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ON A PROBLEM OF W. J. FIREY IN CONNECTION WITH THE CHARACTERIZATION OF SPHERES

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Abstract: Let F be an arbitrary C_∞-smooth closed strictly convex hypersurface in the euclidean space $\mathbb{R}_d$. We describe characterizations for F to be a sphere or an ellipsoid in the form that the support function of F is a prescribed function of special curvatures of F. The method of proof consists in the application of the theory of evolutions.

1. Introduction

In 1967 U. Simon ([9], Satz 6.1.) proved that *a C_∞-smooth, strictly convex closed hypersurface (ovaloid) in the euclidean space $\mathbb{R}_d$ is a sphere if its k-th normalized elementary symmetric function H_k of the principal curvatures κ_k of F and its support function $h > 0$ with respect to the inner point o of F are related by*

(1) $$H_k = G(h) \quad (1 \leq k \leq d-1)$$

for a C_1-function G with

(2) $$\frac{dG}{dh} \leq 0.$$

Especially F must be a sphere for $k = 1$ with $H_1 =: H =$ mean curvature and $k = d-1$ with $H_{d-1} =: K =$ Gauss Kronecker curvature of

F. 1953 K. P. Grotemeyer ([5], 3.) showed that hereby assumption (2) cannot be dropped in general by indication of a rotational ellipsoid in R_3 with

$$(3) \quad 2H = G(h) := \left(\frac{\alpha}{\beta}\right)^{\frac{1}{4}} h + \beta \left(\frac{\alpha}{\beta}\right)^{\frac{3}{4}} h^3 \quad (\alpha, \beta = \text{const.} > 0)$$

where obviously

$$(4) \qquad \frac{dG}{dh} > 0.$$

Now the question arises whether a sphere also may be characterized by $H_k = G(h)$ $(k = 1, \dots, d-1)$ for special functions G with $\frac{dG}{dh} > 0$ as

$$(5) \qquad H = c^{-2} \cdot h$$

or

$$(6) \qquad K = c^{-d} \cdot h$$

$(c = \text{const.} > 0)$. We want to emphasize that the question of a characterization of a sphere by (6) just occured in a paper of W. J. Firey (1974 [3]). There Firey investigated the evolution procedure $\{F_t\}_{t \geq 0}$ of the surface of a worn stone in $\mathbb{R}_3$, initially being smooth and strictly convex, which is controlled by the evolution equation

$$(7) \qquad \frac{\partial h_t}{\partial t} = -\alpha v_t K_t \quad (\alpha = \text{const.} > 0)$$

(h_t = support function, v_t = volume and K_t = Gauss curvature of F_t). Under the additional assumption that F_0 (and therefore all the F_t for $t \geq 0$) are centrally symmetric with respect to the origin o he was able to prove that the F_t for $t \to \infty$ contract to the "round point" o. In other words this means that the rescaled surfaces

$$(8) \qquad \tilde{F}_t := \left(\frac{v_0}{v_t}\right)^{\frac{1}{3}} \cdot F_t \quad (t \geq 0)$$

with constant volume v_o converge to a sphere F_∞ of radius $\left(\frac{3v_0}{4\pi}\right)^{\frac{1}{3}}$ about o for $t \to \infty$ in the Hausdorff topology.

A basic tool of Firey's was the following

Lemma 1. *If F is a smooth ovaloid in the euclidean space $\mathbb{R}_d$ $(d \geq 3)$, centrally symmetric with respect to the origin o of $\mathbb{R}_d$, whose support function h and Gauss curvature K are related by*

$$(9) \qquad h = c^d \cdot K \quad (c = \text{const.} > 0)$$

(compare (6)) *then F must be the sphere $S_c(o)$ about o with radius $c > 0$* (see [3], Th. 3 in the case $d = 3$).

Proof. If one denotes the unit sphere about o by Ω and its volume resp. surface area element by ω_d resp. $d\omega$ then the volumes v and v^* of F and its polar surface F^* with respect to o may be computed by

$$v = \frac{1}{d}\int_\Omega h\frac{d\omega}{K} = \frac{1}{d}\int_\Omega c^d d\omega = c^d\omega_d \tag{10}$$

(see (9)) and

$$v^* = \frac{1}{d}\int_\Omega (r^*)^d d\omega = \frac{1}{d}\int_\Omega \frac{d\omega}{h^d} \tag{11}$$

if r^* is the radius function of F^* with respect to o. But trivially

$$r \geq h \tag{12}$$

for the radial function r of F, and combining (10), (11) and (12) with the formula

$$\omega_d = \frac{1}{d}\int_\Omega \frac{h d\omega}{K r^d} = \frac{c^d}{d}\int_\Omega \frac{d\omega}{r^d} \tag{13}$$

(see (9)) for the volume ω_d of Ω we get

$$v \cdot v^* \geq \omega_d \cdot \frac{c^d}{d}\int_\Omega \frac{d\omega}{r^d} = \omega_d \cdot \omega_d\,. \tag{14}$$

But on the other hand the Blaschke–Santalo inequality yields for the minimal value of $v \cdot v^*$, attained at the Santalo point of F which is the origin because of the central symmetry of F, the estimate

$$v \cdot v^* \leq \omega_d \cdot \omega_d\,. \tag{15}$$

Now (14) and (15) imply equality in (12) from which the assertion of Lemma 1 immediately follows. ◊

In this proof the assumption that F is centrally symmetric with respect to o is only used to guarantee that o is the Santalo point of F, needed for the validity of (15). Therefore Firey in his paper [3], p.10 settles the conjecture that *his Lemma 1 holds true without this assumption.* This problem has not yet been solved up to now but we are able to prove similar sphere characterizations as well as, by the same method, a local version of Firey's problem expressed in the following two theorems:

2. Characterizations of spheres

Theorem 1. *If F is an arbitrary smooth ovaloid in $\mathbb{R}_d$ with*

$$h = c^2 \cdot H \tag{16}$$

or

$$h = c^2 \cdot K^{\frac{1}{d-1}} \tag{17}$$

($c = \text{const.} > 0$) *then F must be a sphere about o of radius c:*

$$F = S_c(o)\,.^* \tag{18}$$

Theorem 2. *Let F be an arbitrary smooth ovaloid in $\mathbb{R}_d$ $(d \geq 3)$ the shape of which is sufficiently close to a sphere in the sense of the validity of the inequalities*

$$\kappa_k \geq C(\beta)(\kappa_1 + \ldots + \kappa_{d-1}) \quad (k = 1, \ldots, d-1) \tag{19}$$

for its principal curvatures where $C(\beta)$ is a suitable constant depending only on $\beta > \frac{1}{d-1}$ with

$$C(\beta) < \frac{1}{d-1} \tag{20}$$

and

$$\lim_{\beta \to +\infty} C(\beta) = \frac{1}{d-1}\,. \tag{21}$$

If moreover F fulfils the condition

$$h = c^{(d-1)\beta+1} \cdot K^{\beta} \tag{22}$$

(generalizing (17)) *then F must be a sphere about o with radius c:*

$$F = S_c(o). \tag{23}$$

Remark 1. *In the special case $\beta = 1$* Th. 2 *provides the fact that the sphere locally is the only solution of Firey's problem in the case of no symmetry assumptions for F.*

Proof of Th. 1. We consider the array

$$F_\tau := \gamma(t) \cdot F \tag{24}$$

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*Addendum after submission: the first part of the theorem has just been proven more generally for hypersurfaces with nonnegative mean curvature H by G. Huisken (Asymptotic behaviour for singularities of the mean curvature flow, *J. Diff. Geom.* **31** (1990), 285–299, Th. 4.1).

of hypersurfaces homothetic to F with respect to the origin o where $\gamma(t)$ is a suitable C_∞-function of the parameter $\tau \geq 0$ which fulfils the initial condition

$$\gamma(0) = 1\,. \tag{25}$$

Then we easily compute

$$h_\tau = \gamma(\tau) \cdot h \tag{26}$$

as well as

$$H_\tau = (\gamma(\tau))^{-1} \cdot H \tag{27}$$

and

$$K_\tau = (\gamma(\tau))^{-(d-1)} \cdot K \tag{28}$$

for the support function h_τ, the mean curvature H_τ and the Gauss curvature K_τ of F_τ.

The idea of the proof of Th. 1 is now to choose the factor $\gamma(\tau)$ in (24) in such a way that F_τ is the solution of the well known evolution equation for F:

$$\frac{\partial h_\tau}{\partial \tau} = -H_\tau \tag{29}$$

or

$$\frac{\partial h_\tau}{\partial \tau} = -\left(K_\tau\right)^{\frac{1}{d-1}}\,. \tag{30}$$

Indeed, using (16) and (27) or (17) and (28) we find

$$\frac{\partial h_\tau}{\partial \tau} = \frac{d\gamma(\tau)}{d\tau} \cdot h = \frac{d\gamma(\tau)}{d\tau} \cdot c^2 H = \frac{d\gamma(\tau)}{d\tau} c^2 \gamma(\tau) \cdot H_\tau \tag{31}$$

or

$$\frac{\partial h_\tau}{\partial \tau} = \frac{d\gamma(\tau)}{d\tau} \cdot h = \frac{d\gamma(\tau)}{d\tau} \cdot c^2 K^{\frac{1}{d-1}} = \frac{d\gamma(\tau)}{d\tau} c^2 \gamma(\tau) \cdot (K_\tau)^{\frac{1}{d-1}} \tag{32}$$

such that we have to solve the ordinary differential equation

$$\frac{d\gamma(\tau)}{d\tau} c^2 \gamma(\tau) = -1 \tag{33}$$

under the initial condition (25) in order to obtain (29) or (30). The solution of (33) and (25) is

$$\gamma(\tau) = \left(1 - 2c^{-2}\tau\right)^{\frac{1}{2}} \qquad \left(0 \leq \tau < \frac{1}{2}c^2 =: T\right) \tag{34}$$

and therefore we have to consider the array

$$F_\tau := \left(1 - 2c^{-2}\tau\right)^{\frac{1}{2}} \cdot F \tag{35}$$

during the time interval $[0,T)$ with

(36) $$T := \frac{1}{2}c^2 > 0 \ .$$

The array (35) represents the solution of the evolution procedures (29) and (30) which contract to the origin o. As Firey did in his paper [3] it is important to introduce the "normalized procedures"

(37) $$\hat{F}_\tau := \left(\frac{A_0}{A_\tau}\right)^{\frac{1}{d-1}} \cdot F_\tau$$

(A_τ = total area of F_τ) respectively

(38) $$\tilde{F}_\tau := \left(\frac{v_0}{v_\tau}\right)^{\frac{1}{d}} \cdot F_\tau$$

(v_τ = total volume of F_τ). Obviously in our case the rescaled hypersurfaces $\hat{F}_\tau$ resp. $\tilde{F}_\tau$ coincide with F:

(39) $$\hat{F}_\tau = \tilde{F}_\tau = F \quad (0 \le \tau < T)$$

because of (35).

We can now apply a theorem Gage and Hamilton ([4], p. 70) that says that in the case $d = 2$ the coinciding conditions (29) and (30) imply (in the C_∞-topology)

(40) $$\lim_{\tau \to T} \tilde{F}_\tau = S_c(o)$$

as a circle obeying (16) or (17) so that (18) holds because of (39). Moreover, after another theorem of G. Huisken ([6], p. 238) we have in the case $d > 2$ because of (29) (again in the C_∞-topology)

(41) $$\lim_{\tau \to T} \hat{F}_\tau = S_c(o)$$

as a sphere obeying (16) which yields (18) in connection with (39). Last not least condition (30) in the case $d > 2$ provides

(42) $$\lim_{\tau \to T} \tilde{F}_\tau = S_c(o)$$

(in the C_∞-topology) as a sphere obeying (17) after Th. 1.3 of B. Chow in [2] and so again (18) holds true because of (39) as the convergence of the rescaled hypersurfaces $\tilde{F}_\tau$ is an obvious consequence of (39)[1)]. All these facts complete the proof of Th. 1. $\Diamond$

[1)] We need the convergence of the $\tilde{F}_\tau$ because there is a gap in the proof of Chow of this fact.

Proof of Th. 2. In the same manner as in the proof of Th. 1 we consider the array (24) for which we conclude, inserting (22) together with (28),

$$\begin{aligned}\frac{\partial h_\tau}{\partial \tau} &= \frac{d\gamma(\tau)}{d\tau}h = \frac{d\gamma(\tau)}{d\tau}c^{(d-1)\beta+1}K^\beta = \\ &= \frac{d\gamma(\tau)}{d\tau}c^{(d-1)\beta+1}(\gamma(\tau))^{(d-1)\beta}(K_\tau)^\beta\end{aligned} \tag{43}$$

($\beta > \frac{1}{d-1}$) instead of (32). For this reason we solve the differential equation

$$\frac{d\gamma(\tau)}{d\tau}c^{(d-1)\beta+1}(\gamma(\tau))^{(d-1)\beta} = -1 \tag{44}$$

(compare (33)) together with (25) in order to get the evolution equation

$$\frac{\partial h_\tau}{\partial \tau} = -(K_\tau)^\beta \tag{45}$$

for the hypersurfaces F_τ. The solution of (44) has the form

$$\gamma(\tau) = \left(1 - ((d-1)\beta+1)c^{-((d-1)\beta+1)}\tau\right)^{\frac{1}{(d-1)\beta+1}} \tag{46}$$

($0 \le \tau < \frac{c^{(d-1)\beta+1}}{(d-1)\beta+1} =: T$). Again we have (39) and a generalization of Th. 1.3 of Chow, namely his Th. 5.1 which is valid because of (45) under the additional assumptions (19), (20) and (21) for F, now implies

$$\lim_{\tau\to T}\tilde{F}_\tau = S_c(o) \tag{47}$$

is a sphere obeying (22). Finally the combination of (39) and (47) provides the assertion (23) of Th. 2. $\Diamond$

3. Characterizations of ellipsoids

At the end of this paper we shall give a new short proof of a characterization of ellipsoids given first by C. Petty (1985 [8]), Def. 7.3 and Lemma 9.6, although under weaker assumptions for the hypersurface F. The reason to do so is the fact that our proof works with the same method also used for the proofs of Ths. 1 and 2. This characterization of ellipsoids may be formulated as follows in

Theorem 3. *If F is an arbitrary smooth ovaloid in $\mathbb{R}_d$ with*

$$h = c^{\frac{2d}{d+1}} \cdot K^{\frac{1}{d+1}} \quad (c = \text{const.} > 0) \tag{48}$$

(compare (22) for $\beta = \frac{1}{d+1} < \frac{1}{d-1}$ which was excluded in Th. 2) *then* F *must be an ellipsoid about* o *of volume* $c^d\omega_d$:

$$F = E_c(o). \tag{49}$$

Remark 2. *Without loss of generality we may refer the support function* h *of* F *in Th. 3 to the Santalo point* s *of* F *instead to* o *(as Petty did) because* (48) *and Minkowski's relation*

$$\int_\Omega n \frac{d\omega(n)}{K(n)} = 0 \tag{50}$$

imply

$$\int_\Omega n \frac{d\omega(n)}{(h(n))^{d+1}} = 0 \tag{51}$$

being characteristic for

$$s = o \tag{52}$$

(compare [8] (3.1)).

Proof of Th. 3. As before we see as a consequence of assumption (48) that the array of homothetic hypersurfaces

$$F_\tau = \gamma(\tau)\cdot F := \left(1 - \frac{2d}{d+1}c^{-\frac{2d}{d+1}}\tau\right)^{\frac{d+1}{2d}} \cdot F \tag{53}$$

$(0 \le \tau < \frac{d+1}{2d}c^{\frac{2d}{d+1}} =: T)$ fulfils the evolution equation

$$\frac{\partial h_\tau}{\partial \tau} = -(K_\tau)^{\frac{1}{d+1}} \tag{54}$$

(compare (46) with $\beta = \frac{1}{d+1}$!). But it is well known that the evolution controlled by (54) is equivalent to an affine evolution controlled by

$$\frac{\partial x_\tau}{\partial \tau} = y_\tau := \frac{1}{d-1}\Delta^a x_\tau \tag{55}$$

(x_τ = position vector of a point of F_τ with the affine normal vector y_τ and the affine Beltrami operator Δ^a; see [1], (1-1) and (1-2) as well as [7]) because of

$$(K_\tau)^{\frac{1}{d+1}} =< y_\tau, n_\tau > \tag{56}$$

(n_τ = euclidean inner unit normal vector of F_τ). Therefore we have especially for $\tau = 0$

$$y = \frac{d\gamma(\tau)}{d\tau}(0)\cdot x \tag{57}$$

(see (55) and (53)). But (57) characterizes F as a smooth strictly convex closed affine hypersphere which must be an ellipsoid about o, the intersection point of the affine normals of F. Finally a trivial computation shows that F must have the volume $c^d\omega_d$ such that we are sure that (49) holds. $\Diamond$

It is possible that our method of proof also is applicable for other characterization problems where the support function and a special curvature are involved.

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