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L-MULTIFUNCTIONS AND THEIR PROPERTIES

Marian **Matłoka**

Department of Mathematics, University of Economics, 6-967 Poznań, Poland

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Abstract: In this paper a concept of L-multifunction is introduced and other related objects are defined. Next their properties are presented.

1. Introduction

The notion of Cartesian product plays an important role in the usual theory of functions and multifunctions. The Cartesian product of two fuzzy subsets $A \in I^X$ and $B \in I^Y$ may be defined as the subset $A \times B$ of $X \times Y$ characterized by $(A \times B)(x, y) = \min(A(x), B(y))$. This definition has the inconvenience that when $A \times B$ is known and $A \times B \neq \neq \emptyset$, it is impossible to retrieve again the subsets A and B. The notion of fuzzy Cartesian product which is introduced in paper [1] is free from this inconvenience. The L-multifunctions which are introduced in this paper are the subsets of a special case of Cartesian product and also free from this inconvenience. We will introduce and develop basic ideas of the L-multifunction theory, necessary for our further considerations on economical systems.

2. Introduction and general properties of L-multifunctions

Let X, Y and Z denote arbitrary but for further considerations fixed reference spaces. Next $\mathcal{P}(X)$, $\mathcal{P}(Y)$ and $\mathcal{P}(Z)$ denote respectively the families of all non-void subsets of X, Y and Z.

Definition 2.1. An *L-multifunction*, $F : X \to \mathcal{P}(Y)$ say, is a subset of the Cartesian product $X \times \mathcal{P}(Y) \times I \times I^Y$ satisfying the following conditions:

(i) if $(x, B, r, f) \in F$, then $\operatorname{supp} f = B$,

(ii) if $(x, B, r, f) \in F$, then $r = 0$ implies $B = \emptyset$ and $r = 1$ implies $f(y) = 1$ for any $y \in B$,

(iii) if $(x, B, r_1, f_1) \in F$ and $(x, B, r_2, f_2) \in F$, then $r_1 > r_2$ implies $f_1 \geq f_2$.

Let $\{x, r\}$ denote a fuzzy singleton in X with support x and value r. This fuzzy singleton is now transformed by F to a family of the fuzzy subsets in Y.

Definition 2.2. A *converse L-multifunction*, F^{-1} say, to an L-multifunction $F : X \to \mathcal{P}(Y)$ is a subset of the Cartesian product $Y \times \mathcal{P}(X) \times \times I \times I^X$ satisfying the following condition:

— $(y, A, t, h) \in F^{-1}$ if there exists $(x, B, r, f) \in F$ such that $x \in A$, $y \in B$, $f(y) = t$, $h(x) = r$.

Definition 2.3. A *composite*, $G \circ F : X \to \mathcal{P}(Z)$ say, of two L-multifunctions $F : X \to \mathcal{P}(Y)$ and $G : Y \to \mathcal{P}(Z)$ is an L-multifunction such that

— $(x, C, r, h) \in G \circ F$ iff there exist $(x, B, r, f) \in F$ and $(y, C, t, h) \in \in G$ such that $Y \in B$, $f(y) = t$.

Let X, Y and Z denote the linear spaces.

Definition 2.4. An L-multifunction, $F : X \to \mathcal{P}(Y)$ say, is called *conical* iff for any $(x, B, r, f) \in F$ and for any $\alpha > 0$ $(\alpha x, \alpha B, r, \alpha f) \in \in F$, where $(\alpha f)(y) = f\left(\frac{1}{\alpha} y\right)$ for any $y \in Y$.

Theorem 2.1. *If an L-multifunction is conical, then its converse L-multifunction is conical too.*

Proof. As a matter of fact, let F be a conical L-multifunction. Let $(y, A, t, h) \in F^{-1}$. So, taking into account Def. 2 there exists $(x, B, r, f) \in F$ such that $x \in A$, $y \in B$, $f(y) = t$, $h(x) = r$. So, with respect to Def. 4 for any $\alpha > 0$ we have $(\alpha x, \alpha B, r, \alpha f) \in F$.

Moreover $\alpha x \in \alpha A$, $\alpha y \in \alpha B$, $\alpha f(\alpha y) = f(y) = t$, $\alpha h(\alpha x) = h(x) = r$. This means that $(\alpha y, \alpha A, t, \alpha h) \in F^{-1}$. So, F^{-1} is a conical L-multifunction. $\Diamond$

Theorem 2.2. *If F and G are conical L-multifunctions then $G \circ F$ is a conical L-multifunction too.*

Proof. Let $(x, C, r, h) \in G \circ F$. So, taking into account Def. 3 there exist $(x, B, r, f) \in F$ and $(y, C, t, h) \in G$ such that $y \in B$, $f(y) = t$. F and G are conical L-multifunctions, so for any $\alpha > 0$ we have $(\alpha x, \alpha B, r, \alpha f) \in \in F$ and $(\alpha y, \alpha C, t, \alpha h) \in G$. Because $y \in B$, $f(y) = t$, so $\alpha y \in \alpha B$, $\alpha f(\alpha y) = f(y) = t$. This means that $(\alpha x, \alpha C, r, \alpha h) \in G \circ F$. $\Diamond$

Definition 2.5. An L-multifunction, $F : X \to \mathcal{P}(Y)$ say, is called *superadditive* iff for any $(x_1, B_1, r_1, f_1) \in F$ and $(x_2, B_2, r_2, f_2) \in F$, we have $(x_1 + x_2, B_1 + B_2, \min(r_1, r_2), f_1 + f_2) \in F$, where $(f_1 + f_2)(y) = = \sup_{y_1 + y_2 = y} \min(f_1(y_1), f_2(y_2))$ for any $y \in Y$.

Theorem 2.3. *If an L-multifunction is superadditive then its converse L-multifunction is superadditive as well.*

Proof. In point of fact, let an L-multifunction, $F : X \to \mathcal{P}(Y)$ say, satisfy the assumption of the theorem. Let

$$(y_1, A_1, t_1, h_1) \in F^{-1} \quad \text{and} \quad (y_2, A_2, t_2, h_2) \in F^{-1}.$$

Then from the Def. 2 it follows that there exist

$$(x_1, B_1, r_1, f_1) \in F \quad \text{and} \quad (x_2, B_2, r_2, f_2) \in F$$

such that

$$x_1 \in A_1, \quad y_1 \in B_1, \quad x_2 \in A_2, \quad y_2 \in B_2,$$
$$h_1(x_1) = r_1, \quad h_2(x_2) = r_2, \quad f_1(y_1) = t_1, \quad f_2(y_2) = t_2.$$

We can assume that x_1, x_2, f_1 and f_2 are such that

$$h_1(x_1) = \sup_{x \in A_1} h_1(x) = r_1, \quad h_2(x_2) = \sup_{x \in A_2} h_2(x) = r_2$$

and

$$f_1(y_1) = \sup_{y \in B_1} f_1(y) = t_1, \quad f_2(y_2) = \sup_{y \in B_2} f_2(y) = t_2.$$

Because F is superadditive L-multifunction, so

$$(x_1 + x_2, B_1 + B_2, \min(r_1, r_2), f_1 + f_2) \in F.$$

Moreover

$$x_1 + x_2 \in A_1 + A_2, (h_1 + h_2)(x_1 + x_2) = \min(r_1, r_2)$$

and

$$y_1 + y_2 \in B_1 + B_2, (f_1 + f_2)(y_1 + y_2) = \min(t_1, t_2)\,.$$

This means that

$$(y_1 + y_2, A_1 + A_2, \min(t_1, t_2), h_1 + h_2) \in F^{-1}\,.$$

So, F^{-1} is a superadditive L-multifunction. $\Diamond$

Theorem 2.4. *If F and G are superadditive L-multifunctions then $G \circ F$ is a superadditive L-multifunction too.*

Proof. Let $(x_1, C_1, r_1, h_1) \in G \circ F$ and $(x_2, C_2, r_2, h_2) \in G \circ F$. Then, from the Def. 3 it follows that there exist $(x_1, B_1, r_1, f_1) \in F$, $(y_1, C_1, t_1, h_1) \in G$ such that $y_1 \in B_1$, $f_1(y_1) = t_1$ and there exist $(x_2, B_2, r_2, f_2) \in F$, $(y_2, C_2, t_2, h_2) \in G$ such that $y_2 \in B_2$, $f_2(y_2) = t_2$. We can assume that

$$f_1(y_1) = \sup_{y \in B_1} f_1(y) = t_1 \quad \text{and} \quad f_2(y_2) = \sup_{y \in B_2} f_2(y) = t_2\,.$$

Because F and G are superadditive L-multifunctions, so

$$(x_1 + x_2, B_1 + B_2, \min(r_1, r_2), f_1 + f_2) \in F$$

and

$$(y_1 + y_2, C_1 + C_2, \min(t_1, t_2), h_1 + h_2) \in G\,.$$

Moreover

$$y_1 + y_2 \in B_1 + B_2 \quad \text{and} \quad (f_1 + f_2)(y_1 + y_2) = \min(t_1, t_2)\,.$$

This means that

$$(x_1 + x_2, C_1 + C_2, \min(r_1, r_2), h_1 + h_2) \in G \circ F\,,$$

i.e. $G \circ F$ is superadditive L-multifunction. $\Diamond$

Definition 2.6. By the *graph* of L-multifunction, $F : X \to \mathcal{P}(Y)$ say, it is understood a set W_F of the elements $(x, y, r, t) \in X \times Y \times I \times I$ such that there exist $B \in \mathcal{P}(Y)$ and $f \in I^Y$ satisfying the following conditions:

(i) $y \in B$,
(ii) $t = f(y)$,
(iii) $(x, B, r, f) \in F$.

Definition 2.7. Let $\alpha, \beta \in I$. An *α, β-cut* of W_F, $W_F^{\alpha,\beta}$ in symbol, is a set of the elements $(x, y) \in X \times Y$ such that for $r \geq \alpha$ and $t \geq \beta$ $(x, y, r, t) \in W_F$.

Theorem 2.5. *If $F : X \to \mathcal{P}(Y)$ is a conical L-multifunction then for any $\alpha, \beta \in I$, the α, β-cut of W_F is a cone.*

Proof. Let $(x, y) \in W_F^{\alpha,\beta}$. Then for $r \geq \alpha$ and $t \geq \beta$, $(x, y, r, t) \in$ $\in W_F$. This means that there exist $B \in \mathcal{P}(Y)$ and $f \in I^Y$ such that

$y \in B$, $t = f(y)$ and $(x, B, r, f) \in F$. Because F is a conical L-multifunction, so for any $\lambda > 0$, $(\lambda x, \lambda B, r, \lambda f) \in F$. Moreover $\lambda y \in \lambda B$, $t = \lambda f(\lambda y) = f(y)$. This means that $(\lambda x, \lambda y, r, t) \in W_F$ and finally $(\lambda x, \lambda y) \in W_F^{\alpha,\beta}$. $\Diamond$

Theorem 2.6. *If F is a conical and superadditive L-multifunction then for any $\alpha, \beta \in I$, $W_F^{\alpha,\beta}$ is a convex set.*

Proof. Let (x_1, y_1), $(x_2, y_2) \in W_F^{\alpha,\beta}$. Then for any $r_1, r_2 \geq \alpha$ and $t_1, t_2 \geq \beta$

$$(x_1, y_1, r_1, t_1) \in W_F\,, \quad (x_2, y_2, r_2, t_2) \in W_F\,.$$

This means that there exist $B_1, B_2 \in \mathcal{P}(Y)$ and $f_1, f_2 \in I^Y$ such that $y_1 \in B_1$, $t_1 = f_1(y_1)$, $y_2 \in B_2$, $f_2(y_2) = t_2$ and $(x_1, B_1, r_1, f_1) \in F$, $(x_2, B_2, r_2, f_2) \in F$. Because F is a conical and superadditive L-multifunction so for any $\lambda > 0$

$$(\lambda x_1, \lambda B_1, r_1, \lambda f_1) \in F\,, \quad ((1-\lambda)x_2(1-\lambda)B_2\,, r_2\,, (1-\lambda)f_2) \in F\,,$$

and

$$(\lambda x_1 + (1-\lambda)x_2\,, \lambda B_1 + (1-\lambda)B_2\,, \min(r_1, r_2)\,, \lambda f_1 + (1-\lambda)f_2) \in F\,.$$

This means that

$$(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2, \min(r_1, r_2), \min(t_1, t_2)) \in W_F\,.$$

Because $\min(r_1, r_2) \geq \alpha$ and $\min(t_1, t_2) \geq \beta$ so

$$(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2) \in W_F^{\alpha,\beta}\,,$$

i.e. a set $W_F^{\alpha,\beta}$ is convex. $\Diamond$

3. Some topological properties of L-multifunctions

Now, let us assume that the reference spaces X, Y and Z are finite dimensional Euclidean spaces.

Definition 3.1. An L-multifunction, $F : X \to \mathcal{P}(Y)$ say, is called *closed* iff its graph W_F is a closed set.

Corollary. *For any closed L-multifunction its converse L-multifunction is closed.*

Definition 3.2. An L-multifunction $F : X \to \mathcal{P}(Y)$ say, is *sequentially bounded* iff for any bounded sequence $S = \{x_n\}$ and any sequence $R = \{r_n\}$, $x_n \in X$, $r_n \in (0, 1\rangle$, the set

$$\{(y, t) \in Y \times I : (x_n, y, r_n, t) \in W_F, x_n \in S, r_n \in R\}$$

is bounded.

Theorem 3.1. *If $F : X \to \mathcal{P}(Y)$ and $G : Y \to \mathcal{P}(Z)$ are closed L-multifunctions and F is sequentially bounded, then $G \circ F$ is a closed L-multifunction.*

Proof. Let $(x_n, z_n, r_n, p_n) \in W_{G\circ F}$ and let $(x_n, z_n, r_n, p_n) \to (x_0, z_0, r_0, p_0)$ as $n \to \infty$ (the convergence may be taken with respect to each coordinate separately), $x_n \in X$, $z_n \in Z$, $r_n, p_n \in (0,1\rangle$. We will prove that $(x_0, z_0, r_0, p_0) \in W_{G\circ F}$. In fact, for any n, (x_n, z_n, r_n, p_n) belongs to $W_{G\circ F}$ iff there exist $C_n \in Z$, $h_n \in I^Z$ such that $(x_n, C_n, r_n, h_n) \in \in G \circ F$ and $z_n \in C_n$, $h_n(z_n) = p_n$. An element $(x_n, C_n, r_n, h_n) \in G \circ F$ iff there exist $(x_n, B_n, r_n, f_n) \in F$ and $(y_n, C_n, t_n, h_n) \in G$ such that $y_n \in B_n$ and $f_n(y_n) = t_n$. From the above conditions it follows that

$$(x_n, y_n, r_n, t_n) \in W_F \quad \text{and} \quad (y_n, z_n, t_n, p_n) \in W_G\,.$$

Because F is a sequentially bounded and closed L-multifunction we observe that the sequences $\{y_n\}$ and $\{t_n\}$ are bounded and without losing generality we may assume that $y_n \to y_0$ and $t_n \to t_0$ as $n \to \infty$. Moreover

$$(x_0, y_0, r_0, t_0) \in W_F \quad \text{and} \quad (y_0, z_0, t_0, p_0) \in W_G\,.$$

This means that there exist $B_0 \in \mathcal{P}(Y)$, $f_0 \in I^Y$ such that $y_0 \in \in B_0$, $f_0(y_0) = t_0$, $(x_0, B_0, r_0, f_0) \in F$ and there exist $C_0 \in \mathcal{P}(Z)$, $h_0 \in I^Z$ such that $z_0 \in C_0$, $h_0(z_0) = p_0$, $(y_0, C_0, t_0, h_0) \in G$. This means that $(x_0, C_0, r_0, h_0) \in G \circ F$. Because $z_0 \in C_0$, $h_0(z_0) = p_0$, so $(x_0, z_0, r_0, p_0) \in W_{G\circ F}$. $\Diamond$

Theorem 3.2. *If an L-multifunction $F : X \to \mathcal{P}(Y)$ is closed and conical and for any $r, t \in I$, $(0, y, r, t) \notin W_F$ for $y \neq 0$, then F is a sequentially bounded L-multifunction.*

Proof. According to Def. 3.2 it suffices to show that for any bounded sequence $S = \{x_n\}$ and any sequence $R = \{r_n\}$, $x_n \in X$, $r_n \in (0,1\rangle$ the set $T = \{(y,t) \in Y \times I : (x_n, y, r_n, t) \in W_F,\ x_n \in S,\ r_n \in R\}$ is bounded. Suppose that the set T is unbounded for some S and some R. Then there exist the sequences $\{y_n\}$, $\{t_n\}$, $(y_n, t_n) \in T$ such that $||y_n|| \to \infty$ as $n \to \infty$. But $(x_n, y_n, r_n, t_n) \in W_F$ and F is a conical L-multifunction, so $(x_n/||y_n||, y_n/||y_n||, r_n, t_n) \in W_F$. Hence, there exist subsequences x_{n_k} y_{n_k}, r_{n_k}, t_{n_k} such that

$$(x_{n_k}/||y_{n_k}||, y_{n_k}/||y_{n_k}||, r_{n_k}, t_{n_k}) \to (0, y_0, r_0, t_0)$$

as $k \to \infty$, where $y_0 \neq 0$ because $\lim\limits_n y_{n_k}/||y_{n_k}|| = 1 = y_0$. Because F is a closed L-multifunction, so $(0, y_0, r_0, t_0) \in W_F$ for $y_0 \neq 0$, a contradiction. $\Diamond$

Definition 3.3. A *fixed point* of the L-multifunction $F : X \to \mathcal{P}(X)$ is an element $\overline{x} \in X$ such that there exist $r, t \in I$ such that $(\overline{x}, \overline{x}, r, t) \in \in W_F$.

Theorem 3.3 (Fixed point theorem). *Let C be a nonempty, convex and compact subset of X. If $F : C \to \mathcal{P}(C)$ is a closed, conical and superadditive L-multifunction, then F has a fixed point in C.*

Proof. Let us consider a point-to-set mapping $\hat{F} : C \to \mathcal{P}(C)$ such that for any $x \in C$

$$\hat{F}(x) = \{y \in C : \exists r, t, \in I, (x, y, r, t) \in W_F\}.$$

First we will prove that $\overline{x}$ is a fixed point of F iff $\overline{x}$ is a fixed point of $\hat{F}$. If $\overline{x}$ is a fixed point of $\hat{F}$, then $\overline{x} \in \hat{F}(\overline{x})$. This means that there exist $r, t \in I$ such that $(\overline{x}, \overline{x}, r, t) \in W_F$, i.e. $\overline{x}$ is a fixed point of F. Now, if $\overline{x}$ is a fixed point of F then from Def. 3.3 it follows that there exist $r, t \in I$ such that $(\overline{x}, \overline{x}, r, t) \in W_F$. This means that $\overline{x} \in \hat{F}(\overline{x})$, i.e. $\overline{x}$ is a fixed point for $\hat{F}$. $\Diamond$

Now, we will show that $\hat{F}$ satisfies the hypothesis of Kakutani fixed point theorem, i.e. that $\hat{F}$ is a closed mapping and for any $x \in C$ $\hat{F}(x)$ is a convex set.

Let y_1, y_2 be elements from $\hat{F}(x)$. From the definition of $\hat{F}$ it follows that there exist elements $r_1, r_2, t_1, t_2 \in I$ such that $(x, y_1, r_1, t_1) \in \in W_F$ and $(x, y_2, r_2, t_2) \in W_F$. This means that there exist $B_1, B_2 \in \in \mathcal{P}(C)$ and $f_1, f_2 \in I^C$ such that

$$(x, B_1, r_1, f_1) \in F, \quad (x, B_2, r_2, f_2) \in F$$

and

$$y_1 \in B_1, \quad y_2 \in B_2, \quad f_1(y_1) = t_1, \quad f_2(y_2) = t_2.$$

Because F is a conical and superadditive L-multifunction, so for any $\alpha \geq 0$

$$(x, \alpha B_1 + (1-\alpha)B_2, \min(r_1, r_2), \alpha f_1 + (1-\alpha)f_2) \in F.$$

This means that

$$(x, \alpha y_1 + (1-\alpha)y_2, \min(r_1, r_2), t) \in W_F,$$

where

$$t = (\alpha f_1 + (1-\alpha)f_2)(\alpha y_1 + (1-\alpha)y_2),$$

i.e. $\alpha y_1 + (1-\alpha)y_2 \in \hat{F}(x)$.

Now, let us consider a sequence $\{x_n\}$, $x_n \in C$ such that $x_n \to x_0$ as $n \to \infty$. Let $y_n \in \hat{F}(x_n)$ and $y_n \to y_0$ as $n \to \infty$. We will prove that $y_0 \in \hat{F}(x_0)$. If $y_n \in \hat{F}(x_n)$ then for any n there exist $r_n, t_n \in I$ such

that $(x_n, y_n, r_n, t_n) \in W_F$. Without losing generality we may assume that $r_n \to r_0$, $t_n \to t_0$ as $n \to \infty$. Because F is a closed L-multifunction, so $(x_0, y_0, r_0, t_0) \in W_F$. This means that $y_0 \in \hat{F}(x_0)$, i.e. $\hat{F}$ is a closed mapping. So, according to the Kakutani theorem there exists $\overline{x} \in C$ such that $\overline{x} \in \hat{F}(\overline{x})$. This means that an L-multifunction F has a fixed point $\overline{x}$ in C. $\Diamond$

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