Mathematica Pannonica
6/1 (1995), 133 – 143

3D-DARBOUX MOTIONS IN 4-DIMENSIONAL EUCLIDEAN SPACE

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Herrn o. Univ. Prof. Dr. Hans Vogler zu seinem sechzigsten Geburtstag gewidmet

Received October 1994

MSC 1991: 53 A 17

Keywords: Differential geometry, kinematics, Darboux motions

Abstract: In this paper we classify all Euclidean motions in E_4 such that the trajectory of every point lies in a three-dimensional subspace of E_4. We show that all trajectories of such a motion are affinely equivalent with respect to the motion parameter. There exist two classes of such motions, the general type and paratactic motions. In the general class these motions are products of plane motions in orthogonal planes. Some special cases are discussed.

1. Introduction

G. Darboux made several fascinating discoveries in geometry and kinematics about one hundred years ago. One of them was the discovery of a space motion, which has all trajectories ellipses, which do not lie in parallel planes. This motion is now called the Darboux motion (see for instance [3] and [4] for more details about the history of this motions). The Darboux motion has several interesting properties:

1) All trajectories are planar curves.
2) All trajectories are affinely equivalent (with respect to the motion parameter, see [5]).
3) The motion is cylindrical (it means that it is a product of a plane motion with a translation in the direction perpendicular to the plane, see [3], [4]).

Because of its interesting properties it drew the attention of many geometers, who tried to generalize some of its properties. W. Blaschke in [1] gave an analytic description of the Darboux motion and from his considerations it is obvious that this motion is the only space motion which has plane trajectories (apart from trivial cases including rotation composed with a translation along the axis of rotation). This means that properties 2) and 3) are a consequence of the property listed as 1). Many authors generalized properties of the Darboux motion, [2] gives a generalization of the property 1) to E_n, H. Vogler in [4] and [5] contributed to the problem by giving new points of view, specially by the study of property 2).

In the present paper we shall study the problem of Darboux in the Euclidean space of dimension higher by one.We shall suppose that the Darboux motion in E_3 is defined as the only motion in E_3 (up to trivial cases) such that the trajectory of any point lies in a two-dimensional subspace. From this point of view it is natural to consider as Darboux motions in E_n such one-parametric motions in E_n which have all trajectories contained in subspaces of codimension 1. The aim of this paper is to classify all such motions in E_4. We shall see that this problem can be solved, but neverthless by adding one more dimension the nature of the problem changes drastically. We shall use methods of differential geometry and we shall see that the order of corresponding differential equations is raised by one — instead of solving ordinary differential equations of second order, we are faced with third order equations, which brings a new quality to the problem.

Another new quality of the problem is the size of all equations — we shall have to deal with large and complicated equations which cannot be simplified. As a consequence the use of a computer is necessary and a solution of the presented problem was quite impossible several years ago. All computations were performed on a DEC 3100 work station using Mathematica. The use of a computer has also an influence at the final form of the result — it makes quite unnecessary to print large formulas in the manuscript, we shall usually present only the algorithm which generates necessary expressions and we shall suppose that anybody interested can generate all expressions on his own computer. To give an information about the size of our computations we shall use the function Lenght[ ] in Mathematica, which gives the number of terms of an expression, which are divided by + or − and which will be given in square brackets.

2. Definition of the Darboux motion

Let us consider a sufficiently differentiable Euclidean motion $g(t)$, $t \in I$ of the moving Euclidean space $\bar{E}_4$ in the fixed Euclidean space E_4, which by definition is a one-parametric system of congruences of $\bar{E}_4$ onto E_4. By a lift of the motion $g(t)$ we understand any pair $(\bar{R}(t), R(t))$ of orthonormal frames $\bar{R}(t)$ and $R(t)$ such that

(1) $$g(t)\bar{R}(t) = R(t) \qquad \text{for all} \quad t \in I,$$

where $\bar{R}(t) = \{\bar{A}(t), \bar{e}_1(t), \dots, \bar{e}_4(t)\}$ is an orthonormal frame in the moving space E_4, $R(t) = \{A(t), e_1(t), \dots, e_4(t)\}$ is an orthonormal frame in the fixed space E_4.

Let $(\bar{R}(t), R(t))$ be any lift of a motion $g(t)$. Let us denote

(2) $$\bar{R}' = \bar{R}.\psi, \quad R' = R.\phi, \quad \omega = \phi - \psi, \eta = \phi + \psi,$$

where ϕ and ψ are 5 x 5 matrices. If the lift of the motion $g(t)$ is changed, we obtain new matrices $\tilde{\omega}$ and $\tilde{\eta}$ instead of ω and η by the following rule

(3) $$\tilde{\omega} = h^{-1}\omega h, \qquad \tilde{\eta} = h^{-1}\eta h + 2h^{-1}h',$$

where $h = h(t)$, is the matrix of the change of the lift (see for instance [2]). Let Ω_k be the matrix operator of the k-th derivative of the trajectory of a point; let $X \in E_4$, $X(t) = g(t)\bar{X}$, $X^{(k)}(t) = \Omega_k(t)X(t)$. Then (see [2])

(4) $$\Omega_1 = \omega, \qquad \Omega_{k+1} = \phi\Omega_k - \Omega_k\psi + \Omega_k'.$$

Definition. A Euclidean motion $g(t)$ in E_4 is called a *3D-Darboux motion* iff the trajectory of every point of E_4 lies in a 3-dimensional subspace of E_4.

Remark. A Euclidean motion $g(t)$ in E_4 is a 3D-Darboux motions iff

(5) $$Q = |\Omega_1 X, \Omega_2 X, \Omega_3 X, \Omega_4 X| = 0 \qquad \text{for all} \quad X \in E_4,$$

where vertical lines denote the determinant of the 4×4 matrix of coordinates of vectors from the first up to fourth derivative of the trajectory of a point. This means that (5) yields an algebraic surface of degree 4 in E_4 which must disappear as a necessary and sufficient condition for 3D-Darboux motions. Before solving equation (5) we shall try to find such a lift of the motion, for which ω and η will have the simplest possible form. We shall do it by the usual way of defining the canonical (Frenet) lift and invariants of the motion.

3. Invariants of Euclidean motions in E_4

Let us write

$$\omega = \begin{pmatrix} 0 & 0 \\ \omega_o & \omega_1 \end{pmatrix}, \quad \eta = \begin{pmatrix} 0 & 0 \\ \eta_o & \eta_1 \end{pmatrix}, \tag{6}$$

where ω, η are 5×5 matrices, ω_o, η_o are 4-columns, ω_1, η_1 are 4×4 skew-symmetric matrices. Let ω_1 be a regular matrix, let $h(t) = \begin{pmatrix} 1 & 0 \\ P & \gamma \end{pmatrix}$ be the matrix of the change of the lift, where $P \in \mathbb{R}^4$ is a 4-dimensional vector, $\gamma \in SO(4)$. Then

$$\tilde{\omega}_o = \gamma^T(\omega_o + \omega_1 P), \qquad \tilde{\omega}_1 = \gamma^T \omega_1 \gamma. \tag{7}$$

We can choose γ in such a way that $\tilde{\omega}_1$ takes the Jordan normal form (as ω_1 is skew-symmetric, the Jordan normal form can be obtained by using orthogonal matrices only) and P can be choosen in such a way that $\tilde{\omega}_o = 0$. ω and η are 5×5 matrices and it is inconvenient to write them in full, specially because many of their entries yield no information. It is convenient to write them in 2×2 block matrices. Therefore, let us suppose that we already have a lift of the motion which satisfies $\omega_o = 0$ and ω_1 is in the Jordan normal form and we shall consider only lifts with this property.

Then

$$\omega_1 = \begin{pmatrix} u_1(t)J & 0 \\ 0 & u_2(t)J \end{pmatrix}, \quad \eta_1 = \begin{pmatrix} v_1(t)J & -A(t)^T \\ A(t) & v_2(t)J \end{pmatrix}, \tag{8}$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$; $\eta_o = \begin{pmatrix} P_1(t) \\ P_2(t) \end{pmatrix}$, where $P_1(t), P_2(t)$ are 2-columns, $u_1(t), u_2(t), v_1(t), v_2(t)$ are functions, $u_1(t)u_2(t) \neq 0$ and $A(t) = \begin{pmatrix} a(t)+b(t) & c(t)+d(t) \\ c(t)-d(t) & a(t)-b(t) \end{pmatrix}$ is a 2×2 matrix.

We have to consider two cases. a) $u_1^2 \neq u_2^2$. Let us denote $r(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$. Then the remaining changes of lift which preserve (8) and $\omega_o = 0$ are of the form

$$P = 0, \qquad \gamma = \begin{pmatrix} r(\alpha) & 0 \\ 0 & r(\beta) \end{pmatrix} \tag{9}$$

(up to some permutations of vectors in the basis, which are easy to describe). Let us change the lift of the motion by (9). Then $\tilde{A} =$

$= r(-\beta)Ar(\alpha)$. Computation yields

$$\tilde{a} = a\cos\mu + d\sin\mu, \quad \tilde{b} = b\cos\nu + c\sin\nu, \quad \mu = \alpha - \beta, \ \nu = \alpha + \beta,$$
$$\tilde{d} = -a\sin\mu + d\cos\mu, \quad \tilde{c} = -b\sin\nu + c\cos\nu.$$

This means that we can change the lift of the motion in such a way that $c = d = 0$, if $a \cdot b \neq 0$, the lift is fixed up to a finite group. We can change the parameter of the motion to have $u_1(t) = 1$ and this proves the following theorem.

Theorem 1. *Let $g(t)$ be a Euclidean motion in E_4 such that ω_1 is regular, $a^2+d^2 \neq 0$, $b^2+c^2 \neq 0$. Then there is a unique lift of the motion such that $c = d = 0$. If we choose the parameter of the motion in such a way that $u_1(t) = 1$, functions $u_2(t), v_1(t), v_2(t), a(t), b(t), P_1(t), P_2(t)$ constitute a complete systems of invariants of the motion.*

Remarks. 1) Unique means up to a finite group. Invariants of the motion define the motion uniquely up to a change of frame in the moving and fixed spaces. 2) If we neglect the translational part, we obtain a corresponding Theorem for spherical motions. We can formulate corresponding statement for the special cases $a \cdot b = 0$ as well. We use the remaining isotropy group to specialize P_1 and P_2, we leave details out.

4. Conditions for 3D-Darboux motions in the general case

Let $u_1 = 1, c = d = 0$, let us denote $a+b = f, a-b = m, u_2 = u$ and let us substitute into (5). At first we shall consider only terms of degree 4 in Q. Let us denote by $G(i,j,k,l)$ the coefficient at $x^i y^j z^k w^l$ in Q, where x, y, z, w are coordinates in E_4. Then $G(4,0,0,0) + G(0,4,0,0) - G(2,2,0,0)$ yields

(10) $$(f^2 - m^2)^2(u^2 - 1)^2(v_2 + 3u) = 0.$$

Similarly $G(0,0,4,0) + G(0,0,0,4) - G(0,0,2,2)$ yields

(11) $$(f^2 - m^2)^2(u^2 - 1)^2(v_1 + 3) = 0.$$

We consider two cases

a1) $f^2 - m^2 \neq 0$. Then $v_2 = -3u$, $v_1 = -3$. $G(1,3,0,0) - G(3,1,0,0)$ yields

(12) $$(mf' - fm')(1 - u^2) + 2(f^2 - m^2)u' = 0.$$

As $f^2 + m^2 \neq 0$, we can interchange f with m by the isotropy group if necessary to have $m \neq 0$. We write $f = m \cdot h$ and (12) yields

$$h' = 2u'(1-h^2)/(1-u^2). \tag{13}$$

We substitute into $G(0,0,1,3) + uG(3,1,0,0)$ from (13) and obtain

$$\begin{aligned} u'\{(hm^3 + 2mu)(1+u^4) - 2mu^2(h+2mu)+ \\ +2(m'u' - mu'')(u^2-1) + 4m(u')^2(h+u)\} = 0. \end{aligned} \tag{14}$$

If $u' \neq 0$, we express u'' from (14), m' from $G(0,0,1,3)$. From $G(2,2,0,0) - 2G(4,0,0,0)$ and $G(0,0,2,2) - 2G(4,0,0,0)$ we obtain $u^2(u^2-1) = 0$, which is impossible. Therefore we have $u' = 0$. From $G(3,1,0,0)$ we obtain

$$m'(hm^2 - 10u) = 0. \tag{15}$$

Similarly as above we show that $m' = 0$ is impossible. If $hm^2 = 10u$, we see from (13) that h is constant and so m is also constant, which is impossible.

a2)$f^2 = m^2 \neq 0$. We change the lift to have $f = m$. Let us denote $v_1 = r + s$, $v_2 = r - s$. We express Q from (5) and terms of degree 4 give 10 equations, denoted by $J_1, \dots, J_{10}$, as follows:
$J_1 = G(4,0,0,0)$, $J_2 = G(0,0,4,0)$, $J_3 = G(3,0,1,0)$, $J_4 = G(3,0,0,1)$, $J_5 = G(1,0,0,3)$, $J_6 = G(0,1,0,3)$, $J_7 = G(2,0,2,0)$, $J_8 = G(2,0,0,2)$, $J_9 = G(2,0,1,1)$, $J_{10} = G(1,1,1,1)$.
We have to solve the system of 10 differential equations

$$J_i = 0, \ i = 1, \dots, 10 \tag{16}$$

of order 3 in unknown functions u, s, m (r has disappeared), which are very long and complicated. We shall show that this system can be brought to a contradiction in the following way. We shall try to eliminate all derivatives from (16) to obtain 3 algebraic equations in u, s, m which do not have a common factor. This follows that u, s, m must be constant and it is easy to show that (16) has no solution by constants. We proceed as follows. At first me make the following substitution:

$$m^2 = h, \quad m' = h'/(2m), m'' = (h'' - 2(m')^2)/(2m). \tag{17}$$

As a result m will disappear and we obtain equations (16) of the following lenght (in succession): 60, 68, 70, 97, 115, 82, 86, 87, 61, 62. From (4) we see that equations (16) are linear in highest derivatives, which are s', h'', u'''. We can eliminate them by using linear operations — we express s' from $J_1 = 0$, h'' from $J_{10} = 0$, we denote $K_1 = J_2 + J_1 u^2$, $K_2 = J_5 - J_4 u^2$, we observe that $J_8 = J_7$. We eliminate u''' using J_1 and obtain new expressions from J_3, J_4, J_6, J_7, we denote them $L_3[407], L_4[397], L_6[426], L_7[362]$, we define $L_8 = L_6 - L_3u^2$. The coefficient at u'' in K_1 is equal to $3(1-u)+4s$.

α) Let $3(1-u)+4s \neq 0$. We compute u'' from K_1 and substitute into K_2. We see that K_2 splits into two factors, the first of which is a sum of squares and at least one of them is not equal to zero.Therefore the second factor must be equal to zero, we express $(u')^2$ from this factor and substitute into remaining equations. Remaining equations are linear in u' and algebraic in u, s, h, h'. We obtain new equations by taking the derivative of the expressions for $(u')^2$ and u''. By substitution we eliminate all derivatives except h' and u', all equations are linear in u'. Elimination of u' leads to equations, which are linear in $(h')^2$ and elimination of it yields equations in h, s, u only. Unfortunately they have a common factor which is equal to

$$[9(1-u)+8s]\,[3(1+u)-2s]\,[3(1+u)+2s]\cdot R = 0, \tag{18}$$

where R is a quadratical equation for s in terms of u. The first three factors lead to a contradiction, $R=0$ yields

$$s = \frac{27+39u-63u^2-3u^3+3(u-1)W}{4(-3-14u+u^2)}, \tag{19}$$

where $W^2 = 45+180u+278u^2+4u^3+5u^4$. The derivative of (19) and substitution leads to equations which contradict $(u')^2 > 0$ or yield algebraic equations for u and s only, which follows $u = \text{const}$, $s = \text{const}$. u and s constant contradicts $h > 0$.

β) Let $3(1-u)+4s = 0$. We substitute into (16) and we obtain a contradiction to $h > 0$. This finishes the most complicated part of the classification procedure, the rest is relatively simple. Let us formulate now the obtained results.

We have two classes of motions according to the character of the instantaneous motion. We shall call a Euclidean motion in E_4 a *motion of the general type* if the instantaneous motion has exactly two mutually orthogonal invariant two-dimensional planes passing through the instantaneous pole of the motion (ω_1 has only simple eigenvalues). This means that the instantaneous motion is a composition of rotations in mutually orthogonal two-dimensional planes with different angular velocities. We shall call a Euclidean motion in E_4 *paratactic*, if each instantaneous motion is a composition of two rotations in mutually perpendicular two-dimensional planes with the same angular velocity (ω_1 has double eigenvalues). Invariant two-dimensional planes are not uniquely determined, there are infinitely many. Cases where ω_1 is singular lead to trivial cases of 3D-Darboux motions — motions in parallel three- or two-dimensional spaces, we leave this case out.

Theorem 2. *A spherical 3D-Darboux motion in E_4 of general type is a composition of rotations in fixed orthogonal two-dimensional planes, where the ratio of angular velocities satisfies the differential equation*

$$\begin{aligned} 3uu'u''' - 4u(u'')^2 + u''[-3u(u')^2 + 5u^2(u^2-1)] - \\ -15u^3(u')^2 - u^3(u^2-1)^2 = 0, \end{aligned} \tag{20}$$

where $u(t)$ is the angular velocity of the second rotation as a function of the first angle of rotation t.

Theorem 3. *3D-Darboux motion in E_4 of general type is cylindrical - it preserves two two-dimensional directions.*

Theorem 4. *3D-Darboux motion in E_4 of general type is a composition of two plane motions in orthogonal two-dimensional planes.*

Proof of Theorems 2, 3, 4. For the corresponding spherical motion we have

$$\omega = \begin{pmatrix} J & 0 \\ 0 & uJ \end{pmatrix}, \quad \eta = \begin{pmatrix} v_1 J & 0 \\ 0 & v_2 J \end{pmatrix},$$

which shows that Frenet formulas (2) split into two parts which integrate separately. ◊

Theorem 5. *Let $g(t)$ be a Euclidean motion in E_4 of general type such that $A = 0$. Then there is a lift such that $P_1 = (p(t), 0)^T, P_2 = (q(t), 0)^T$. If $p(t)q(t) \neq 0$, such a lift is unique (up to a finite group), invariants of such a motion are $u(t), v_1(t), v_2(t), p(t), q(t)$.*

Proof. The group (9) remains. We use it to specialize η_0 to the given form. ◊

Theorem 6. *The set of all 3D-Darboux motions of general type in E_4 depends on 8 arbitrary constants and it is obtained by the solution of a system of ordinary differential equations for invariants $u(t)$, $v_1(t)$, $v_2(t)$, $p(t)$, $q(t)$. All 3-dimensional trajectories of such a motion are affinely equivalent to a curve which lies on a cylinder of revolution.*

Proof. We expand (5) and we obtain exactly one equation for each of the highest derivatives $u''', v_1', v_2', p'', q''$, which can be explicity solved with respect to them. The general solution depends on 9 constants, one constant is absorbed by the translation of the dependent variable. For matrices $\Omega_1, \ldots, \Omega_4$ we obtain the following identity

$$\Omega_4 + \Omega_2 + k(\Omega_1 + \Omega_3) = 0, \tag{21}$$

where $k = [u^2(1-u^2) + 3(u')^2 + 4uu'']/(3uu')$. This means that the trajectory $X(t)$ of every point satisfies the following differential equation $k(X' + X''') + X'' + X^{(iv)} = 0$, which has $X(t) = f_1 \cos t + f_2 \sin t$ as solutions, f_1, f_2 are constant vectors. ◊

5. Paratactic 3D-Darboux motions

b) $u_1 = u_2 = 1$. It remains to deal with the special case where ω_1 has double characteristic roots. This case is much simpler than the general one and therefore we shall proceed more quickly.
At first we shall find invariants of paratactic motions. For any lift of such a motion we have $\omega_1 = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}$. At first we have to describe the isotropy group of ω_1. Let $Y = \begin{pmatrix} x_1 J & -K^T \\ K & x_6 J \end{pmatrix}$, where $K = \begin{pmatrix} x_2 & x_4 \\ x_3 & x_5 \end{pmatrix}$, be an element of the Lie algebra of the Lie group S0(4). Y belongs to the isotropy algebra of ω_1 iff the commutator $[\omega_1, Y] = 0$, which yields $x_4 = -x_3$, $x_2 = x_5$. This means that ω_1 has a four-dimensional isotropy group. We shall represent elements of this isotropy group by the following procedure. Let as define matrices Y_1, Y_2, Y_3 in such a way that
Y_1 is equal to Y, where $x_1 = \alpha_1$, $x_6 = \alpha_2$, other elements are equal to zero,
Y_2 is equal to Y with $x_3 = -x_4 = \alpha_3$, other elements are equal to zero,
Y_3 is equal to Y with $x_2 = x_3 = \alpha_4$, other elements are zero.

Let us denote $g_i = \exp(Y_i)$. Elements of the isotropy group can be locally represented as the product

$$g(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = g_1(\alpha_1, \alpha_2) g_2(\alpha_3) g_3(\alpha_4).$$

Let Z be an element of a complementary subspace to the isotropy algebra in the Lie algebra of S0(4). Then $x_1 = x_6 = 0$, $x_2 = -x_5$, $x_3 = x_4$ will be a natural choice and the action of the isotropy group on Z is described by

$$\tilde{x}_3 = x_3 \cos(\alpha_1 + \alpha_2) - x_2 \sin(\alpha_1 + \alpha_2),$$
$$\tilde{x}_2 = x_3 \sin(\alpha_1 + \alpha_2) + x_2 \cos(\alpha_1 + \alpha_2),$$

g_2, g_3 act trivially. Therefore we can change the lift of the motion to have $x_3 = 0$ and a three-dimensional isotropy group remains. This group is locally isomorphic to 0(3) and it acts in the four-dimensional vector space $\mathbb{R}^4$ represented by (P_1^T, P_2^T), see (8). This action is transitive on directions and therefore we may choose such a lift that $P_2 = 0$, $P_1 = (a(t), 0)^T$. This proves the following theorem.

Theorem 7. *For a paratactic motion in E_4 there exists a lift such that*

$$\eta_o = (a(t), 0, 0, 0)^T,$$

$$\eta_1 = \begin{pmatrix} (r+s)J & -A^T \\ A & (r-s)J \end{pmatrix}, \quad \text{where} \quad A = \begin{pmatrix} f+m & -v \\ v & f-m \end{pmatrix}.$$

Functions a, v, s, v, f, m are invariants of the motion (in the general case).

Theorem 8. *The set of paratactic 3D-Darboux motions depends on 5 arbitrary constants and one arbitrary function. All three-dimensional trajectories are affinely equivalent to a fixed space curve.*

Proof. We expand (5) and we obtain

$$2(m')^2 r - mm'' r + m^2 r^3 + mm' r' = 0 \tag{22}$$

together with equations for f', v', a'', s' which can be solved with respect to them. This means that we can chose $m(t)$ arbitrarily and the statement follows from existence theorem for systems of ordinary differential equations. Computation shows that $\Omega_4 = w_1\Omega_1 + w_2\Omega_2 + w_3\Omega_3$, where $w_2 = -1 - m^2$, $w_3 = (2m'r + mr')/(mr)$, $w_1 = w_3(1 + m^2) - 3mm'$. ◊

Remark. Paratactic 3D-Darboux motions are not cylindrical if $A \neq 0$. Trajectories are in general not equivalent to cylindrical curves. To give an example of 3D-Darboux motion, we shall consider motions with constant invariants. Such 3D-Darboux motions can be explicitly given as we shall see from the following theorem.

Theorem 9. *3D-Darboux motions in E_4 with constant invariants are paratactic. Their invariants are as follows: $r = -2$, $m = 0$, $f^2 + s^2 + v^2 = 1$ or $m = f = v = 0$, $r + s = -1$ or $v = r = 0$, $f = m$, $s = -1$. Trajectories of these motions are affinely equivalent to the curve $x = \cos t, y = \sin t, z = e^{\lambda t}$ or $z = \lambda t$.*

Example. For illustrations we shall present an example of a paratactic 3D-Darboux motion ($v = r = f = m, s = -1$). We have $g(t) = \begin{pmatrix} 1 & 0 \\ T & \gamma \end{pmatrix}$, where

$$T = (a(t - \sin t), a(\cos t - 1), 0, 0)^T, \gamma = \begin{pmatrix} r(t) & 0 \\ 0 & r(t) \end{pmatrix}.$$

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