NORMS OF HYPERCYCLIC SEQUENCES

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Abstract: In this note, it is proved that every member of a wide class of Banach spaces supports a sequence (T_n) of operators with a hereditarily hypercyclic subsequence (T_{n_k}) such that (T_n) itself does not satisfy the so-called Hypercyclicity Criterion and such that, in addition, the norms of the T_n's are controlled in a certain natural sense.

1. Introduction

Assume that X is an F-space (= completely metrizable topological vector space) over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $L(X)$ denote the space of all operators on X, that is, all continuous linear mappings $X \to X$. Then an operator $T \in L(X)$ is called hypercyclic whenever there exists some vector $x \in X$ – called hypercyclic for T – such that the orbit $\{T^n x : n \in \mathbb{N}\}$ of x under T is dense in X. The theory of hypercyclic operators has recently been studied intensively. We refer to the comprehensive survey [16], see also [12, Sect. 1], [17] and [22]. More generally, a sequence (T_n) of operators on X is called hypercyclic provided there exists some $x \in X$ – called hypercyclic for (T_n) – such that its orbit

$\{T_n x : n \in \mathbb{N}\}$ under (T_n) is dense in X. Observe that X must be separable in order to support such a sequence. This more general notion of hypercyclicity is also sometimes referred to as universality, see [16, Sect. 1].

Moreover, the sequence $(T_n) \subset L(X)$ is called densely hypercyclic whenever the set of its hypercyclic vectors is dense in X. It is called hereditarily hypercyclic whenever each subsequence (T_{n_k}) is hypercyclic. Finally, it is called densely hereditarily hypercyclic whenever each subsequence is densely hypercyclic. See [4] and [16, Sect. 2], but note that Bès and Peris [7] use a different notion of hereditary hypercyclicity. Corresponding concepts can be defined for a single operator $T \in L(X)$ by looking at its sequence of iterates.

The "Hypercyclicity Criterion", which gives sufficient conditions under which a sequence (T_n) is hypercyclic, has turned out to be extremely useful in applications.

Definition 1.1. A sequence $(T_n) \subset L(X)$ satisfies the Hypercyclicity Criterion provided there exist dense subsets X_0 and Y_0 of X and an increasing sequence (n_k) of positive integers satisfying the following two conditions:
  (i) $T_{n_k} x \to 0$ $(k \to \infty)$ for all $x \in X_0$;
  (ii) for any $y \in Y_0$ there is a sequence (u_k) in X such that $u_k \to 0$ and $T_{n_k} u_k \to y$ $(k \to \infty)$.

Note that this is an equivalent reformulation of the Hypercyclicity Criterion as stated in [7, Def. 1.2 and Rem. 2.6]. Earlier versions of it are due to Kitai [19] and Gethner and Shapiro [11, Rem. 2.3], see also [15] and [12, Cor. 1.4]. As before, an operator T is said to satisfy the Hypercyclicity Criterion provided the sequence (T^n) of its iterates satisfies it. It is still unknown whether any hypercyclic operator must satisfy the Hypercyclicity Criterion.

Bès and Peris [7, Th. 2.3] have recently shown that an operator T satisfies the Hypercyclicity Criterion if and only if some subsequence (T^{n_k}) is hereditarily hypercyclic, and if and only if the operator $T \oplus T :$ $: (x_1, x_2) \in X \times X \mapsto (Tx_1, Tx_2) \in X \times X$ is hypercyclic. They have generalized their result to sequences (T_n) satisfying the condition that each T_n is an operator with dense range and the sequence is commuting, that is, $T_n T_m = T_m T_n$ for all $m, n \in \mathbb{N}$ [7, Rem. 2.6(3)].

Inspired by this result, Grosse-Erdmann and the author have recently proved [6] the statement contained in Th. 1.2, see below. Before this, we recall that a sequence (T_n) in $L(X)$ is called almost-commuting

[5] if and only if $\lim_{n\to\infty} (T_n T_m - T_m T_n)x = 0$ for every $m \in \mathbb{N}$ and every $x \in X$.

Theorem 1.2. *Let (T_n) be a sequence of operators on X. Then the following assertions are equivalent:*

(A) *(T_n) satisfies the Hypercyclicity Criterion.*

(B) *(T_n) has a densely hereditarily hypercyclic subsequence.*

(C) *For every $N \in \mathbb{N}$, the sequence $(T_n \oplus \cdots \oplus T_n)$ (N-fold) is densely hypercyclic on X^N.*

If, in addition, the sequence (T_n) is almost-commuting, then the preceding assertions are equivalent to each of the following:

(D) *$(T_n \oplus T_n)$ is densely hypercyclic on $X \times X$.*

(E) *(T_n) has a hereditarily hypercyclic subsequence.*

(F) *$(T_n \oplus T_n)$ is hypercyclic on $X \times X$.*

In view of Th. 1.2 one might believe that, even with no hypothesis of commutativity, the assumptions of density in conditions (B) and (C) could be dropped. This is not true: A concrete example on l^2 is constructed in Rem. 2.3(c) of [6]. Here we will extend highly this construction. In fact, we give in Sect. 2 a sufficient 'soft' geometric condition for a Banach space to support a sequence (T_n) of operators having a hereditarily hypercyclic subsequence (T_{n_k}) but such that (T_n) itself is not densely hypercyclic, so not satisfying the Hypercyclicity Criterion. For this, we introduce the concept of "shrinkable" Banach spaces. In addition, a rather natural 'control' property of the norms of the members of such sequence is obtained.

2. Norms of hypercyclic sequences of operators and shrinkable spaces

In this section we shall study Banach spaces X. Since a hypercyclic operator T on X cannot be a contraction we have that $\|T\| > 1$. On the other hand, Rolewicz proved in 1969 ([23], see also [12, Sect. 2]) that if $B : (x_1, x_2, \dots) \mapsto (x_2, x_3, \dots)$ is the backward shift on the sequence space $l^2 = \{(x_n) \in \mathbb{K}^{\mathbb{N}} : \sum_{n=1}^{\infty} |x_n|^2 < \infty\}$ then λB is hypercyclic for every scalar λ with $|\lambda| > 1$. Evidently, $\|\lambda B\| = |\lambda|$. Hence, if H is a separable infinite-dimensional Hilbert space then for any $\alpha > 1$ there is a hypercyclic operator T on H with $\|T\| = \alpha$; T may even be chosen to satisfy the Hypercyclicity Criterion because the λB do.

Turning to general separable infinite-dimensional Banach spaces

X, we will show in the next auxiliary result that, again, any value $\alpha > 1$ may be attained as norm.

Lemma 2.1. *Let X be a separable infinite-dimensional Banach space. Given $\alpha > 1$ there exists an operator T on X with the following properties:*

  (i) *T satisfies the Hypercyclicity Criterion.*

  (ii) *$\|T\| = \alpha$.*

Proof. By [1] or [3] there exists an operator $K \in L(X)$ such that $T_\lambda :$ $:= I + \lambda K$ is a hypercyclic operator for each $\lambda > 0$, and by [20, p. 530] each T_λ may even satisfy the Hypercyclicity Criterion. Now consider the mapping

$$h : \lambda \mapsto \|T_\lambda\| \quad (\lambda > 0).$$

Since $|h(\lambda) - h(\mu)| \le |\lambda - \mu| \|K\|$ we have that h is continuous. But $\lim_{\lambda \to 0} h(\lambda) = 1$ and $\lim_{\lambda \to \infty} h(\lambda) = \infty$. Hence, given $\alpha > 1$, there must be at least one $\lambda > 0$ with $h(\lambda) = \alpha$, that is, $\|T_\lambda\| = \alpha$, as required. $\Diamond$

Let us briefly consider the possible spectral radii of hypercyclic operators, where we now assume that X is a complex Banach space. By a result of Kitai [19] each component of the spectrum of any hypercyclic operator T on a complex Banach space X meets the unit circle, hence its spectral radius satisfies $\rho(T) \ge 1$. Now, on a Hilbert space any value $\alpha \ge 1$ can be attained as spectral radius because Rolewicz' operators satisfy $\rho(\lambda B) = |\lambda| > 1$, and by Chan and Shapiro [9, p. 1446] there is a hypercyclic operator with spectral radius one. On the other hand, there are complex Banach spaces on which each hypercyclic operator has spectral radius one. Indeed, every hypercyclic operator on a Banach space with hereditarily indecomposable dual has finite spectrum (see the proof of [8, Th. 1]) and hence has spectral radius one by Kitai's result.

We now turn to sequences of operators. As we mentioned earlier it is an open problem if every hypercyclic operator satisfies the Hypercyclicity Criterion. On the other hand, it is not surprising that there are hypercyclic sequences (T_n) that do not satisfy the Hypercyclicity Criterion. Indeed, it suffices to consider the following example borrowed from [12]: If $X = \mathbb{R}^2$, (x_n) is a dense sequence in $\mathbb{R}^2$ and for each $n \in \mathbb{N}$ the vector y_n is orthogonal to x_n with norm n, then the sequence $T_n : X \to X$ given by $T_n(a, b) = ax_n + by_n$ is hypercyclic but not densely hypercyclic and, in addition, no subsequence (T_{n_k}) is

hereditarily hypercyclic.

We shall show next that in 'most' separable Banach spaces there is a hypercyclic sequence (T_n) that has a hereditarily hypercyclic subsequence but that is not densely hypercyclic such that, in addition, $\lim_{n\to\infty} \|T_n\|^{1/n}$ equals a prescribed number $\alpha \in [1, \infty]$. This condition was suggested by Gelfand's formula by which $\rho(T) = \lim_{n\to\infty} \|T^n\|^{1/n}$ for operators T on complex Banach spaces, see [24]. Note that, trivially, any hypercyclic sequence (T_n) satisfies $\limsup_{n\to\infty} \|T_n\|^{1/n} \geq 1$, and if, more, (T_n) is hereditarily hypercyclic then $\liminf_{n\to\infty} \|T_n\|^{1/n} \geq 1$.

To specify the Banach spaces for which the result holds we have to consider the following geometric concept. In the sequel, a subspace always refers to a closed linear submanifold.

Definition 2.2. We say that a Banach space X is *shrinkable* whenever it is isomorphic to some proper complemented subspace.

Clearly, such a space X must be infinite-dimensional. We now establish an elementary characterization of shrinkable spaces: A Banach space X is shrinkable if and only if there is a non-trivial subspace $Y \subset$ $\subset X$ such that X is isomorphic to $X \times Y$. Indeed, if X is isomorphic to $X \times Y$ then it is shrinkable because X is a proper complemented subspace of $X \times Y$. Conversely, if X is shrinkable then $X = Z \oplus Y$ with subspaces Z and $Y \neq \{0\}$ of X such that Z is isomorphic to X. Hence X is isomorphic to $X \times Y$.

In the case $\mathbb{K} = \mathbb{R}$, we will consider in the proof of Th. 2.3 the complexification of X and of an operator $S \in L(X)$. A thorough treating of the complexification problem can be found in [21] and [18]. If X is a general topological vector space over $\mathbb{R}$ then its complexification is the product space $\widetilde{X} = X \times X = X + iX = \{(x,y) = x + iy : x, y \in$ $\in X\}$, endowed with the sum in each coordinate and with the scalar multiplication

$$(\alpha + i\beta)(x + iy) = \alpha x - \beta y + i(\beta x + \alpha y) \quad (\alpha, \beta \in \mathbb{R}, \ x, y \in X).$$

Then $\widetilde{X}$ is a topological vector space over $\mathbb{C}$. Assume that $S \in L(X)$. Then the complexification $\widetilde{S}$ of S is defined as $\widetilde{S}(x + iy) = Sx + iSy$. We have that $\widetilde{S} \in L(\widetilde{X})$. If X is normable and $\|\cdot\|$ is a norm generating its topology then $\widetilde{X}$ is also normable and, in fact, there are plenty of norms $\|\|\cdot\|\|$ on $\widetilde{X}$ generating its topology. If, in addition, $(X, \|\cdot\|)$ is a Banach space over $\mathbb{R}$ then $(\widetilde{X}, \|\|\cdot\|\|)$ is a Banach space over $\mathbb{C}$. One of the most useful norms on $\widetilde{X}$ is the Taylor norm $\|\cdot\|_\tau$ given by

$$\|x + iy\|_\tau = \sup_{0 \le t \le 2\pi} \|x \cos t - y \sin t\|.$$

If $S \in L(X)$ and $\|S\|$ ($\|\widetilde{S}\|$) denotes the norm of S (of $\widetilde{S}$) as a member of $L(X)$ (of $L(\widetilde{X})$, respectively) – where $\widetilde{X}$ is assumed to be endowed with $\| \cdot \|_\tau$ – then $\|\widetilde{S}\| = \|S\|$.

Now we can state our main result in this section.

Theorem 2.3. *Assume that X is a separable shrinkable Banach space and let $\alpha \in [1, \infty]$. Then there exists a sequence $(T_n) \subset L(X)$ with the following properties:*

(a) *(T_n) has a hereditarily hypercyclic subsequence.*
(b) *For every $N \in \mathbb{N}, (T_n \oplus \cdots \oplus T_n)$ (N-fold) is hypercyclic on X^N.*
(c) *(T_n) is not densely hypercyclic.*
(d) *$\lim_{n \to \infty} \|T_n\|^{1/n} = \alpha$.*

In particular, (T_n) does not satisfy the Hypercyclicity Criterion.

Proof. It follows from the hypothesis that X is a separable infinite-dimensional Banach space.

Assume first that $\alpha \in (1, \infty)$, and fix $\beta \in (1, \alpha)$. Then by Lemma 2.1 there is an operator A on X with $\|A\| = \beta$ that satisfies the Hypercyclicity Criterion. Since X is shrinkable we can choose a non-trivial subspace Y of X and an isomorphism

$$S : X \to X \times Y, \ x \mapsto (S_1 x, S_2 x).$$

We define the sequence of mappings $T_n : X \to X$ as

(1) $T_n x = A^n S_1 x + \alpha^n S_2 x \quad (n \in \mathbb{N}, \ x \in X).$

It is clear that $(T_n) \subset L(X)$. By continuity of S_2 the set $\mathrm{Ker}\, S_2 = \{x \in X : S_2 x = 0\}$ is a closed subset of X, and we have $\mathrm{Ker}\, S_2 \ne X$ since $S_2(X) = Y \ne \{0\}$. Fix $x \in X \setminus \mathrm{Ker}\, S_2$. Since $\|S_2 x\| > 0$ and $\alpha > \beta$, (1) implies that

$$\|T_n x\| \ge \alpha^n \|S_2 x\| - \|A^n\| \|S_1\| \|x\|$$
(2) $$\ge \alpha^n \|S_2 x\| - \|A\|^n \|S_1\| \|x\|$$
$$= \alpha^n \|S_2 x\| - \beta^n \|S_1\| \|x\| \to \infty$$

as $n \to \infty$. Thus x cannot be hypercyclic for (T_n). Hence all hypercyclic vectors for (T_n) are contained in $\mathrm{Ker}\, S_2$, a proper closed subset of T. Consequently, (T_n) is not densely hypercyclic.

Next, since A satisfies the Hypercyclicity Criterion there is an increasing sequence (n_k) of positive integers such that (A^{n_k}) is hereditarily hypercyclic. Let (m_k) be a subsequence of (n_k) and let x be

a hypercyclic vector for (A^{m_k}). For the vector $z \in X$ with $Sz = = (S_1 z, S_2 z) = (x, 0)$ we have that $T_{m_k} z = A^{m_k} x$ for all $k \in \mathbb{N}$, which tells us that z is hypercyclic for (T_{m_k}). Thus, (T_{n_k}) is hereditarily hypercyclic.

Similarly, since A satisfies the Hypercyclicity Criterion we have that, for each $N \in \mathbb{N}$, $A \oplus \cdots \oplus A$ (N-fold) is hypercyclic on X^N. Let $(x_1, \ldots, x_N)$ be hypercyclic for $A \oplus \cdots \oplus A$ and choose $z_j \in X$ with $Sz_j = (S_1 z_j, S_2 z_j) = (x_j, 0)$ for $1 \leq j \leq N$. Then it follows as above that $(z_1, \ldots, z_N)$ is hypercyclic for $(T_n \oplus \cdots \oplus T_n)$.

As for the behaviour of the norms, consider any vector $x \in X \setminus \mathrm{Ker}\, S_2$ with $\|x\| = 1$. By (2) we have

$$\|T_n\| \geq \|T_n x\| \geq \alpha^n \|S_2 x\| - \beta^n \|S_1\|.$$

Since $\|S_2 x\| > 0$ and $\alpha > \beta$ this implies that

$$\liminf_{n \to \infty} \|T_n\|^{1/n} \geq \alpha.$$

By (1) we also obtain for all x with $\|x\| = 1$ that

$$\|T_n x\| \leq \beta^n \|S_1 x\| + \alpha^n \|S_2 x\| \leq (\|S_1\| + \|S_2\|)\, \alpha^n \quad (n \in \mathbb{N}),$$

so that

$$\limsup_{n \to \infty} \|T_n\|^{1/n} \leq \alpha,$$

which gives part (d) of the theorem.

In the case $\alpha = \infty$ we simply replace α^n by n^n in the above reasoning.

As for the case $\alpha = 1$, we have to distinguish the real and complex scalar case. If $\mathbb{K} = \mathbb{C}$, we take an operator A on X that satisfies the Hypercyclicity Criterion and such that $\rho(A) = 1$, cf. our remarks after Lemma 2.1. By Gelfand's formula we have $\lim_{n \to \infty} \|A^n\|^{1/n} = 1$. When we define operators T_n as in (1) with $n(\|A^n\| + 1)$ in place of α^n the result follows essentially as before.

Finally, if $\mathbb{K} = \mathbb{R}$, we take an operator A that is a compact perturbation of the identity, that is, $A = I + K$, and that satisfies the Hypercyclicity Criterion. Consider the complexification $\widetilde{X}$ of X – endowed, for instance, with the Taylor norm – and the complexification $\widetilde{A}$ of A. Then $\widetilde{A}$ satisfies the Hypercyclicity Criterion [7, Cor. 2.8] and hence is hypercyclic. Since $\widetilde{A}$ is also a compact perturbation of the identity $\widetilde{I}$ on $\widetilde{X}$, we get $\sigma(\widetilde{A}) = \{1\}$ (see again [9, p. 1446]), which implies that $\lim_{n \to \infty} \|(\widetilde{A})^n\|^{1/n} = 1$ by Gelfand's formula. Thus,

$\lim_{n\to\infty} \|A^n\|^{1/n} = 1$ because $\|A^n\| = \|\widetilde{A^n}\| = \|(\widetilde{A})^n\|$ for every n. Now the proof in the real case can be obtained as in the complex case. This completes the proof. $\Diamond$

The result leaves open the possibility that every densely hypercyclic sequence (T_n) satisfies the Hypercyclicity Criterion. We give a simple example to show that this is not the case. Recall that a Fréchet space is a locally convex F-space.

Proposition 2.4. *Let X be a separable infinite-dimensional Fréchet space. Then there exists a densely hypercyclic sequence $(T_n) \subset L(X)$ so that no subsequence (T_{n_k}) is hereditarily hypercyclic. In particular, (T_n) does not satisfy the Hypercyclicity Criterion.*

Proof. Let (x_n) be a dense sequence in X and f a non-trivial continuous linear functional on X. Define operators $T_n \in L(X)$ by $T_n x = f(x)x_n$ ($x \in X$). Then, clearly, (T_n) has the desired properties. $\Diamond$

Concerning Th. 2.3 one may wonder whether infinite-dimensional non-shrinkable Banach spaces can exist. This is indeed so. In [13], Gowers was able to construct an infinite-dimensional Banach space X that is not isomorphic to any of its hyperplanes, and X even has an unconditional basis. In fact, it turned out that X is not isomorphic to any proper subspace and hence cannot be shrinkable. Moreover, Gowers and Maurey showed that every (real or complex) hereditarily indecomposable Banach space, whose existence they also demonstrated, is not isomorphic to any proper subspace, see [14, Cor. 19 and Th. 21].

Next we provide a sufficient condition for a Banach space X to have the desired property. Such condition is satisfied, for instance, by the sequence spaces c_0 and l^p ($1 \le p < \infty$) with respect to their canonical bases. We recall that two basic sequences (x_n), (y_n) in a Banach space are called equivalent if, for every sequence (a_n) of scalars, the series $\sum_{n=1}^{\infty} a_n x_n$ converges if and only if the series $\sum_{n=1}^{\infty} a_n y_n$ converges, see [2]. We have that X is shrinkable if it admits a Schauder basis (e_n) which is equivalent to the shifted sequence (e_{n+1}). For the proof one need only to observe that by the Closed Graph Theorem the mapping $\sum_{n=1}^{\infty} a_n e_n \mapsto \sum_{n=1}^{\infty} a_n e_{n+1}$ defines an isomorphism of X onto the complemented subspace $Y := \overline{\text{span}}\{e_n : n \ge 2\}$.

If the basis (e_n) is unconditional one may replace the shift $\sum_{n=1}^{\infty} a_n e_n \mapsto \sum_{n=1}^{\infty} a_n e_{n+1}$ in the last paragraph by $\sum_{n=1}^{\infty} a_n e_n \mapsto \sum_{n=1}^{\infty} a_n e_{\pi(n)}$ with an arbitrary one-to-one non-onto self-mapping π on $\mathbb{N}$. It should also be noted that a Schauder basis need not to be equivalent to a shifted one: consider, for instance, the canonical

basis (e_n) of the Hilbert sequence space $X = \{(\lambda_n) \in \mathbb{R}^{\mathbb{N}} : \|(\lambda_n)\| :=$
$:= (\sum_{n=1}^{\infty} |\lambda_n|^2 e^{n^2})^{1/2} < \infty\}$ and the sequence of scalars $a_n = \frac{1}{n} e^{-n^2/2}$.

Finally, let us recall that an infinite-dimensional Banach space X is said to be prime whenever every complemented infinite-dimensional subspace is isomorphic to X. As for examples, the sequence spaces c_0, l_p $(1 \le p < \infty)$ and l_∞ are prime, see for instance [10] and the references contained in it. Hence, in view of Def. 2.2, one might believe that there is a strong relationship between shrinkable spaces and prime spaces. Obviously, any prime space is shrinkable. But that a shrinkable space is prime is far from being true. For instance, the space $X = l_2 \oplus c_0$ is clearly shrinkable but it is not prime since l_2 is reflexive while X is not.

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