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CONTINUA WITH THE PERIODIC-RECURRENT PROPERTY

To the memory of Professor Victor Neumann-Lara

Gerardo **Acosta**

Instituto de Matemáticas, UNAM, Ciudad Universitaria, 04510 México, D. F., México

Janusz J. **Charatonik**

Mathematical Institute, University of Wrocław, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland

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Abstract: A space X has the periodic-recurrent (PR-)property if for each self-mapping the closures of the sets of periodic points and of recurrent points are equal. It is known that the Gehman dendrite does not have the PR-property. First, various consequences of this result are proved. Second, the PR-property is studied for some continua obtained as compactifications of trees with a finite number of points deleted. As a particular case we investigate the property of compactifications of the ray and of the line.

1. Introduction and preliminaries

Let X be a compact Hausdorff space, and let $f : X \to X$ be a mapping (i.e., a continuous function) of X to itself. We denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{C}$ the set of complex numbers. For any $n \in \mathbb{N}$ let $f^n : X \to X$ denote the n-th composition *of* f.

E-mail addresses: gacosta@matem.unam.mx, jjc@matem.unam.mx

A point x of X is said to be:
— a fixed point of f if $f(x) = x$;
— a periodic point of f provided that there is $n \in \mathbb{N}$ such that $f^n(x) = x$; if, moreover, $f^k(x) \neq x$ for all integers k with $1 \leq k < n$, then x is called a periodic point of period n;
— a recurrent point of f, provided that for each open set U containing x there is $n \in \mathbb{N}$ such that $f^n(x) \in U$.

The sets of fixed points, periodic points and recurrent points of a mapping $f : X \to X$ will be denoted by $F(f)$, $P(f)$ and $R(f)$, respectively. Notice that the following are consequences of the definitions (compare e.g. [3, p. 77]).

$$(*) \qquad F(f) \subset P(f) \subset R(f), \quad P(f) = f(P(f)), \quad R(f) = f(R(f)).$$

Definition 1.1. A space X is said to have the *periodic-recurrent property (surjective periodic-recurrent property)*, abbreviated *PR-property* (*SPR-property*, respectively), provided that each mapping (each surjective mapping) $f : X \to X$ satisfies

$$\mathrm{cl}_X(P(f)) = \mathrm{cl}_X(R(f)).$$

Note that the first inclusion in $(*)$ and Def. 1.1 imply the following statement.

Statement 1.2. *A space X has the (S)PR-property if and only if for every (surjective) mapping $f : X \to X$ the inclusion $\mathrm{cl}_X(R(f)) \subset \mathrm{cl}_X(P(f))$ holds.*

The *PR*-property implies the *SPR*-property just by the definitions, while the opposite implication need not be true in general. However, we do not have any example showing this.

Question 1.3. Does there exist a space which has the *SPR*-property while it does not have the *PR*-property?

The aim of the paper is to present a further study of the *PR*-property for various spaces. After the first, preliminary section, auxiliary general properties are collected in the second one. The third section concerns the *PR*-property for some special continua. It contains also several open problems related to the subject. In the fourth section we give various consequences and applications of the fact that the Gehman dendrite does not have the *PR*-property. In particular, at the end of this section, we apply the quoted result on the Gehman dendrite to show that many hyperspaces for continua do not have the property. Finally, in the last section, we investigate the *PR*-property for some

λ-dendroids, namely for those which are obtained as compactifications of trees with a finite set deleted. Studying the *PR*-property for those λ-dendroids we have observed that in the proof of [6, Th. 5.7, p. 116] (as given there) not all mappings are considered, but only surjective ones, so to show the result an additional argument is needed. In Sec. 5 a complete proof of the result is presented.

We would like to underline that the results presented in the paper do not form any closed and/or complete theory. Our knowledge on the subject is close rather to the beginning of the way than to the end, and for today we have much more open problems than final results.

A mapping $r : X \to Y$ between continua X and Y is called a retraction if $Y \subset X$ and the partial mapping $r|Y : Y \to Y$ is the identity. In this case Y is called a retract of X.

The following lemma on compositions of mappings is known (see [4, Lemma 3.1, p. 136] and compare [10, Lemma 2.9]).

Lemma 1.4. *Let X and Y be spaces with Y being a closed subset of X, and let $g : Y \to Y$ be a mapping. If $r : X \to Y$ is a retraction and $f = g \circ r : X \to Y$, then:*
(1.4.1) *$f^n = g^n \circ r$ for each $n \in \mathbb{N}$;*
(1.4.2) *$P(f) = P(g)$;*
(1.4.3) *$R(f) = R(g)$.*

As a consequence of the above lemma we get a corollary, see [4, Prop. 3.2, p. 136].

Corollary 1.5. *The PR-property is preserved under retractions, i.e., if a space X having the PR-property contains a closed subspace Y which is a retract of X, then Y has the PR-property, too.*

Question 1.6. Is the *SPR*-property preserved under retractions?

In connection with Cor. 1.5 the following question is of some interest.

Question 1.7. What mappings do preserve the *PR*-property?

For partial answers see below, Prop. 4.11.

A continuum means a compact connected metric space. Given a continuum X and a sequence $\{A_n\}_{n=1}^{\infty}$ of subsets of X, we let $\operatorname{Lim} A_n$ denote the limit of the sequence as defined in [22, §29, VI, p. 339] or in [31, Def. 4.8, p. 56]. By a ray and a line we mean a space homeomorphic to the real half-line $[0, \infty)$ and to the real line $(-\infty, \infty)$, respectively. Given two points p and q in the Euclidean space, we denote by $\overline{pq}$ the straight line segment joining them.

Given $\varepsilon > 0$ a mapping $f : X \to Y$ is called an ε-mapping provided that each fiber $f^{-1}(y)$ of Y has diameter less than ε. Recall that a continuum X is said to be

— a graph if it can be written as the union of finitely many arcs any two of which are either disjoint or intersect only at one or both of its end points;

— a tree if it is a graph containing no simple closed curves;

— a solenoid if it is the inverse limit of an inverse sequence of simple closed curves with surjective open bonding mappings;

— a tree-like if it is the inverse limit of an inverse sequence of trees with surjective bonding mappings (equivalently, if for each $\varepsilon > 0$ there is an ε-mapping from X onto a tree);

— an arc-like if it is the inverse limit of an inverse sequence of arcs with surjective bonding mappings (equivalently, if for each $\varepsilon > 0$ there is an ε-mapping from X onto an arc);

— a circle-like if it is the inverse limit of an inverse sequence of circles with surjective bonding mappings (equivalently, if for each $\varepsilon > 0$ there is an ε-mapping from X onto a circle);

— a dendrite if it is locally connected and contains no simple closed curve;

— hereditarily unicoherent provided that the intersection of any two subcontinua of X is connected;

— hereditarily decomposable provided that every subcontinuum of X is the union of two of its proper subcontinua;

— hereditarily indecomposable provided that no subcontinuum of X can be written as the union of two of its proper subcontinua;

— a dendroid if it is hereditarily unicoherent and arcwise connected;

— a λ-dendroid if it is hereditarily unicoherent and hereditarily decomposable.

Let $\mathcal{T}$ denote the class of trees, $\mathcal{D}_0$ — the class of dendrites, $\mathcal{D}$ — the class of dendroids, $\lambda\mathcal{D}$ — of λ-dendroids, $\mathcal{T}L$ — of tree-like continua, and $\mathcal{H}U$ — the class of hereditarily unicoherent ones. Then we have the following inclusions.

$$\mathcal{T} \subset \mathcal{D}_0 \subset \mathcal{D} \subset \lambda\mathcal{D} \subset \mathcal{T}L \subset \mathcal{H}U.$$

We will use a concept of an order of a point p in a continuum X in the sense of Menger–Urysohn, written $\operatorname{ord}(p, X)$, as defined in [31, 9.3, p. 141] or in [23, §51, I, p. 274]. For a dendrite X points of order 1 are called end points of X, and points of order at least 3 are called branch points of X. We denote the sets of end points of X and of branch points of X by $E(X)$ and $B(X)$, respectively.

A compact space K is said to be an absolute retract (written AR) provided that whenever K is embedded in a normal space X, the embedded copy of K is a retract of X. In particular, the following result is known, see e.g. [23, §53, III, Th. 16, p. 344].

Proposition 1.8. *Every dendrite is an* AR.

Several results of this paper are connected with properties of the Gehman dendrite. For the reader convenience we recall its construction here.

Example 1.9. The Gehman dendrite.

Construction. Let C denote the Cantor middle-third set lying in the closed unit interval $I = [0,1] \times \{0\}$ in the plane. Put $v = (\frac{1}{2}, \frac{1}{2})$ and join the end points $a = (0,0)$ and $b = (1,0)$ of I with v by the straight line segments $\overline{av}$ and $\overline{bv}$. Note that the slope of the former segment is 1 and the slope of the latter one is -1. Next take the end points $(\frac{1}{3}, 0)$ and $(\frac{2}{3}, 0)$ of the biggest component of $I \setminus C$ and construct two perpendiculars, one from $(\frac{1}{3}, 0)$ to $\overline{av}$ and the other from $(\frac{2}{3}, 0)$ to $\overline{bv}$. Again slopes of these perpendiculars are ± 1. Proceeding in this way, we construct countably many segments of slope ± 1, each of which starts from an end point of a component of $I \setminus C$ and is perpendicular to some suitably chosen straight line segment previously constructed. The Gehman dendrite G is the closure of the union of the constructed segments (see [31, Ex. 10.39, p. 186 and Fig. 10.39, p. 187]; compare [32, p. 422–424] for a detailed description).

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2. Generalities

We start this section with the following result.

Proposition 2.1. *For a mapping* $f: X \to X$ *and* $k \in \mathbb{N}$ *it follows that* $P(f) = P(f^k)$ *and* $R(f) = R(f^k)$.

Proof. The inclusion $R(f) \subset R(f^k)$ is shown in [14, Th. I]. The other inclusion and the equality $P(f) = P(f^k)$ follows from the definitions of $R(f)$ and $P(f)$, respectively. $\Diamond$

The following proposition is an immediate consequence of definitions.

Proposition 2.2 *Let A be a subspace of a space X, and let a mapping $f : X \to X$ be such that $f(A) \subset A$. Then*
(2.2.1) $P(f|A) = A \cap P(f)$;
(2.2.2) $R(f|A) = A \cap R(f)$.

The next result is formulated in [6, Prop. 4.1, p. 113], however it is presented in that paper almost without proof (only the definition of the continuum in matter is given, without verifying its properties). Since the result has some important consequences, in particular in the present paper, we decided to attach a full proof here. We also present the statement of the proposition in the way we are going to use it.

Proposition 2.3. *Let $f : X \to X$ be a mapping of a continuum X to itself. Then $\{f^n(X)\}_{n=1}^{\infty}$ is a decreasing sequence of subcontinua of X and $M(X, f) = \bigcap\{f^n(X) : n \in \mathbb{N}\}$ is a subcontinuum of X such that*
(2.3.1) $M(X, f) = \mathrm{Lim} f^n(X)$;
(2.3.2) $f|M(X, f) : M(X, f) \to M(X, f)$ *is a surjection;*
(2.3.3) $P(f) \subset R(f) \subset M(X, f)$;
(2.3.4) $P(f) = P(f|M(X, f))$ *and* $R(f) = R(f|M(X, f))$;
(2.3.5) $M(X, f)$ *is a maximal subcontinuum of X satisfying* (2.3.2).

Proof. It is easy to see that $\{f^n(X)\}_{n=1}^{\infty}$ is a decreasing sequence of subcontinua of X, so the intersection of all its terms, which is $M(X, f)$, is a continuum, see [31, Th. 1.8, p. 6]. Further, since any decreasing sequence of continua has its intersection as the limit, see [22, §29, VI, 8, p. 339], we get (2.3.1). Clearly
(2.3.6) $f(M(X, f)) \subset M(X, f)$.

To show (2.3.2) take a point $y \in M(X, f)$. Then $y \in f^n(X)$ for all $n \in \mathbb{N}$. Hence, for a given $n \in \mathbb{N}$, there is a point $y_n \in X$ such that $y = f^n(y_n)$. Put $x_n = f^{n-1}(y_n)$ and note that $x_1 = f^0(y_1) = y_1$. By compactness of X there is a convergent subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ of the sequence $\{x_n\}_{n=1}^{\infty}$. Put $x = \lim_{i\to\infty} x_{n_i}$. Since $x_{n_i} \in f^{n_i-1}(X)$ and since the sequence of continua $\{f^{n_i-1}(X)\}_{i=1}^{\infty}$ converges to $M(X, f)$ by (2.3.1), it follows that $x \in M(X, f)$. By continuity of f we have

$$f(x) = f(\lim_{i\to\infty} x_{n_i}) =$$
$$= \lim_{i\to\infty} f(x_{n_i}) = \lim_{i\to\infty} f(f^{n_i-1}(y_{n_i})) = \lim_{i\to\infty} f^{n_i}(y_{n_i}) = y,$$

since $f^{n_i}(y_{n_i}) = y$ for each $i \in \mathbb{N}$. This completes the proof of (2.3.2).

Since $P(f) \subset R(f)$, to prove (2.3.3) it is enough to show that $R(f) \subset M(X, f)$. To do so let $x \in R(f)$ and suppose, on the contrary, that $x \notin M(X,f)$. Thus $X \setminus M(X,f)$ is an open set containing x. Since $x \in R(f)$, there is $n_1 \in \mathbb{N}$ such that $f^{n_1}(x) \notin M(X,f)$, and thus there in $m \in \mathbb{N}$ such that $f^{n_1}(x) \notin f^m(X)$. Note that $m > n_1$, because otherwise $f^{n_1}(x) \in f^{n_1}(X) \subset f^m(X)$, a contradiction. Thus there is a positive integer k with $m = n_1 + k$. Since $f(R(f)) \subset R(f)$ according to $(*)$, it follows that $f^{n_1}(x) \in R(f)$. Note that $X \setminus f^m(X)$ is an open set containing $f^{n_1}(x)$, and since $f^{n_1}(x) \in R(f)$, there is $n_2 \in \mathbb{N}$ such that $f^{n_1+n_2}(x) \notin f^m(X)$. As previously, it follows that $m > n_1 + n_2$. Repeating this argument $k+1$ times we find numbers $n_1, n_2, \ldots, n_{k+1} \in \mathbb{N}$ such that $f^{n_1+n_2+\cdots+n_{k+1}}(x) \notin f^m(X)$, whence $n_1 + n_2 + \cdots + n_{k+1} < m$. But

$$n_1 + n_2 + \cdots + n_{k+1} \geq n_1 + 1 + \cdots + 1 = n_1 + k = m,$$

a contradiction. Thus $x \in M(X,f)$, and then $R(f) \subset M(X,f)$.

By (2.3.6) Prop. 2.2 can be applied with $A = M(X,f)$, whence by (2.3.3) we get (2.3.4). To show (2.3.5) consider a subcontinuum N of X such that $f|N : N \to N$ is a surjection. Thus $f(N) = N$, whence $f^2(N) = f(N) = N$, and further, inductively, $f^n(N) = N$ for each $n \in \mathbb{N}$. Therefore

$$N = \bigcap\{f^n(N) : n \in \mathbb{N}\} \subset \bigcap\{f^n(X) : n \in \mathbb{N}\} = M(X,f),$$

as required. Thus (2.3.5) is shown.

The proof is finished. $\Diamond$

Remark 2.4. Prop. 2.3 shows that, when a mapping f from a continuum X into itself is investigated, the whole dynamics for f is on the subcontinuum $M(X,f)$ of X. So, in such situation, we can replace X by $M(X,f)$ and then assume that f is a surjection. However, contrary to how it was considered in [6] after Cor. 4.5, p. 113, from Prop. 2.3 we cannot reduce the study of the *PR*-property of X to the study of the *SPR*-property of X, since for a given mapping f from X to itself, the sets X and $M(X,f)$ do not have to be homeomorphic and, in general, a given map g from $M(X,f)$ onto itself cannot be extended to a map from X to itself. In the next example we see that the *PR*-property on some set $M(X,f)$ for some mapping f from X to itself, does not imply the *PR*-property of X.

Example 2.5. *There exists a continuum X which does not have the PR-property, and a mapping f from X to itself, such that the subcontinuum $M(X,f)$ has the PR-property.*

Proof. The continuum X is the Gehman dendrite G, as described in Ex. 1.9. The mapping $f : G \to \overline{av}$ from G onto the segment $\overline{av} \subset G$ is the projection parallel to the x-axis, that is, for each point $p \in G$ and its image $q = f(p) \in \overline{av}$ both p and q have the same y-coordinate. Then $M(G, f) = \overline{av}$. It is known that G does not have the PR-property (see [20, Sec. 2, p. 460]), while $M(G, f)$, being homeomorphic to $[0, 1]$, has the property, see Statement 3.1 below. ◊

The following question is related to the above example; it is also connected with Question 3.3.

Question 2.6. Does there exist a continuum X having the PR-property, and a map f from X to itself such that $M(X, f)$ does not have the PR-property?

3. Some special continua

Topological dynamics on various spaces started with its study on the interval (see for example an expository paper [34] and references therein). In particular, the following statement has been shown in [12, Th. 1, p. 316].

Statement 3.1. *The closed unit interval* $[0, 1]$ *has the PR-property.*

The result has been extended to mappings of trees in [35, Th. 2.6, p. 349]. In [6, Cor. 5.10, p. 117] it has been shown that the $\sin \frac{1}{x}$-curve has the SPR-property. In this paper we show that this continuum also has the PR-property (see Th. 5.8).

We say that a continuum X has the PR-property hereditarily provided that each subcontinuum of X has the property. Observe that each subcontinuum of a tree is a tree, and that each subcontinuum of the $\sin \frac{1}{x}$-curve S either is an arc or is homeomorphic to S. Thus these continua have the PR-property hereditarily. So, the following problem is natural.

Problem 3.2. Characterize continua that have the PR-property hereditarily.

But at the moment the authors do not have any example of a continuum X that has the PR-property not hereditarily, i.e., such that X has, while a proper subcontinuum of X does not have the PR-property. So, the next question is a particular case of the previous problem.

Question 3.3. Does there exist a continuum X having the PR-property and containing a subcontinuum without this property?

Both the closed unit interval $[0,1]$ and the $\sin\frac{1}{x}$-curve S are examples of arc-like continua. Since these two continua have the *PR*-property, it is natural to ask if all arc-like continua enjoy the property. The answer is negative, since a class $\mathcal{K}$ of Knaster-type (thus arc-like) continua has been constructed in [27, p. 426] such that each member X of $\mathcal{K}$ admits a self-mapping f with the property that $\mathrm{cl}_X(P(f))$ is a singleton, while $\mathrm{cl}_X(R(f)) = X$. Therefore we have arc-like continua with the *PR*-property, as an arc or S, and the ones without the property, as members of $\mathcal{K}$. The next question is of a particular interest.

Question 3.4. Does the pseudo-arc have the *PR*-property?

Therefore the following problem is natural.

Problem 3.5. Characterize arc-like continua having the *PR*-property.

Remark 3.6. The result saying that the closed unit interval has the *PR*-property cannot be extended to all graphs because the unit circle $\mathbb{S} = \{z \in \mathbb{C} : |z| = 1\}$ does not have the property. Indeed, if $f : \mathbb{S} \to \mathbb{S}$ is an irrational rotation (i.e., a rotation by an angle α such that α/π is irrational), then $P(f) = \emptyset$, while $R(f) = \mathbb{S}$.

4. Gehman dendrite and its applications

For the definition of the Gehman dendrite G see Ex. 1.9. In other words, G is a dendrite having the Cantor ternary set in $[0,1]$ as the set $E(G)$ of its end points, such that all branch points are of order 3 and are situated in such a way that $E(G) = \mathrm{cl}_G(B(G)) \setminus B(G)$, see [31, Ex. 10.39, p. 186], where $B(G)$ stands for the set of branch points of G. Note that the infinite binary tree is another name of this dendrite, see e.g. [16, Ex. 1.6, p. 45] and [20, p. 461]. Recall that G can be characterized as the only dendrite whose set of end points is homeomorphic to the Cantor set, and whose branch points are of order 3 only, [33, p. 100].

It is known that the result saying that trees have the *PR*-property, [35, Th. 2.6, p. 349], cannot be generalized to dendrites, because of the following result, see [20, Sec. 2, p. 460] and [4, Cor. 3.4, p. 136].

Proposition 4.1. *The Gehman dendrite G (and any dendrite containing G) does not have the PR-property.*

Moreover, the following characterization is known, see [18, Th. 2, p. 222].

Theorem 4.2. *A dendrite has the PR-property if and only if it does not contain any copy of the Gehman dendrite.*

Note that the unit circle $\mathbb{S}$ is an example of a continuum that does not have the PR-property, see Rem. 3.6, while each of its proper subcontinua, being an arc, has the property. Other examples of such continua are members of the above mentioned class $\mathcal{K}$ of Knaster-type continua described in [27]. These continua are indecomposable. Thus we have the next questions.

Questions 4.3. Is the simple closed curve the only (a) decomposable, (b) hereditarily decomposable, continuum X without the PR-property such that each proper subcontinuum of X has the PR-property?

The next result is related to the above questions. In its proof we use Th. 4.2.

Theorem 4.4. *The simple closed curve is the only locally connected continuum X without the PR-property such that each proper subcontinuum of X has the PR-property.*

Proof. Let a continuum X be locally connected without the PR-property and such that each proper subcontinuum of X has the PR-property. Since a simple closed curve does not have the PR-property, it follows that X does not contain any simple closed curve as a proper subset. Thus, if X itself is not a simple closed curve, it follows that X is a dendrite without the PR-property. Hence, by Th. 4.2, X contains a copy G of the Gehman dendrite. Clearly G contains proper subcontinua without the PR-property. This shows that X is a simple closed curve, as required. $\Diamond$

Note that a solenoid is a continuum such that each of its nondegenerate proper subcontinua is an arc, so each of them has the PR-property. In the next result we show that no solenoid has the PR-property.

Theorem 4.5. *Solenoids do not have the PR-property.*

Proof. Let Σ be a solenoid. Assume that $\Sigma = \varprojlim\{S_i, f_i\}$, where, for each $i \in \mathbb{N}$, the space S_i is the unit circle $\mathbb{S}$ and the mapping $f_i :$ $: S_{i+1} \to S_i$ is defined by $f_i(z) = z^{p_i}$, with a positive integer p_i for any $z \in S_{i+i}$. Let $\varphi_1 : S_1 \to S_1$ be a rotation by an angle α such that $\frac{\alpha}{\pi}$ is irrational. To have the diagram

$$\begin{array}{ccc} S_i & \xleftarrow{\;f_i\;} & S_{i+1} \\ {\scriptstyle\varphi_i}\downarrow & & \downarrow{\scriptstyle\varphi_{i+1}} \\ S_i & \xleftarrow[\;f_i\;]{} & S_{i+1} \end{array}$$

commutative for each $i \in \mathbb{N}$, we have to define $\varphi_i : S_i \to S_i$ for each

$i \geq 2$, as a rotation by an angle $\frac{\alpha}{p_1 p_2 \cdots p_{i-1}}$. Then $\varphi_i \circ f_i = f_i \circ \varphi_{i+1}$. Let $\varphi : \Sigma \to \Sigma$ be the limit mapping induced by the sequence $\{\varphi_i\}_{i=1}^{\infty}$, i.e. the mapping defined by

$$\varphi((x_1, x_2, x_3, \dots)) = (\varphi_1(x_1), \varphi_2(x_2), \varphi_3(x_3), \dots)$$

for any point $(x_1, x_2, x_3, \dots) \in \Sigma$. Note that φ is a surjection and that $\varphi^n((x_1, x_2, x_3, \dots)) = (\varphi_1^n(x_1), \varphi_2^n(x_2), \varphi_3^n(x_3), \dots)$ for each $n \in \in \mathbb{N}$ and each $(x_1, x_2, x_3, \dots) \in \Sigma$. Then, since each φ_i is an irrational rotation, it follows that $P(\varphi_i) = \emptyset$, whence $P(\varphi) = \emptyset$ as well. Note that $x_0 = (1, 1, 1, \dots) \in \Sigma$. We will show that $x_0 \in R(\varphi)$. So, let U be an open subset of Σ such that $x_0 \in U$. Then there are $j \in \mathbb{N}$ and an open subset U_j of S_j such that $x_0 \in \pi_j^{-1}(U_j) \subset U$, where $\pi_j : \Sigma \to S_j$ is the j-th projection mapping. Since $R(\varphi_j) = S_j$ and $1 = \pi_j(x_0) \in U_j$, there is an $n \in \mathbb{N}$ such that $\varphi_j^n(1) \in U_j$. Then $\varphi^n(x_0) = (\varphi_1^n(1), \varphi_2^n(1), \varphi_3^n(1), \dots) \in \Sigma$ and $\varphi^n(x_0) \in \pi_j^{-1}(U_j) \subset U$. Hence $x_0 \in R(\varphi)$. This shows that Σ does not have the *PR*-property. $\Diamond$

Note that $\mathbb{S}$ is a light open image of any solenoid. Thus, in connection with Question 1.7, we have the following question.

Question 4.6. Is the *PR*-property preserved under light open maps?

The result showed in Th. 4.2 has many other consequences, not mentioned in [20], in [18] or in [10]. For example, it follows from Th. 4.2 that if a dendrite has finitely many branch points, then it has the *PR*-property. The next proposition also presents such a consequence.

Proposition 4.7. *If a normal space contains the Gehman dendrite, then it does not have the PR-property.*

Proof. If a normal space X contains the Gehman dendrite G, then by Prop. 1.8 there exist a retraction $r : X \to G$. Thus the conclusion holds by Cor. 1.5 using Prop. 4.1 (or Th. 4.2). $\Diamond$

Corollary 4.8. *If a normal space contains a 2-cell, then it does not have the PR-property.*

Proof. Let X be a normal space that contains a 2-cell D. Clearly D is homeomorphic to $[0, 1] \times [0, 1]$, so D contains a topological copy of the Gehman dendrite G. Thus the result follows from Prop. 4.7. $\Diamond$

Corollary 4.9. *The PR-property is not productive, i.e., the Cartesian product of spaces having the property need not have it.*

Proof. Indeed, $[0, 1]$ has the *PR*-property according to Statement 3.1, while the unit square $[0, 1] \times [0, 1]$ does not have it by Cor. 4.8. $\Diamond$

Spaces that contain no 2-cells are not necessarily 1-dimensional.

It is known that there exist hereditarily indecomposable continua of any dimension (see [2, Ths. 4 and 5, p. 270]) and, clearly, any such continuum contains no 2-cells. Thus trying to detect 1-dimensionality on continua with the *PR*-property the following questions are natural and, indeed, of a special interest.

Question 4.10. Let X be a locally connected continuum with the *PR*-property. Must X be 1-dimensional? Must X be a dendrite?

A positive answer to the second question will generalize the result showed in Th. 4.2 as follows: a locally connected continuum has the *PR*-property if and only if it does not contain any copy of the Gehman dendrite. Also a positive answer to this question will imply that locally connected continua with the *PR*-property have the *PR*-property hereditarily (see Problem 3.2). In other words, under the assumption that locally connected continua with the *PR*-property are dendrites, it follows that the only locally connected continua having the *PR*-property hereditarily are dendrites that contain no copy of the Gehman dendrite.

Prop. 4.7 was applied in [5, Th. 3.11] to show the following result.

Proposition 4.11. *Let a dendrite X have the PR-property. If a surjective mapping from X onto Y is one of the following:*

(4.11.1) *monotone, open, OM-mapping, confluent, locally confluent, confluent over locally connected continua, or quasi-monotone,*

then Y also is a dendrite having the PR-property.

As before, if locally connected continua with the *PR*-property are dendrites, then we can replace the term "dendrite" in the previous proposition by "locally connected continuum".

Let us recall that in [20, p. 460] some kinds of dendrites are constructed, applying the general method of [21] in the following way. For an inverse sequence $\mathbf{X} = \{X_n, p_{n,n+1} : n \in \mathbb{N}\}$ of compact polyhedra X_n such that X_1 is a singleton, with bonding mappings $p_{n,n+1} : X_{n+1} \to \to X_n$ and the inverse limit $X = \varprojlim \mathbf{X}$ consider an infinite telescope $T(\mathbf{X})$ (see [21] and [20, p. 460] for details) and define $Z(\mathbf{X}) = X \cup T(\mathbf{X})$. It is known that $Z(\mathbf{X})$ is a compact absolute retract, and that each mapping $g : X \to X$ with $\mathrm{cl}_X(P(g)) \neq \mathrm{cl}_X(R(g))$ can be extended to a mapping $f : Z(\mathbf{X}) \to Z(\mathbf{X})$ satisfying $\mathrm{cl}_{Z(\mathbf{X})}(P(f)) \neq \mathrm{cl}_{Z(\mathbf{X})}(R(f))$. Then Th. 4.2 implies the following observation.

Observation 4.12. *If, for an inverse system $\mathbf{X}$, the constructed continuum $Z(\mathbf{X})$ is a dendrite, then it necessarily contains a copy of the Gehman dendrite.*

We close this section with some results (also being consequences of Th. 4.2) concerning hyperspaces. Some definitions are in order first.

Given a continuum X with a metric d, we let 2^X denote the hyperspace of all nonempty closed subsets of X equipped with the Hausdorff metric H (see e.g. [30, (0.1), p. 1 and (0.12), p. 10]). We denote by $F_1(X)$ the hyperspace of all singletons of X and, for each $m \in \mathbb{N}$, we denote by $F_m(X)$ the hyperspace composed of sets of cardinality at most m, and we put $F_\infty(X) = \bigcup\{F_m(X) : m \in \mathbb{N}\}$. Thus $F_1(X)$ is homeomorphic to X, and $F_\infty(X)$ consists of finite subsets of X. Further, we denote by $C(X)$ the hyperspace of all subcontinua of X, i.e., of all connected elements of 2^X and, for a given $m \in \mathbb{N}$, we let $C_m(X)$ denote the hyperspace of all elements of 2^X having at most m components. Note that $C(X) = C_1(X)$. Let $C_\infty(X) = \bigcup\{C_m(X) : : m \in \mathbb{N}\}$ be the hyperspace of all elements of 2^X having finitely many components. All these hyperspaces are equipped with the inherited topology (thus induced by the Hausdorff metric H). Therefore the following statement holds by the definitions.

Statement 4.13. *For each continuum X and each $m \in \mathbb{N}$,*

$$F_1(X) \subset \ldots \subset F_m(X) \subset F_{m+1}(X) \subset \ldots \subset F_\infty(X) \subset 2^X; \tag{4.13.1}$$

$$C_1(X) \subset \ldots \subset C_m(X) \subset C_{m+1}(X) \subset \ldots \subset C_\infty(X) \subset 2^X; \tag{4.13.2}$$

$$F_m(X) \subset C_m(X) \ \textit{and}\ F_\infty(X) \subset C_\infty(X). \tag{4.13.3}$$

It is known that, for each continuum X and for each $m \in \mathbb{N}$, the hyperspaces $F_m(X)$, $C_m(X)$ and 2^X are continua, whence it follows that $F_\infty(X)$ and $C_\infty(X)$ are connected subsets of 2^X, see [24, p. 238 and 239]. The reader is referred to [30] and [19] for more information on hyperspaces. In particular, the papers [24] and [25] are devoted to the hyperspaces $C_n(X)$.

As a consequence of Prop. 4.7 we have the following result.

Theorem 4.14. *If, for some fixed $k \in \mathbb{N}$, the hyperspace $F_k(X)$ contains a copy of the Gehman dendrite, then for each integer $m \geq k$ the hyperspaces $F_m(X)$ and $F_\infty(X)$ do not have the PR-property.*

A particular case of the above, for $k = 1$, is of a special importance.

Corollary 4.15. *If a continuum X contains a copy of the Gehman dendrite, then for each $m \in \mathbb{N}$ the hyperspaces $F_m(X)$ and $F_\infty(X)$ do not have the PR-property.*

As a consequence of Cor. 4.8 we obtain the following result.

Theorem 4.16. *If, for some fixed $k \in \mathbb{N}$, with $k \geq 2$ the hyperspace $F_k(X)$ contains a 2-cell, then for each $m \in \mathbb{N}$ with $m \geq k$ the hyperspaces $F_m(X)$ and $F_\infty(X)$ do not have the PR-property.*

The next proposition indicates the only way how the assumption of Th. 4.16 can be realized.

Proposition 4.17. *Let X be a continuum, and let $k \in \mathbb{N}$, with $k \geq 2$, be given. Then the hyperspace $F_k(X)$ contains a 2-cell if and only if the continuum X contains an arc.*

Proof. Let C be a 2-cell contained in $F_k(X)$, for $k \geq 2$. Since C is a compact and locally connected subset of $F_k(X)$, it follows by [13, Lemma 2.2, p. 252] that $\cup C$ is a locally connected subcontinuum of X. Thus $\cup C$ is an arcwise connected subset of X, so it contains an arc, as required.

Conversely, if A is an arc in X, then $F_2(A)$ is a 2-cell, and we have $F_2(A) \subset F_2(X) \subset F_k(X)$ for any $k \geq 2$. The argument is complete. ◊

Corollary 4.18. *If a continuum X contains an arc A, then for each $m \in \mathbb{N}$ with $m \geq 2$ the hyperspaces $F_m(X)$ and $F_\infty(X)$ do not have the PR-property.*

Proof. Apply Prop. 4.17 and Th. 4.16. ◊

Corollary 4.19. *If a continuum X is locally connected, then for each $m \in \mathbb{N}$ with $m \geq 2$ the hyperspaces $F_m(X)$ and $F_\infty(X)$ do not have the PR-property.*

Remark 4.20. Note that in Cor. 4.18 one cannot replace the arc A by a hereditarily decomposable continuum A, since hereditary decomposability of A does not imply that $F_2(A)$ contains a 2-cell. Indeed, note that there are hereditarily decomposable continua Z containing no arcs, thus containing no locally connected subcontinua (see e.g. the continuum Σ in [26, Sec. 2, Part B, p. 14–16]; containing no arcs follows from [26, (2.9), p. 16]; compare also [26, Main Th. (6.1), p. 30]). For each such continuum Z the hyperspace $F_k(Z)$, where $k \geq 2$, does not contain a 2-cell by Prop. 4.17.

The study of the *PR*-property on the hyperspaces $C_m(X)$ reduces to the following result, see [30, Th. 1.74.1, p. 120].

Theorem 4.21. *For each continuum X any dendrite can be embedded in the hyperspace $C(X)$.*

Theorem 4.22. *For each continuum X and for each $m \in \mathbb{N}$ the hyperspaces $C_m(X)$, $C_\infty(X)$ and 2^X do not have the PR-property.*

Proof. By Th. 4.21 the hyperspace $C(X)$ contains the Gehman dendrite G, whence it follows that G is a subset of any of the mentioned hyperspaces according to inclusions (4.13.2) of Statement 4.13. Thus these hyperspaces do not have the *PR*-property by Prop. 4.7. ◊

In connection with Th. 4.16 the following questions are of some interest.

Question 4.23. Does there exist a continuum X such that the hyperspaces $F_m(X)$ (for some natural $m \geq 2$) and/or $F_\infty(X)$ have the *PR*-property? Note that, by Cor. 4.18, the continuum X must contain no arc.

Question 4.24. Does the continuum Σ of [26, Sec. 2, Part B, p. 14–16] (mentioned in Rem. 4.20 above) have the *PR*-property? Note that the continuum Σ is arc-like.

5. λ-dendroids as compactifications

Given a λ-dendroid X we denote by $\mathcal{P}(X)$ the family of all subcontinua S of X such that for each finite cover of X the elements of which are subcontinua of X, the continuum S is contained in a member of the cover. A (transfinite) well-ordered sequence (numbered with ordinals α) of nondegenerate subcontinua X_α of a λ-dendroid X is said to be normal provided that the following conditions are satisfied:

$$X_1 = X;$$

$$X_{\alpha+1} \in \mathcal{P}(X_\alpha);$$

$$X_\beta = \bigcap \{X_\alpha : \alpha < \beta\} \quad \text{for each limit ordinal } \beta.$$

The depth $k(X)$ of a λ-dendroid X is defined as the minimum ordinal number η such that the order type of each normal sequence of subcontinua of X is not greater than η. The reader is referred to [17] and [28] for an additional information related to this concept. The following three assertions concerning the depth are known, [17, Ths. 1, 2 and 3, p. 94 and 95].

Statement 5.1. *Let X and Y be λ-dendroids.*

(5.1.1) *If $Y \subset X$, then $k(Y) \leq k(X)$.*

(5.1.2) *X is locally connected (i.e., it is a dendrite) if and only if $k(X) = 1$.*

(5.1.3) *If Y is a continuous image of X, then $k(Y) \leq k(X)$.*

A subcontinuum Q of a continuum X is said to be terminal provided that for every subcontinuum K of X if $K \cap Q \neq \emptyset$ then either

$K \subset Q$ or $Q \subset K$. We need the following result, see [1, Th., p. 35] and [6, Th. 3.1, p. 111].

Theorem 5.2. *If X is a locally compact, noncompact metric space, then each continuum is a remainder of X in some compactification of X as a terminal subcontinuum of the compactification.*

To make the paper self-contained and to formulate the needed results we recall a construction described in [6, Sec. 3, p. 111]. Let a tree T and points $q_1, \ldots, q_n$ of T be given for some positive integer n. Let $Q_1, \ldots, Q_n$ be continua. Choose in T closed connected and mutually disjoint neighborhoods $U_1, \ldots, U_n$ of points $q_1, \ldots, q_n$. Then for each $i \in \{1, \ldots, n\}$ the sets $U_i \setminus \{q_i\}$ are locally compact and noncompact, thus applying Th. 5.2 to each of them we construct in a standard way a compactification

$$(\gamma) \qquad \gamma : (T \setminus \{q_1, \ldots, q_n\}) \to \gamma(T \setminus \{q_1, \ldots, q_n\})$$

such that:

(γ.1) $X = \mathrm{cl}_X(\gamma(T \setminus \{q_1, \ldots, q_n\}))$ is a continuum;

(γ.2) the remainder of X, i.e. the set $X \setminus \gamma(T \setminus \{q_1, \ldots, q_n\})$, consists of n components $Q_1, \ldots, Q_n$;

(γ.3) for each index $i \in \{1, \ldots, n\}$ the continuum Q_i is a terminal subcontinuum of X.

The next observation is in [6, Obs. 3.6, p. 112].

Observation 5.3. *If the inserted continua Q_i are λ-dendroids, then the resulting continuum X satisfying (γ.1)-(γ.3) is a λ-dendroid, too.*

Thus the concept of the depth $k(X)$ is well defined for such X (and for all subcontinua of X). We say that the inserted continua Q_i have the same finite depth provided that there is $m \in \mathbb{N}$ such that for each $i \in \{1, \ldots, n\}$ we have $k(Q_i) = m$.

To make formulation of the forthcoming results shorter accept the following definition.

Definition 5.4. Let $\mathcal{F}$ be the class of all λ-dendroids X that can be obtained from some tree T, called the base of X, by replacing finitely many of its points $q_1, \ldots, q_n$ by λ-dendroids $Q_1, \ldots, Q_n$ of the same finite depth using a compactification γ with (γ.1)-(γ.3). Further, for any member X of $\mathcal{F}$, let $A(X)$ stand for the union $\bigcup\{Q_i : i \in \{1, \ldots, n\}\}$.

Note that different elements of $\mathcal{F}$ have, in general, different trees as its base. The next theorem is proved in [6, Th. 5.7, p. 116].

Theorem 5.5. *Let a λ-dendroid X belong to the class $\mathcal{F}$. If all continua Q_i for $i \in \{1, \ldots, n\}$ have the PR-property, then X has the SPR-*

property.

In [6, Th. 5.7, p. 116] it is written, as the conclusion of the theorem, that X has the *PR*-property (instead of the *SPR*-property as written in Th. 5.5). Such statement is correct but it does not follow from the proof of [6, Th. 5.7, p. 116], since it is based on the incorrect statement that the study of the *PR*-property reduces to the study of the *SPR*-property (see Rem. 2.4 and Ex. 2.5 above). In the next theorem we complete the proof of the result that any continuum X, constructed as in Th. 5.5, has the *PR*-property.

Theorem 5.6. *Let a λ-dendroid X belong to the class $\mathcal{F}$. If all continua Q_i for $i \in \{1, \dots, n\}$ have the PR-property, then X has the PR-property.*

Proof. Take $X \in \mathcal{F}$ and a mapping $f : X \to X$. Let $M(X, f) = \bigcap\{f^i(X) : i \in \mathbb{N}\}$ and consider two cases.

CASE 1. For some $k \in \mathbb{N}$ we have $f^k(X) \cap A(X) = \emptyset$.

Then $f^k(X)$ is a tree and $f|f^k(X) : f^k(X) \to f^{k+1}(X) \subset f^k(X)$ is a mapping from $f^k(X)$ to itself. Since trees have the *PR*-property, it follows that $\mathrm{cl}_{f^k(X)}(P(f|f^k(X))) = \mathrm{cl}_{f^k(X)}(R(f|f^k(X)))$. By (2.3.3) we have $P(f) \subset R(f) \subset M(X, f) \subset f^k(X)$. Then, using (2.2.1) and (2.2.2) we get

$$\begin{aligned}\mathrm{cl}_X(P(f)) &= f^k(X) \cap \mathrm{cl}_X(P(f)) = \mathrm{cl}_{f^k(X)}(P(f)) = \\ &= \mathrm{cl}_{f^k(X)}(P(f) \cap f^k(X)) = \mathrm{cl}_{f^k(X)}(P(f|f^k(X))) = \\ &= \mathrm{cl}_{f^k(X)}(R(f|f^k(X))) = \mathrm{cl}_{f^k(X)}(f^k(X) \cap R(f)) = \\ &= \mathrm{cl}_{f^k(X)}(R(f)) = f^k(X) \cap \mathrm{cl}_X(R(f)) = \mathrm{cl}_X(R(f)).\end{aligned}$$

Thus, in this case, we conclude that X has the *PR*-property.

CASE 2. For each $k \in \mathbb{N}$ we have $f^k(X) \cap A(X) \neq \emptyset$.

Since we have finitely many sets $Q_1, \dots, Q_n$ and since each of them is terminal in X according to (γ.3), there is $i \in \{1, \dots, n\}$ such that either $f^k(X) \subset Q_i$ or $Q_i \subset f^k(X)$ for infinitely many $k \in \mathbb{N}$.

SUBCASE 2A. $Q_i \subset f^k(X)$ for infinitely many k.

Then $Q_i \subset M(X, f)$, so $M(X, f) \in \mathcal{F}$. Moreover, by (2.3.2) and (2.3.4) it follows that $f|M(X, f) : M(X, f) \to M(X, f)$ is a surjection such that $P(f) \subset R(f) \subset M(X, f)$. Thus, by Th. 5.5, $M(X, f)$ has the *SPR*-property. To simplify notation, put $M = M(X, f)$. Then we have

$$\begin{aligned}\mathrm{cl}_X(P(f)) &= M \cap \mathrm{cl}_X(P(f)) = \mathrm{cl}_M(P(f)) = \mathrm{cl}_M(P(f) \cap M) = \\ &= \mathrm{cl}_M(P(f|M)) = \mathrm{cl}_M(R(f|M)) = \mathrm{cl}_M(M \cap R(f)) = \\ &= \mathrm{cl}_M(R(f)) = M \cap \mathrm{cl}_X(R(f)) = \mathrm{cl}_X(R(f)).\end{aligned}$$

Thus, in this subcase, we conclude that X has the PR-property.

SUBCASE 2B. $f^k(X) \subset Q_i$ for infinitely many k.

Notice that, to complete the proof, it is enough to assume that for some $k \in \mathbb{N}$ we have $f^k(X) \subset Q_i$. Since $f^k(Q_i) \subset f^k(X) \subset Q_i$, it follows that $f^k|Q_i : Q_i \to Q_i$ is a mapping from Q_i to itself. Since Q_i has the PR-property, we have $\mathrm{cl}_{Q_i}(P(f^k|Q_i)) = \mathrm{cl}_{Q_i}(R(f^k|Q_i))$. Moreover, $P(f) \subset R(f) \subset M(X, f) \subset f^k(X) \subset Q_i$ and, by Prop. 2.1, $R(f) = R(f^k)$ and $P(f) = P(f^k)$. Thus

$$\begin{aligned}\mathrm{cl}_X(P(f)) &= Q_i \cap \mathrm{cl}_X(P(f)) = \mathrm{cl}_{Q_i}(P(f)) = \mathrm{cl}_{Q_i}(P(f) \cap Q_i) = \\ &= \mathrm{cl}_{Q_i}(P(f^k) \cap Q_i) = \mathrm{cl}_{Q_i}(P(f^k|Q_i)) = \mathrm{cl}_{Q_i}(R(f^k|Q_i)) = \\ &= \mathrm{cl}_{Q_i}(R(f^k) \cap Q_i) = \mathrm{cl}_{Q_i}(R(f) \cap Q_i) = \mathrm{cl}_{Q_i}(R(f)) = \\ &= Q_i \cap \mathrm{cl}_X(R(f)) = \mathrm{cl}_X(R(f)).\end{aligned}$$

Then, we again conclude that X has the PR-property. ◇

As a consequence of Ths. 5.6, 4.2 and of (5.1.2) we get the next corollary.

Corollary 5.7 *Let a λ-dendroid X be in the class $\mathcal{F}$, and let Q_i be dendrites none of which contains a copy of the Gehman dendrite. Then X has the PR-property.*

Taking as T an arc with end points q_1 and q_2 note that the inserted continua Q_1 and/or Q_2 are terminal, that is, condition (γ.3) is in this case satisfied automatically (compare [7, Statement 3.18, p. 96]). Hence we obtain very particular but important corollaries on compactifications of the ray and of the real line.

Corollary 5.8. *Let a λ-dendroid $X = S \cup Q$ be a compactification of the ray $S \approx [0, \infty)$ with a remainder Q. If Q has the PR-property, then X has the PR-property.*

Corollary 5.9. *Let a λ-dendroid $X = Q_1 \cup S \cup Q_2$ be a compactification of the real line $S \approx (-\infty, \infty)$ with a remainder being the union of two its components, Q_1 and Q_2, of the same finite depth. If each of Q_1 and Q_2 has the PR-property, then X has the PR-property.*

Questions 5.10. Are the assumptions on the two components of the remainder (a) to have a finite depth, (b) to have the same depth, necessary in Cor. 5.9?

Now we consider the opposite implication: from the *PR*-property of the whole space X to the property of the inserted continua Q_i. The next theorem is a consequence of Cor. 1.5.

Theorem 5.11. *Let a λ-dendroid X be in the class $\mathcal{F}$, and let $i \in \in \{1, \ldots, n\}$ be fixed. If X has the PR-property and if there exists a retraction from X onto Q_i, then Q_i has the PR-property.*

We present several consequences of the above theorem. To this aim recall the needed concepts and results.

A compact space X is called an *absolute retract for a class $\mathcal{K}$ of spaces* (written $\mathrm{AR}(\mathcal{K})$) provided that whenever X is embedded in a space $Y \in \mathcal{K}$ as a closed subset, the embedded copy of X is a retract of Y.

The next result follows from Th. 5.11.

Theorem 5.12. *Let a λ-dendroid X be in the class $\mathcal{F}$, and let $Q_1, \ldots, Q_n \in \mathrm{AR}(\lambda\mathcal{D})$. If X has the PR-property, then each of the continua Q_i has the PR-property.*

Let us discuss a special case when the inserted continua $Q_i \in \in \mathrm{AR}(\lambda\mathcal{D})$ are dendroids. To formulate the next result some definitions are in order first.

A dendroid X is said to be smooth provided that there is a point $v \in X$ (called an initial point of X) such that for each point $x \in X$ and for each sequence $\{x_n\}_{n=1}^{\infty}$ of points of X which tends to x the sequence of arcs $\{vx_n\}_{n=1}^{\infty}$ is convergent, in the Hausdorff metric, and it has the arc vx as its limit (see e.g. [19, p. 194]).

A continuum X is said to have the property of Kelley provided that for each point $p \in X$, for each subcontinuum K of X containing p and for each sequence of points $\{p_n\}_{n=1}^{\infty}$ converging to p there exists a sequence of subcontinua $\{K_n\}_{n=1}^{\infty}$ of Xconverging to the continuum K, in the Hausdorff metric, and such that $p_n \in K_n$ for any $n \in \mathbb{N}$ (see e.g. [19, p. 167]).

The class of all smooth dendroids has a universal element, i.e., there is a smooth dendroid that contains all other smooth dendroids, see [11, Cor. 2, p. 165], [15, Th. 3.1, p. 992] and [29]. It is known that each member of $\mathrm{AR}(\mathcal{D})$ is a smooth dendroid having the property of Kelley, [9, Cor. 3.6, p. 59]. Further, we have the following characterizations (see [8, Th. 3.12, Cors. 3.14 and 4.6, p. 97 and 101]).

Theorem 5.13. *The following conditions are equivalent for a dendroid Q.*

(5.13.1) *Q is a member of* $\mathrm{AR}(\mathcal{D})$ *(equivalently:* $\mathrm{AR}(\lambda\mathcal{D})$ *or* $\mathrm{AR}(\mathcal{H}U)$*);*

(5.13.2) *Q is a retract of the Mohler–Nikiel universal smooth dendroid;*
(5.13.3) *Q is a retract of the inverse limit of trees with open (equivalently: with confluent) bonding mappings.*

As consequences of the above results we get the following equivalences.

Theorem 5.14. *Let a λ-dendroid X be in the class $\mathcal{F}$, and let continua $Q_1, \dots, Q_n$ be locally connected. Then the following conditions are equivalent.*
(5.14.1) *X has the PR-property;*
(5.14.2) *each of the continua $Q_1, \dots, Q_n$ has the PR-property;*
(5.14.3) *each of the continua $Q_1, \dots, Q_n$ is a dendrite which does not contain any copy of the Gehman dendrite.*

In particular, the equivalences are true if X is a compactification of either the ray or the line, i.e., if T is an arc and either $n = 1$ with q_1 being an end point of T, or $n = 2$ with q_1 and q_2 being the end points of T.

Theorem 5.15. *Let a λ-dendroid X be in the class $\mathcal{F}$, and let each of the continua $Q_1, \dots, Q_n$ be a dendroid which is a member of* $\mathrm{AR}(\lambda\mathcal{D})$ *(equivalently* $\mathrm{AR}(\mathcal{D})$ *or* $\mathrm{AR}(\mathcal{HU})$*). Then X has the PR-property if and only if each of the dendroids $Q_1, \dots, Q_n$ has the PR-property.*

In particular, the equivalence is true if X is a compactification of either the ray or the line, i.e., if T is an arc and either $n = 1$ with q_1 being an end point of T, or $n = 2$ with q_1 and q_2 being the end points of T.

Theorem 5.16. *Let a λ-dendroid X be in the class $\mathcal{F}$, and let each of the continua $Q_1, \dots, Q_n$ be a λ-dendroid which is a member of* $\mathrm{AR}(\lambda\mathcal{D})$*. Then X has the PR-property if and only if each of the λ-dendroids $Q_1, \dots, Q_n$ has the PR-property.*

In particular, the equivalence is true if X is a compactification of either the ray or the line, i.e., if T is an arc and either $n = 1$ with q_1 being an end point of T, or $n = 2$ with q_1 and q_2 being the end points of T.

In the particular (final) parts of Ths. 5.14, 5.15 and 5.16 properties of λ-dendroids X as compactifications of the ray and the line are discussed. The assumption that X is a λ-dendroid implies that each component of the remainder in the compactification must be a λ-dendroid. The general case, when no extra conditions are assumed on the obtained compactification of either the ray or the line, remains open. Thus we have the following problems.

Problems 5.17. Let a continuum X be obtained as a compactification of either a ray or a line. Under what conditions one of the conditions below implies the other one?
(5.17.1) X has the PR-property;
(5.17.2) each component of the remainder of X has the PR-property.

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