h$ is computable over the sets Ω and $\mathcal{I}$ defined as follows:

$$\begin{gathered} \Omega = \{t_q\}_{q=1}^{m+1}, \quad \mathfrak{I} = \{\mathfrak{I}_q\}_{q=0}^{m+1}, \\ \mathfrak{I}_0 = (-\infty, 0), \quad \mathfrak{I}_q = [t_{q-1}, t_q), \ \ q = 1, \dots, m, \quad \mathfrak{I}_{m+1} = [t_m, T], \\ t_0 = 0, \ \ t_1 = {}^a\gamma_1, \ \ t_q = {}^a h^{-1}(t_{q-1}), \ \ q = 2, \dots, m, \end{gathered} \tag{3.5}$$

where m is such that both conditions $t_m < T$ and ${}^a h^{-1}(t_m) \ge T$ are fulfilled.

(ii) The function h is defined by the equality

$$\begin{gathered} h(t) = \begin{cases} c_0 + c_1 t, & t \in [0, \gamma_3), \\ 0, & t \in [\gamma_3, T], \end{cases} \quad t \in [0, T], \\ c_1 = \frac{\gamma_4}{\gamma_3 - \gamma_1}\,, \quad c_0 = -\frac{\gamma_1 \gamma_4}{\gamma_3 - \gamma_1}\,, \end{gathered} \tag{3.6}$$

where γ_1, γ_3 and γ_4 are from (2.3). The numbers γ_1, γ_3 and γ_4 are approximated by rational numbers ${}^a\gamma_1$, ${}^a\gamma_3$ and ${}^a\gamma_4$, respectively, as follows:

$$\begin{gathered}
\tau_* < {}^a\gamma_1 < T, \quad {}^a\gamma_1 \le \gamma_1 \le {}^a\gamma_1 + \varepsilon, \\
T > {}^a\gamma_3 > {}^a\gamma_1 + \varepsilon, \quad {}^a\gamma_3 \ge \gamma_3 \ge {}^a\gamma_3 - \varepsilon, \\
{}^a\gamma_3 > {}^a\gamma_4 > \varepsilon, \quad {}^a\gamma_4 \ge \gamma_4 \ge {}^a\gamma_4 - \varepsilon.
\end{gathered} \tag{3.7}$$

Define the approximation function ${}^a h$:

$$\begin{gathered}
{}^a h(t) = \begin{cases} {}^a c_0 + {}^a c_1 t, & t \in [0, {}^a\gamma_3), \\ 0, & t \in [{}^a\gamma_3, T], \end{cases}, \quad t \in [0, T], \\
{}^a c_1 = \frac{{}^a\gamma_4}{{}^a\gamma_3 - {}^a\gamma_1}\,, \quad {}^a c_0 = -\frac{{}^a\gamma_1 {}^a\gamma_4}{{}^a\gamma_3 - {}^a\gamma_1}\,.
\end{gathered} \tag{3.8}$$

From the above, we get the rational estimates

$$|h(t) - {}^a h(t)| \le {}^a\gamma_3 {}^v c_1 + {}^v c_0 \stackrel{\text{def}}{=} {}^v h, \ \ t \in [0, T], \tag{3.9}$$

here,

$$|c_r - {}^a c_r| \le \max\left\{|{}^a c_r - \underline{C}_r|, |{}^a c_r - \overline{C}_r|\right\} \stackrel{\text{def}}{=} {}^v c_r, \ \ r = 0, 1,$$

the intervals $\mathbf{C}_r = [\underline{C}_r, \overline{C}_r]$ are the solutions of the system

$$\begin{gathered}
\mathbf{C}_0 + \mathbf{C}_1[{}^a\gamma_1, {}^a\gamma_1 + \varepsilon] = 0, \\
\mathbf{C}_0 + \mathbf{C}_1[{}^a\gamma_3 - \varepsilon, {}^a\gamma_3] = [{}^a\gamma_4 - \varepsilon, {}^a\gamma_4].
\end{gathered}$$

Remark 3.2. We observe that the function ${}^a h$ is computable over the sets Ω and $\mathcal{I}$ defined as follows:

$$\begin{gathered}
\Omega = \{t_q\}_{q=1}^{m+1}, \quad \mathfrak{I} = \{\mathfrak{I}_q\}_{q=0}^{m+1}, \\
\mathfrak{I}_0 = (-\infty, 0), \quad \mathfrak{I}_q = [t_{q-1}, t_q), \ \ q = 1, \dots, m, \quad \mathfrak{I}_{m+1} = [t_m, T], \\
t_0 = 0, \ \ t_1 = {}^a\gamma_1, \ \ t_q = {}^a h^{-1}(t_{q-1}), \ \ q = 2, \dots, m-1, \ \ t_m = {}^a\gamma_4,
\end{gathered} \tag{3.10}$$

here, m is such that both conditions $t_{m-1} < \gamma_4$ and ${}^a h^{-1}(t_{m-1}) \ge {}^a\gamma_4$ are fulfilled.

(iii) The function h is given by the equality

$$\begin{gathered}
h(t) = \begin{cases} c_0 + c_1 t, & t \in [\gamma_1, \gamma_3), \\ 0, & t \notin [\gamma_1, \gamma_3), \end{cases} \quad t \in [0, T], \\
c_1 = \frac{\gamma_4 - \gamma_2}{\gamma_3 - \gamma_1}\,, \quad c_0 = \frac{\gamma_2\gamma_3 - \gamma_1\gamma_4}{\gamma_3 - \gamma_1}\,,
\end{gathered} \tag{3.11}$$

where γ_1, γ_3 and γ_4 are from (2.3). The numbers γ_1, γ_3 and γ_4 are approximated by rational numbers ${}^a\gamma_1$, ${}^a\gamma_3$ and ${}^a\gamma_4$, respectively, as follows:

$$\begin{gathered}
\tau_* < {}^a\gamma_1 < T, \quad {}^a\gamma_1 \le \gamma_1 \le {}^a\gamma_1 + \varepsilon, \\
{}^a\gamma_1 > {}^a\gamma_2 > \varepsilon, \quad {}^a\gamma_2 \ge \gamma_2 \ge {}^a\gamma_2 - \varepsilon, \\
T > {}^a\gamma_3 > {}^a\gamma_1 + \varepsilon, \quad {}^a\gamma_3 \ge \gamma_3 \ge {}^a\gamma_3 - \varepsilon, \\
{}^a\gamma_3 > {}^a\gamma_4 > {}^a\gamma_2, \quad {}^a\gamma_4 \ge \gamma_4 \ge {}^a\gamma_4 - \varepsilon.
\end{gathered} \tag{3.12}$$

Define the approximation function ${}^a h$ as

$$\begin{gathered}
{}^a h(t) = \begin{cases} {}^a c_0 + {}^a c_1 t, & t \in [0, {}^a\gamma_3), \\ 0, & t \in [{}^a\gamma_3, T], \end{cases}, \quad t \in [0, T], \\
{}^a c_1 = \frac{{}^a\gamma_4}{{}^a\gamma_3 - {}^a\gamma_1}\,, \quad {}^a c_0 = \frac{{}^a\gamma_1\, {}^a\gamma_4}{{}^a\gamma_3 - {}^a\gamma_1}\,.
\end{gathered} \tag{3.13}$$

Similarly to the previous, we obtain

$$\begin{gathered}|h(t) - {}^{a}h(t)| \le {}^{a}\gamma_3 {}^{v}c_1 + {}^{v}c_0 \overset{\text{def}}{=} {}^{v}h, \;\; t \in [0, T],\\ |c_r - {}^{a}c_r| \le \max\left\{|{}^{a}c_r - \underline{C}_r|, |{}^{a}c_r - \overline{C}_r|\right\} \overset{\text{def}}{=} {}^{v}c_r, \;\; r = 0, 1,\end{gathered} \tag{3.14}$$

the intervals $\mathbf{C}_r = [\underline{C}_r, \overline{C}_r]$ are the solutions of the interval system

$$\begin{gathered}\mathbf{C}_0 + \mathbf{C}_1\,[{}^{a}\gamma_1,\, {}^{a}\gamma_1 + \varepsilon] = [{}^{a}\gamma_2 - \varepsilon, {}^{a}\gamma_2],\\ \mathbf{C}_0 + \mathbf{C}_1[{}^{a}\gamma_3 - \varepsilon, {}^{a}\gamma_3] = [{}^{a}\gamma_4 - \varepsilon, {}^{a}\gamma_4].\end{gathered}$$

Remark 3.3. It can easily be checked that the function ${}^{a}h$ is computable over the sets Ω and $\mathcal{I}$ defined similarly to (3.10).

Next, consider the approximation of the function h from Section 2.2. We have the following cases:

- The function h is defined as follows:

$$\begin{gathered}h(t) = \begin{cases} 0, & t \in [0, \gamma_3),\\ c_0 + c_1 t, & t \in [\gamma_3, T], \end{cases} \quad t \in [0, T],\\ c_1 = \frac{\gamma_4}{\gamma_3 - \gamma_1}, \quad c_0 = -\frac{\gamma_1\gamma_4}{\gamma_3 - \gamma_1},\end{gathered} \tag{3.15}$$

where γ_1, γ_3 and γ_4 are from (2.12). The numbers γ_1, γ_3 and γ_4 are approximated by rational numbers ${}^{a}\gamma_1$, ${}^{a}\gamma_3$ and ${}^{a}\gamma_4$, respectively, such that

$$\begin{gathered}\tau_* < {}^{a}\gamma_3 < T, \quad {}^{a}\gamma_3 - \varepsilon \le \gamma_3 \le {}^{a}\gamma_3,\\ T > {}^{a}\gamma_1 > {}^{a}\gamma_3 + \varepsilon, \quad {}^{a}\gamma_1 \ge \gamma_1 \ge {}^{a}\gamma_1 - \varepsilon,\\ {}^{a}\gamma_3 > {}^{a}\gamma_4 > \varepsilon, \quad {}^{a}\gamma_4 \ge \gamma_4 \ge {}^{a}\gamma_4 - \varepsilon.\end{gathered} \tag{3.16}$$

Denote by ${}^{a}h$ the approximation function

$$\begin{gathered}{}^{a}h(t) = \begin{cases} 0, & t \in [0, {}^{a}\gamma_3),\\ c_0 + c_1 t, & t \in [{}^{a}\gamma_3, T], \end{cases} \quad t \in [0, T],\\ c_1 = \frac{{}^{a}\gamma_3}{{}^{a}\gamma_3 - {}^{a}\gamma_1}, \quad c_0 = -\frac{{}^{a}\gamma_1\, {}^{a}\gamma_4}{{}^{a}\gamma_3 - {}^{a}\gamma_1}\,.\end{gathered} \tag{3.17}$$

We have

$$\begin{gathered}|h(t) - {}^{a}h(t)| \le {}^{a}\gamma_1\, {}^{v}c_1 + {}^{v}c_0 \overset{\text{def}}{=} {}^{v}h, \;\; t \in [0, T],\\ |c_r - {}^{a}c_r| \le \max\left\{|{}^{a}c_r - \underline{C}_r|, |{}^{a}c_r - \overline{C}_r|\right\} \overset{\text{def}}{=} {}^{v}c_r, \;\; r = 0, 1,\end{gathered} \tag{3.18}$$

the intervals $\mathbf{C}_r = [\underline{C}_r, \overline{C}_r]$ are the solutions of the interval system

$$\begin{gathered}\mathbf{C}_0 + \mathbf{C}_1[{}^{a}\gamma_3 - \varepsilon, {}^{a}\gamma_3] = [{}^{a}\gamma_4 - \varepsilon,\, {}^{a}\gamma_4],\\ \mathbf{C}_0 + \mathbf{C}_1[{}^{a}\gamma_3 - \varepsilon, {}^{a}\gamma_3] = 0.\end{gathered}$$

Remark 3.4. The function ${}^{a}h$ is computable over the sets Ω and $\mathcal{I}$ defined as

$$\begin{gathered}\Omega = \{0, t_1, t_2, T\}, \quad \mathcal{I} = \{\mathcal{I}_q\}_{q=1}^{3},\\ \mathfrak{I}_0 = (-\infty, 0), \quad \mathfrak{I}_1 = [0, t_1], \quad \mathfrak{I}_2 = (t_1, t_2], \quad \mathfrak{I}_3 = (t_2, T],\end{gathered} \tag{3.19}$$

where $t_1 = {}^{a}\gamma_3$, $t_2 = {}^{a}\gamma_1$.

- The function h is expressed as

$$\begin{gathered}h(t) = \begin{cases} 0, & t \in [0, \gamma_3),\\ c_0 + c_1 t, & t \in [\gamma_3, T], \end{cases} \quad t \in [0, T],\\ c_1 = \frac{\gamma_2 - \gamma_4}{T - \gamma_3}, \quad c_0 = \frac{T\,\gamma_4 - \gamma_2\gamma_3}{T - \gamma_3},\end{gathered} \tag{3.20}$$

where γ_2, γ_3 and γ_4 are from (2.14). The numbers γ_2, γ_3 and γ_4 are approximated by the rational numbers ${}^a\gamma_2$, ${}^a\gamma_3$ and ${}^a\gamma_4$ as follows:

$$\begin{gathered}\tau_* < {}^a\gamma_3 < T, \quad {}^a\gamma_3 - \varepsilon \le \gamma_3 \le {}^a\gamma_3,\\ {}^a\gamma_4 - \varepsilon > {}^a\gamma_2 > \varepsilon, \quad {}^a\gamma_2 \ge \gamma_2 \ge {}^a\gamma_2 - \varepsilon,\\ {}^a\gamma_3 > {}^a\gamma_4 > \varepsilon, \quad {}^a\gamma_4 \ge \gamma_4 \ge {}^a\gamma_4 - \varepsilon.\end{gathered} \tag{3.21}$$

Denote by ${}^ah$ the approximation function

$$\begin{gathered}{}^ah(t) = \begin{cases} 0, & t \in [0, {}^a\gamma_3),\\ c_0 + c_1 t, & t \in [{}^a\gamma_3, T], \end{cases} \quad t \in [0,T],\\ {}^ac_1 = \frac{{}^a\gamma_2 -{}^a \gamma_4}{T - {}^a\gamma_3}\,, \quad {}^ac_0 = \frac{T\,{}^a\gamma_4 - {}^a\gamma_2{}^a\gamma_3}{T - {}^a\gamma_3}\,.\end{gathered} \tag{3.22}$$

Next, we obtain

$$\begin{gathered}|h(t) - {}^ah(t)| \le {}^a\gamma_1\,{}^vc_1 + {}^vc_0 \overset{\text{def}}{=} {}^vh, \ \ t \in [0,T],\\ |c_r - {}^ac_r| \le \max\big\{|{}^ac_r - \underline{C}_r|, |{}^ac_r - \overline{C}_r|\big\} \overset{\text{def}}{=} {}^vc_r, \ \ r = 0,1,\end{gathered} \tag{3.23}$$

the intervals $\mathbf{C}_r = [\underline{C}_r, \overline{C}_r]$ are the solutions of the interval system

$$\begin{aligned}\mathbf{C}_0 + \mathbf{C}_1[{}^a\gamma_3 - \varepsilon, {}^a\gamma_3] &= [{}^a\gamma_4 - \varepsilon, {}^a\gamma_4],\\ \mathbf{C}_0 + \mathbf{C}_1 T &= [{}^a\gamma_2 - \varepsilon, {}^a\gamma_2].\end{aligned}$$

Remark 3.5. The function ${}^ah$ is computable over the sets Ω and $\mathcal{I}$ given by

$$\begin{gathered}\Omega = \{0, {}^a\gamma_3, T\}, \quad \mathcal{I} = \{\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2\},\\ \mathfrak{I}_0 = (-\infty, 0), \quad \mathfrak{I}_1 = [0, {}^a\gamma_3], \quad \mathfrak{I}_2 = ({}^a\gamma_3, T].\end{gathered} \tag{3.24}$$

- The function h is defined by the equality

$$\begin{gathered}h(t) = \begin{cases} 0, & t \in [0, \gamma_3),\\ c_0 + c_1 t, & t \in [\gamma_3, \gamma_1),\\ 0, & t \in [0, \gamma_1, T], \end{cases} \quad t \in [0,T],\\ c_1 = \frac{\gamma_2 - \gamma_4}{\gamma_1 - \gamma_3}\,, \quad c_0 = \frac{\gamma_1\,\gamma_4 - \gamma_2\gamma_3}{\gamma_1 - \gamma_3}\,,\end{gathered} \tag{3.25}$$

where γ_1, γ_2, γ_3 and γ_4 are from (2.16). The numbers γ_1, γ_2, γ_3 and γ_4 are approximated by the rational numbers ${}^a\gamma_1$, ${}^a\gamma_2$, ${}^a\gamma_3$ and ${}^a\gamma_4$, respectively, as follows:

$$\begin{gathered}\tau_* < {}^a\gamma_3 < T, \quad {}^a\gamma_3 - \varepsilon \le \gamma_3 \le {}^a\gamma_3,\\ T > {}^a\gamma_1 > {}^a\gamma_3 + \varepsilon, \quad {}^a\gamma_1 \ge \gamma_1 \ge {}^a\gamma_1 - \varepsilon,\\ {}^a\gamma_4 - \varepsilon > {}^a\gamma_2 > \varepsilon, \quad {}^a\gamma_2 \ge \gamma_2 \ge {}^a\gamma_2 - \varepsilon,\\ {}^a\gamma_3 > {}^a\gamma_4 > \varepsilon, \quad {}^a\gamma_4 \ge \gamma_4 \ge {}^a\gamma_4 - \varepsilon.\end{gathered} \tag{3.26}$$

Define the approximation function ${}^ah$ as

$$\begin{gathered}{}^ah(t) = \begin{cases} 0, & t \in [0, {}^a\gamma_3),\\ c_0 + c_1 t, & t \in [{}^a\gamma_3, {}^a\gamma_1),\\ 0, & t \in [0, {}^a\gamma_1, T], \end{cases} \quad t \in [0,T],\\ {}^ac_1 = \frac{{}^a\gamma_2 -{}^a \gamma_4}{{}^a\gamma_1 - {}^a\gamma_3}\,, \quad {}^ac_0 = \frac{{}^a\gamma_1\,{}^a\gamma_4 - {}^a\gamma_2{}^a\gamma_3}{{}^a\gamma_1 - {}^a\gamma_3}\,.\end{gathered} \tag{3.27}$$

Thus we obtain

$$\begin{aligned} &|h(t) - {}^a h(t)| \le {}^a\gamma_1 {}^v c_1 + {}^v c_0 \overset{\text{def}}{=} {}^v h, \ \ t \in [0,T], \\ &|c_r - {}^a c_r| \le \max\left\{|{}^a c_r - \underline{C}_r|, |{}^a c_r - \overline{C}_r|\right\} \overset{\text{def}}{=} {}^v c_r, \ \ r = 0,1, \end{aligned} \tag{3.28}$$

the intervals $\mathbf{C}_r = [\underline{C}_r, \overline{C}_r]$ are the solutions of the interval system

$$\begin{aligned} \mathbf{C}_0 + \mathbf{C}_1[{}^a\gamma_3 - \varepsilon, {}^a\gamma_3] &= [{}^a\gamma_4 - \varepsilon, {}^a\gamma_4], \\ \mathbf{C}_0 + \mathbf{C}_1[{}^a\gamma_1 - \varepsilon, {}^a\gamma_1] &= [{}^a\gamma_2 - \varepsilon, {}^a\gamma_2]. \end{aligned}$$

Remark 3.6. The function ${}^a h$ is computable over the sets Ω and $\mathcal{I}$ given by

$$\begin{gathered} \Omega = \{0, t_1, t_2, T\}, \quad \mathcal{I} = \{\mathcal{I}_q\}_{q=1}^3, \\ \mathfrak{I}_0 = (-\infty, 0), \quad \mathfrak{I}_1 = [0, t_1], \quad \mathfrak{I}_2 = (t_1, t_2], \quad \mathfrak{I}_3 = (t_2, T], \end{gathered} \tag{3.29}$$

where $t_1 = {}^a\gamma_3$, $t_2 = {}^a\gamma_1$.

3.2 Construction of computable operators

Let $\mathcal{L} = \operatorname{col}\{\mathcal{L}_1, \ldots, \mathcal{L}_n\} : \mathbb{AC}^n \to \mathbb{L}^n$ be the linear operator given by the equality

$$\begin{gathered} (\mathcal{L}_i x)(t) = \dot{x}_i(t) + \sum_{j=1}^{n} \sum_{k=1}^{n_{ij}} p_{ij}^k(t)\, x_j[h_{ij}^k(t)], \ \ t \in [0,T], \\ x_i(\xi) = 0, \ \ \xi < 0, \end{gathered} \tag{3.30}$$

where $p_{ij}^k \in \mathbb{L}^1$, h_{ij}^k are from (3.1), (3.6) and (3.11), $i = 1, \ldots, n$. Construct the corresponding approximating operator ${}^a\mathcal{L} = \operatorname{col}\{{}^a\mathcal{L}_1, \ldots, {}^a\mathcal{L}_n\} : \mathbb{AC}^n \to \mathbb{L}^n$ as follows:

$$\begin{gathered} ({}^a\mathcal{L}_i x)(t) \equiv \dot{x}_i(t) + \sum_{j=1}^{n} \sum_{k=1}^{n_{ij}} {}^a p_{ij}^k(t) x_j[{}^a h_{ij}^k(t)], \ \ t \in [0,T], \\ x_i(\xi) = 0, \ \ \xi < 0, \end{gathered} \tag{3.31}$$

here, the functions ${}^a p_{ij}^k$ are from (1.6), h_{ij}^k are from (3.3), (3.8) and (3.13), $i = 1, \ldots, n$. Due to Theorem 2.1, there exist a set Ω (2.2) with rational elements and a set $\mathfrak{I}$ (2.3) such that all functions ${}^a h_{ij}^k$ are computable over Ω and $\mathfrak{I}$. This implies that the operator ${}^a\mathcal{L}$ is *computable* over Ω and $\mathfrak{I}$, too. Define the operator $\mathcal{L} = \operatorname{col}\{\mathcal{L}_1, \ldots, \mathcal{L}_n\} : \mathbb{AC}^n \to \mathbb{L}^n$ as follows:

$$\begin{gathered} (\mathcal{L}_i x)(t) \equiv \dot{x}_i(t) + \sum_{j=1}^{n} \sum_{k=1}^{n_{ij}} p_{ij}^k(t)\, x_j[h_{ij}^k(t)], \ \ t \in [0,T], \\ x_i(\xi) = 0, \ \ \xi < 0, \end{gathered} \tag{3.32}$$

where $p_{ij}^k \in \mathbb{L}^1$, h_{ij}^k are from (3.15), (3.20) and (3.25), $i = 1, \ldots, n$. The corresponding approximating operator ${}^a\mathcal{L} = \operatorname{col}\{{}^a\mathcal{L}_1, \ldots, {}^a\mathcal{L}_n\} : \mathbb{AC}^n \to \mathbb{L}^n$ has the form

$$\begin{gathered} ({}^a\mathcal{L}_i x)(t) \equiv \dot{x}_i(t) + \sum_{j=1}^{n} \sum_{k=1}^{n_{ij}} {}^a p_{ij}^k(t)\, x_j[{}^a h_{ij}^k(t)], \ \ t \in [0,T], \\ x_i(\xi) = 0, \ \ \xi < 0, \end{gathered} \tag{3.33}$$

here, the functions ${}^a p_{ij}^k$ are from (1.6), h_{ij}^k are from (3.17), (3.22) and (3.27), $i = 1, \ldots, n$. By virtue of Theorem 2.2, there exist the sets Ω (2.2) and $\mathfrak{I}$ (2.3) such that all functions ${}^a h_{ij}^k$ are computable over Ω and $\mathfrak{I}$. This implies that the operator ${}^a\mathcal{L}$ is computable over Ω and $\mathfrak{I}$.

3.3 Illustrative example

Let us give an example of application of the proposed way of constructing a computable operator for the study of the solvability of one boundary value problem for delay differential equations. Consider the following equation:

$$\begin{gathered}\ddot{x}(t)+\sum_{i=1}^{4}{}^{a}p_i(t)\,x[{}^{a}h_i(t)]=f(t),\ \ t\in[0,1],\\ x(\xi)=0,\ \ \xi<0,\end{gathered}\tag{3.34}$$

where $f\in\mathbb{L}^1$,

$$\begin{gathered}{}^{a}p_1(t)=-3t+\frac{11}{4}\,,\quad {}^{a}h_1(t)=t-\frac{1}{4}\,,\\ {}^{a}p_2(t)=3t^2-\frac{33}{8}t+\frac{109}{32}\,,\quad {}^{a}h_2(t)=\frac{3}{4}t-\frac{3}{8}\,,\\ {}^{a}p_3(t)=\frac{5}{3}\,t,\quad {}^{a}h_3(t)=t-\frac{3}{4}\,,\\ {}^{a}p_4(t)=-5t+\frac{3}{2}\,,\quad {}^{a}h_4(t)=\frac{7}{2}t-\frac{21}{8}\,,\end{gathered}\tag{3.35}$$

and the set of boundary conditions

$$\begin{gathered}x(0)=\alpha,\ \ x(1)=\beta,\ \ x(\tau_1)=\alpha,\ \ x(\tau_2)=\beta,\\ x(0)=\alpha,\ \ \dot{x}(1)=\beta,\ \ x(\tau_1)=\alpha,\ \ \dot{x}(\tau_2)=\beta,\\ \dot{x}(0)=\alpha,\ \ x(1)=\beta,\ \ \dot{x}(\tau_1)=\alpha,\ \ x(\tau_2)=\beta,\\ \dot{x}(0)=\alpha,\ \ \dot{x}(1)=\beta,\ \ \dot{x}(\tau_1)=\alpha,\ \ \dot{x}(\tau_2)=\beta,\end{gathered}\tag{3.36}$$

here, $\alpha,\beta\in\mathbb{R}$, $\tau_1=\frac{2}{10}$, $\tau_2=\frac{7}{10}$. By means of a reliable computing experiment, it is proved that all boundary value problems (3.35), (3.36) are uniquely solvable. In addition, for the equation

$$\begin{gathered}\ddot{x}(t)+\sum_{i=1}^{4}p_i(t)\,x[h_i(t)]=f(t),\ \ t\in[0,1],\\ x(\xi)=0,\ \ \xi<0,\end{gathered}\tag{3.37}$$

where $p_i\in\mathbb{L}^1$, h_i has form (3.1), it is proved that all boundary value problems (3.37), (3.36) are likewise uniquely solvable if the following inequalities are fulfilled:

$$\|p_i-{}^{a}p_i\|_{\mathbb{L}^1}\le 10^{-10},\quad |h_i(t)-{}^{a}h_i(t)|\le 10^{-10},\ \ t\in[0,1].\tag{3.38}$$

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