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RAINBOW HAMILTONIAN PATHS AND CANONICALLY COLORED SUBGRAPHS IN INFINITE COMPLETE GRAPHS

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Abstract: A sufficient condition is given for the existence of a Hamiltonian path all of whose edges have a distinct color, in edge-colored infinite complete graphs. Also, a variant of the Erdős-Rado theorem is presented for canonically colored subgraph.

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0. Introduction

In this note we consider complete graphs $K = (X, E)$ with an infinite vertex set X and edge set $E = \{xx' : x, x' \in X\}$. For a given coloring φ of the *edge set*, a subgraph $G \subset K$ is called a *rainbow subgraph* of K if $G|_\varphi$, the coloring of G induced by φ, contains no monochromatic pair of edges.

If $Y \subset X$ and G is the complete subgraph induced by Y in K, then we write $Y|_\varphi$ instead of $G|_\varphi$.

Our first aim is to find a condition ensuring the existence of a rainbow Hamiltonian path (i.e., a path visiting all vertices of K) when X is countable. As shown in Theorem 1, it is enough to exclude canonically colored infinite subgraphs (see definition below) from $K|_\varphi$, provided that at each vertex, each color class has a finite or 0-measure infinite degree. This result generalizes a theorem of Hahn and Thomassen [6]. Examples show that the condition in Theorem 1 is nearly the best possible; it would be interesting, however, to see an "if and only if"-type characterization, in terms of forbidden subgraphs (cf. Problem 6).

In the second part of the paper we investigate the question how large canonically colored subgraphs exist in K when X is an ordered set of arbitrary cardinality. We consider a particular class of (so-called "properly ordered") colorings and show that if rainbow triangles are forbidden in $K|_\varphi$ then there can be found a canonically colored complete subgraph on a vertex set of cardinality $|X|$ (Theorem 3). The exclusion of a rainbow K_4, however, is not sufficient, as shown by a suitable coloring for $X = \mathbb{R}$ (the set of real numbers).

1. Rainbow Hamiltonian paths in K_ω

Throughout this section, K denotes the *countable* complete graph with vertex set $X = \{x_1, x_2, \ldots\}$ and edge set $E = \{x_i x_j : i \neq j\}$. We assume there is a 0-1 measure μ on X, i.e., for every $Y \subset X$, $\mu(Y) \in \{0, 1\}, \mu$ is finitely additive, $\mu(X) = 1$, and $\mu(Y) = 0$ for all finite $Y \subset X$.

For convenience, we denote the colors by integers 1,2,.... Two colorings φ, φ' of a graph G are said to be isomorphic if φ' can be obtained from φ, as well as φ from φ', by renumbering (but not indentifying) the colors. In this sense, two edge-colored graphs G_1, G_2 are isomorphic if for their colorings φ_1, φ_2 we have $G_1|_{\varphi_1} \cong G_2|_{\varphi_2}$, i.e., there is a one-to one mapping between the vertex sets $V(G_1)$ and $V(G_2)$, yielding the isomorphism of φ_1 und φ_2.

Denote by Z^* the complete graph with a countable vertex set $\{z_0, z_1, z_2, \ldots\}$ and having the (canonical) edge coloring in which $z_i z_j$ has color j whenever $i < j$.

Theorem 1. *Suppose $\varphi : E \to \mathbb{N}$ is a coloring of K, such that for each vertex x_i and each color j, the vertices adjacent to x_i by an edge of color j form a set of measure 0. If $K|_{\varphi}$ contains no subgraph isomorphic to Z^* then K has a one-way infinite and a two-way infinite rainbow Hamiltonian path.*

Proof. We construct a sequence $P_1, P_2, \ldots$ of (finite) rainbow paths with the following properties: $x_i \in P_i$ for all $i \geq 1$, and $P_i \subset P_{i+1}$ in the sense that all edges of P_i are edges of P_{i+1} too. This clearly implies that $\cup P_i$ is a rainbow Hamiltonian path of K.

Let $P_1 = (x_1), P_2 = (x_1 x_2)$. If the Hamiltonian path to be found is one-way infinite then we extend P_i at the end different from x_1; if it should be two-way infinite, we extend P_i at the end being closer to x_1.

Suppose P_i is a rainbow path covering $\{x_1, \ldots, x_i\}$. If $x_{i+1} \in P_i$ define $P_{i+1} = P_i$. Otherwise, denote by y_j the j^{th} vertex of P_i, i.e., $P_i = (y_1 y_2 \ldots y_k)$ where $k = |P_i|$. Set $Y = X \backslash (\{x_{i+l}\} \cup \{y_l, \ldots, y_k\})$.

Delete all vertices y from Y, for which $\varphi(y_k y)$ or $\varphi(x_{i+l} y)$ appears on some edge of P_i. The resulting vertex set Y' has $\mu(Y') = 1$, since each of the $k-1$ colors appearing in P_i defines a neighborhood of x_{i+1} and y_k of measure 0 (and μ is finitely additive). If there is a $y \in Y'$ such that $\varphi(x_{i+1} y) \neq \varphi(y_k y)$ then $P_{i+1} = (y_1 \ldots y_k y x_{i+1})$ is a rainbow path containing x_{i+1}. Otherwise, $\varphi(x_{i+1} y) = \varphi(y_k y)$ for all $y \in Y'$. Let $Y_1 \cup Y_2 \cup \ldots = Y'$ be the partition of Y' in which two vertices y and y' belong to the same class if and only if $\varphi(y_k y) = \varphi(y_k y')$. Then $\mu(Y_m) = 0$ for all $m \geq 1$.

Choose an arbitrary $y' \in Y'$, and delete all y from Y' for which $\varphi(y' y)$ appears in P_i or is identical to $\varphi(y_k y')$. The set of those y

is of measure 0, so that the resulting set Y'' has $\mu(Y'') = 1$. If, for some $y'' \in Y''$, $\varphi(y''y') \neq \varphi(y''y_k)$ then $P_{i+1} = (y_1 \dots y_k y'' y' x_{i+1})$ is a rainbow path containing x_{i+1} and we are home. Otherwise, choose a $y'' \in Y''$ and repeat the same argument. Either a rainbow P_{i+1}, containing x_{i+1}, is found after a finite number of steps, or an infinite sequence y', y'', y''', ... of vertices is defined with the property that $\varphi(y^{(p)}y^{(q)} = \varphi(y_k y^{(q)})$ for all $p < q$. In the latter case, however, those vertices would induce a subgraph isomorphic to Z^*, condradicting our assumptions, so that P_i can be extended to a rainbow path P_{i+1}, for all i.

$\diamondsuit$

Corollary 1.1. (Hahn and Thomassen [6]) *If all monochromatic subgraphs are locally finite in a Z^*-free coloring of K, then K contains a rainbow Hamiltonian path.*

$\diamondsuit$

An interesting particular case is when any two edges of the same color in $K|_\varphi$ are vertex-disjoint. Such a φ is called a *proper edge coloring* of K.

Corollary 1.2. *Every proper edge coloring of K contains a rainbow Hamiltonian path.*

$\diamondsuit$

Though Z^* itself contains a rainbow Hamiltonian path, it is very close to being non-Hamiltonian in the following sense. Denote by Z^Δ the graph which is obtained from Z^* by recoloring the edge z_0z_1 to color 2.

Proposition 2. *The graph Z^Δ contains no rainbow Hamiltonian paths.*

Based on a similar idea, the following more general class of examples can be given. Consider an arbitrary complete graph K_n on n vertices, with a coloring φ_n which does not contain a rainbow Hamiltonian path. Suppose φ_n uses colors 1',2',..., none of them appearing among the colors 1,2,.... Replace z_0 by $K_n|_{\varphi_n}$ in Z^*, and define the edge z_iy to have color i, whenever $y \in V(K_n)$ and $i \geq 1$. Denote this edge-colored graph by $Z^*(\varphi_n)$. Now Proposition 2 can be stated in the following stronger form.

Proposition 2'. *If $K_n|_{\varphi_n}$ contains no rainbow Hamiltonian path then neither does $Z^*(\varphi_n)$.*

Proof. Suppose to the contrary that P is a rainbow Hamiltonian path in $Z^*(\varphi_n)$. Then the vertices of K_n induce at least two subpaths P_1, P_2 (both maximal under inclusion) in P. We may assume all vertices between P_1 and P_2 belong to Z^* z_0. Let z_m be the vertex between P_1 and P_2 in P having maximum subscript. Then the two neighbors of z_m in P are adjacent to z_m by edges of color m, contradicting the assumption that P is rainbow.

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In particular, any coloring of K_n with at most n-2 colors satisfies the assumptions on φ_n.

2. Canonically colored subgraphs

In this section we consider infinite complete graphs $K = (X, E)$ with a vertex set X of arbitrary cardinality. We assume there is an ordering $<$ given on X.

Erdős and Rado [2] proved that every coloring φ of K contains an infinite $Y \subset X$ such that $Y|_\varphi$ is rainbow or monochromatic or, $\varphi(yy') = \varphi(yy'')$ either for all $y < y' < y''$ or for all $y'' < y' < y$ $(y, y', y'' \in Y)$.

Call a $Y \subset X$ *cannonically colored* if for all $y, y', y'' \in Y$, $y < y' < y''$, $\varphi(yy') = \varphi(yy'')$. We are interested in the question how large canonically colored complete subgraphs must exist in $K|_\varphi$. The following particular class of colorings will be considered. We say that φ is *properly ordered* if $\varphi(xx') \neq \varphi(xx'')$ whenever $x'' < x' < x(x, x', x'' \in X)$.

Theorem 3. *Let φ be a properly ordered coloring of K, not containing rainbow triangles. Then there is a $Y \subset X$, $|Y| = |X|$, such that $Y|_\varphi$ is canonically colored.*

Proof. For any three elements $x, y, z \in X$, $x < y < z$, either $\varphi(xy) = \varphi(xz)$ or $\varphi(xy) = \varphi(yz)$, since $\varphi(xz) \neq \varphi(yz)$.

If X contains a maximum element x_0 then set $X' = X \setminus \{x_0\}$; otherwise, $X' = X$. Now any two monochromatic edges of X' share a vertex. Indeed, suppose $\varphi(uv) = \varphi(yz)$. Choose an $x \in X$ such that $x > \max(u, v, y, z)$. Then there is an edge of color $\varphi(uv)$ that joins x to uv and also to yz. Those two edges must coincide, however, since we have a properly ordered coloring.

Thus, each monochromatic subgraph of $X'|_\varphi$ is a star, since monochromatic triangles cannot occur in properly ordered colorings.

Call a monochromatic star non-trivial if it contains at least two edges. Such a star has a (unique) centre, the common vertex of its edges. Observe that every $x \in X'$ is the centre of at most one (non-trivial) star. Otherwise, let $\varphi(xy) = \varphi(xy') \neq \varphi(xz) = \varphi(xz')$. Choose a $w \in X$, $w > \max(x, y, y', z, z')$. Then $\varphi(xy) = \varphi(xw) = \varphi(xz)$ should hold, a contradiction. Since each triangle contains a pair of monochromatic edges, there are at most two vertices x', x'' that are not centres of some star. Set $X'' = X' \setminus \{x', x''\}$.

Thus, each $x \in X''$ is the centre of exactly one non-trivial star S_x. Renumbering the colors, if necessary, we may assume S_x is colored by color x. We define a partition $X_1 \cup X_2 = X''$ as follows: $x \in X_1$ if $y < x$ implies $\varphi(xy) \neq x$; $x \in X_2$ if there is a $y < x$ with $\varphi(xy) = x$. The proof will be done if we show $X_1|_\varphi$ and $X_2|_\varphi$ are both canonically colored.

Suppose $x \in X_2$. If there were a $z > x$ such that $\varphi(xz) \neq x$ then $\varphi(yz) = x$ would follow for any $y, \varphi(xy) = x$, a contradiction as S_y cannot have color x. Hence, X_2 is canonically colored, and $y \in X_1$ whenever $\varphi(xy) \neq y$, $y < x$.

Suppose X_1 is not canonically colored, i.e., there are three elements $x,y,z \in X_1$, $x < y < z$, $\varphi(xy) = a \neq b = \varphi(xz)$. Then $\varphi(yz) = a$ (since φ is properly ordered), so that $y \in X_2$, contrary to our assumption.

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We note that the above argument yields the following result for the finite case.

Theorem 3'. *Every properly ordered coloring of K_n with no rainbow triangle contains a canonicaly colored $K_{\lfloor n/2 \rfloor - 1}$.*

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Instead of K_3, the exclusion of a rainbow K_4 is not sufficient in Theorem 3. This fact can be proved in the following stronger form. ($\mathbb{R}$ denotes the set of real numbers.)

Theorem 4. *For $X = \mathbb{R}$, there exists a properly ordered coloring φ with the following properties:*

(i) *Every canonically colored Y is countable;*

(ii) *$X|_\varphi$ contains no rainbow finite subgraphs of minimum degree greater than 2. (In particular, $X|_\varphi$ is rainbow-K_4-free.)*

Proof. First, consider the properly ordered (canonical) coloring φ^+ defined by $\varphi^+(xy) = x$ for all $x < y$. We modify φ^+ by splitting each color class into two parts, and replacing each color x by two colors x', x''. (Clearly, after any kind of splitting, the obtained coloring remains properly ordered.)

The splitting is based on idea due to Sierpiński [5]. Consider a well-ordering $<_L$ of $\mathbb{R}$. For $x < y$, define $\varphi(xy)$ to be x' if $x <_L y$ and to be x'' if $y <_L x$. Let $Y|_\varphi$ be canonically colored, for some $Y \subset X = \mathbb{R}$. We show Y is countable.

Set $E_x = \{xy : x < y \in Y\}$ for $x \in Y$. If Y is canonically colored then each E_x is monochromatic. Divide Y into two (disjoint) parts Y_1, Y_2 as follows: $x \in Y_1$ if E_x has color x' and $x \in Y_2$ if E_x has color x''. By the definition of $<_L$, for each $x \in Y_1$, the set $\{y \in Y_1 : y > x\}$ contains a minimum element y_x. Picking a rational number from the interval $[x, y_x)$, it follows that Y_1 is countable. By a similar argument, considering the sets $\{y \in Y_2 : y < x\}$ and the intervals $(y_x, x]$, it follows that Y_2 is countable.

Let G be a finite rainbow subgraph of $X|_\varphi$, with a vertex set $\{x_1, \ldots, x_n\}$. Then $x = \min x_i$ has degree at most 2, since all edges incident to x in G have color x' or x''.

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3. Concluding remarks

<u>I.</u> Corollary 1.2 is much easier to prove than Theorem 1. As a matter of fact, in a proper edge coloring, P_i can be extended to a suitable P_{i+1} by adding x_{i+1} and at most one extra vertex. The finite version of Corollary 1.2, however, is unknown. A nice construction of Maamoun and Meyniel [3] shows there is a proper edge coloring of the complete graph K_n on $n = 2^k$ vertices (for all $k \geq 2$) not containing a rainbow Hamiltonian path. It would be interesting to see such colorings for all

even n.

On the other hand, Andersen [1] conjectures that every proper edge coloring of K_n contains a rainbow path covering all vertices but one. Some lower bounds on the length of a maximum rainbow path are given by Rödl und Tuza [4]. Here we raise the following related question.

Problem 5. Find the minimum number $f(n)$ of colors, such that every proper edge coloring of K_n by at least $f(n)$ colors contains a rainbow Hamiltonian path.

The examples of [3] show $f(n) \leq n-1$ does not hold in general. It seems to be reasonable to conjecture, however, that $f(n)$ is very close (or, perhaps, equal) to n.

II. All our examples for colorings of a countable complete graph without a rainbow Hamiltonian path have a canonical structure (cf. Proposition 2). Now the following two problems arise.

Problem 6. (a) Find a class $\underline{F}$ of edge-colored countable complete graphs with the following properties:

(i) No $F \in \underline{F}$ contains a rainbow Hamiltonian path.
(ii) All infinite complete subgraphs of $K|_\varphi$ have a rainbow Hamiltonian path if and only if $K|_\varphi$ contains no subgraph isomorphic to any $F \in \underline{F}$.

(b) Do all $f \in \underline{F}$ have a canononical structure?

III. It is easy to show there is a subset $\{a_1, a_2, \ldots\}$ of the natural numbers such that every positive integer occurs exactly once among the numbers and $|a_i - a_j|$, $1 \leq i < j$. In other words, if color $|i-j|$ is assigned to edge $x_i x_j$ then in this coloring of K_ω some rainbow complete subgraph contains all colors. This observation leads to the following questions.

Problem 7. (a) Under what conditions does a countable (or an arbitrary infinite) complete graph K contain a rainbow complete subgraph involving all colors that appear in K?
(b) Find theorems of this type for *finite* complete graphs.
(c) Let $0 < a_1 < a_2 < \ldots < a_k$, and suppose that for each integer $i, 1 \leq i \leq n$, there is exactly one pair $j, m (1 \leq j < m \leq k)$ such that

$a_m - a_j = i$. Find $a(n) = \min a_k$. Also, find the minimum value of $k = k(n)$, for which such a sequence $a_1, \dots, a_k$ exists.

Note that a greedy argument shows $a(n) \le 0(n^3)$. In fact, there exists an infinite sequence $a_1, a_2, \dots$ with $a_n \le cn^3$ (for some constant c), whose $(2n)^{th}$ slice satisfies the requirements, for all $n \ge 1$.

IV. Concerning Theorem 3, one should ask that, instead of triangles, what sort of rainbow subgraphs F can be excluded so that $X|_\varphi$ must contain a canonically colored subgraph Y of cardinality $|Y| = |X|$. Theorem 4 shows F always has minimum degree at most 2 (when F is finite).

References

[1] ANDERSON, L.D.: Hamilton circuits with many colours in properly edge-coloured complete graphs, *Math. Scandin.* (to appear).

[2] ERDŐS, P. and RADO, R.: A combinatorial problem, *J. London Math. Soc.* **25** (1950), 249-255.

[3] MAAMOUN, M. and MEYNIEL, H.: On a problem of G. Hahn about coloured Hamiltonian paths in K_{2n}, *Discrete Math.* **51** (1984), 213-214.

[4] RÖDL, V. and TUZA, Zs.: Rainbow subraphs in properly edge-colored graphs, (to appear).

[5] SIERPIŃSKI, W.: Sur un problème de la théorie des relations, *Annali R. Scuola Normale Superiore de Pisa*, Ser. 2, **2** (1933), 285-287.

[6] HAHN, G. and THOMASSEN, C.: Path and cycle sub-Ramsey numbers and an edge-colouring conjecture, *Discrete Math.* **62** (1986), 29-33.