Mathematica Pannonica
16/1 (2005), 19–27

# ON THE FUNCTION $\zeta(S)\zeta(S-A)\dots\zeta(S-RA) = \sum \frac{\sigma_{A,R+1}(N)}{N^S}$

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*Received*: November 2004

*MSC 2000*: 11 A 25, 11 M 06, 11 N 37

*Keywords*: Generalized sum of divisors function, mean value, limit distribution.

**Abstract**: Some theorems are proved for $\sigma_{a,r+1}(n)$.

**1. Introduction.** Let $\mathcal{P}$ be the set of primes, $p, q$ with or without suffixes denote general elements of $\mathcal{P}$, $\omega(n)$ = the number of distinct prime divisors of $n$, $P(n)$ = the largest and $p(n)$ the smallest prime divisor of $n$. We shall write $x_1 = \log x$, $x_2 = \log x_1, \dots,$ and $e(\alpha) := e^{i\alpha}$.

Let $\varphi(n)$ = Euler's totient function, $\varphi_k$ its $k$-fold iterate.

Let $\sigma_{a,r+1}(n) = \sum\limits_{d_0 d_1 d_2 \dots d_r = n} d_1^a \cdot d_2^{2a} \dots d_r^{ra}$. Then

$$\sum_n \frac{\sigma_{a,r+1}(n)}{n^s} = \zeta(s)\zeta(s-a)\dots\zeta(s-ra). \tag{1.1}$$

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(1) Research supported by the Applied Number Theory Research Group of the Hungarian Academy of Science and by a grant from OTKA T46993.
(2) Research supported in part by a grant from NSERC of the author.

The special case $r = 1$ gives: $\sigma_{a,2}(n) = \sum_{d_1|n} d_1^a$ which is the same as $\sigma_a(n)$ in the usual notation.

We may assume that $a > 0$. Let us observe that

$$\frac{\sigma_{a,r+1}(n)}{n^r} = \sum_{d_0 d_1 \ldots d_r = n} d_{r-1}^{-a} d_{r-2}^{-2a} \ldots d_0^{-ra} = \sigma_{-a,r+1}(n), \tag{1.2}$$

and so

$$\sum_n \frac{\sigma_{-a,r+1}(n)}{n^s} = \zeta(s+ra)\zeta(s+(r-1)a)\ldots\zeta(s). \tag{1.3}$$

Let $F_r(s) = \zeta(s+ra)\zeta(s+(r-1)a)\ldots\zeta(s)$. Since

$$F_r(s) = \zeta(s+ra)F_{r-1}(s), \tag{1.4}$$

therefore

$$\sigma_{-a,r+1}(n) = \frac{1}{n^{ra}} \sum_{d|n} \sigma_{-a,r}(d) \cdot d^{ra}. \tag{1.5}$$

Since $\sigma_{-a,r+1}(n)$ is multiplicative, therefore

$$\frac{1}{\prod_{\nu=0}^{r} \left(1 - \frac{p^{-\nu a}}{p^s}\right)} = \sum_{\beta=0}^{\infty} \frac{\sigma_{a,r+1}(p^\beta)}{p^{\beta s}}. \tag{1.6}$$

Let $\zeta = 1/p^s$, $\Lambda = \dfrac{1}{p^a}$. From (1.6), by writing it as partial fractions,

(1.7)

$$\frac{1}{(1-x)(1-\Lambda x)\ldots(1-\Lambda^r x)} = \frac{A_0}{1-x} + \frac{A_1}{1-\Lambda x} + \ldots + \frac{A_r}{1-\Lambda^r x},$$

$$= \sum_{k=0}^{\infty} \left(A_0 + A_1 \Lambda^k + \ldots + A_r \Lambda^{rk}\right) x^k,$$

where

$$A_j = \frac{1}{\prod_{\substack{\nu=0 \\ \nu \neq j}}^{r} (1 - \Lambda^{\nu - j})} = \frac{1}{\prod_{\substack{\nu=0 \\ \nu \neq j}}^{r} \left(1 - \left(\frac{1}{p^a}\right)^{\nu - j}\right)}. \tag{1.8}$$

Thus

$$(1.9)\qquad \begin{cases} \sigma_{-a,r+1}(p^\alpha)=A_0+A_1\Lambda^\alpha+A_2\Lambda^{2\alpha}+\dots+A_r\cdot\Lambda^{r\alpha},\\ \Lambda=p^{-a}.\end{cases}$$

Let $\eta_m(p)=\prod_{l=1}^{m}\left(1-\frac{1}{p^{al}}\right)$. By easy calculation we have

$$(1.10)\qquad A_j=\frac{(-1)^j\cdot p^{-\frac{a_j(j+1)}{2}}}{\eta_{r-j}(p)\eta_j(p)},$$

whence especially

$$(1.11)\qquad \sigma_{-a,r+1}(p)=1+\frac{1}{p^a}+O\left(\frac{1}{p^{2a}}\right)$$

follows.

There exists a lot of interesting and important theorem for the function $\sigma_{-a}(n)$:

a. The mean-value of $\sigma_{-a}(n)$ with good remainder term.
b. The mean-value of $\sigma_{-a}$ on some special subsets of integers.
c. The distribution of $\sigma_{-a}(n)$.
d. The maximal order of $\sigma_{-a}(n)$.

**2.** Let $f(n)=\log\sigma_{-a}(n)$ Assume that $N$ is a "champion" in the sense that $f(n)\le f(N)$ if $n<N$. From (1.9) it is obvious that $f(p^k)<$ $<f(p)$ if $k\ge 2$, therefore $N$ should be a square-free integer. Since $f$ is monotonically decreasing on the set of primes, therefore $N=$ $=p_1p_2\dots p_k$ $(p_1<p_2<\dots,p_k)$ is the product of the first $k$ prime numbers, consequently

$$\log N=\log p_1+\dots+\log p_k=p_k+O\left(\frac{p_k}{(\log p_k)^A}\right),$$

$$p_k=\log N+O\left(\frac{\log N}{(\log\log N)^A}\right),$$

$$f(N)=f(p_1)+\dots+f(p_k).$$

Since $f(p_j)=\log\sigma_{-a,r}(p_j)=\frac{1}{p_j^a}+O\left(\frac{1}{p_j^{2a}}\right)$, therefore

$$O(1)+f(N)=\sum_{j=1}^{k}\frac{1}{p_j^a}=\int_2^{p_k}\frac{1}{u^a}d\pi(u)=\int_2^{p_k}\frac{du}{u^a\log u}=\int_2^{\log p_k}\frac{e^{-v(a-1)}}{v}dv.$$

Thus $f(N)=O(1)$ if $a>1$. If $a=1$, then $f(N)=\log\log p_k+O(1)=$ $=\log\log\log N+O(1)$, while in the case $0<a<1$:

$$f(N)=\frac{e^{(1-a)\log\log N}}{(1-a)\log\log N}\left(1+O\left(\frac{1}{\log\log N}\right)\right).$$

Hence we obtain the following

**Theorem 1.** *We have*

(a) $$\sigma_{-a,r+1}(n)=O(1)\quad \text{if } a>1,$$

(b) $$\limsup_{n\to a}\frac{\sigma_{-1,r+1}(n)}{\log\log n}=c,\quad \text{where } 0<c<\infty,$$

(c)
$$\max_{n\le N}\sigma_{-a,r+1}(n)=\exp\left(\frac{(\log N)^{1-a}}{(1-a)\log\log N}\right)\left(1+O\left(\frac{1}{(\log\log N)^A}\right)\right),$$

*holds for every fixed A.*

**3.** A. S. Fajnleib [5] proved the following theorem which is referred now as

**Lemma 1.** *Let $\psi(m)$ be an additive arithmetical function for which:*

1. $\sum\frac{\psi^2(p^k)}{p^k}<\infty$, *the summation is extended for all prime powers $p^k$,*

2. $|\psi(n)-\psi(m)|\ge\frac{1}{(nm)^b}$ if $n\ne m$, *for square-free integers $n,m$, where $b$ is a suitable constant.*

*Then uniformly in $u$,*

$$\frac{1}{x}\#\left\{n\le x|\psi(n)-\sum_{p\le N}\frac{\psi(p)}{p}<u\right\}-F(u)=$$
$$=O\left(\frac{\log\log 1/\rho_x}{\left(\log\frac{1}{\rho_x}\right)\left(\log\log\log\frac{1}{\rho_x}\right)}\right),$$

*where $F$ is the distribution function, the corresponding characteristic* *function of which is*

$$\varphi(t)=\prod_p\left(1-\frac{1}{p}\right)\left(1+\sum_{k=1}^{\infty}\frac{e^{it\psi(p^k)}}{p^k}\right)e^{-it\frac{\psi(p)}{p}},$$

$$\rho_x=\sum_{p>\exp\left(\frac{x_1\cdot x_3}{x_2}\right)}\frac{\psi^2(p)}{p}.$$

Let $\psi(n)=\log\sigma_{-1,r+1}(n)$. Then $\psi(n)=\log\frac{\sigma_{1,r+1}(n)}{n^r}$ (by (1.2)), furthermore $\sigma_{1,r+1}(n)$ is an integer. Let $n,m\le x$, $n$ and $m$ be square free, $n\neq m$.

We would like to estimate from below the quantity

$$|\psi(n)-\psi(m)|=\left|\log\left(\frac{\sigma_{1,r+1}(n)}{n^r}\cdot\frac{m^r}{\sigma_{1,r+1}(m)}\right)\right|. \tag{3.1}$$

We may assume that $(n,m)=1$, $n,m\ge 2$. Let $P(\nu)$ denote the largest prime factor of $\nu$.

Let $P(mn)=p^*$, and $p^*|n$ say. Then $p^*\nmid m$, $p^*\nmid\sigma_{1,r+1}(n)$, therefore the argument on the right hand side of (3.1) is $\neq 1$, and so (3.1) is larger than $\ge\frac{1}{(nm)^{2r}}$, say.

Therefore the condition 2 of Lemma 1 holds.

The fulfilment of condition 1 is obvious.

From (1.11) we have $\psi(p)=\frac{1}{p}+O\left(\frac{1}{p^2}\right)$, and so

$$\rho_x=(1+o_x(1))\sum_{p>\exp\left(\frac{x_1\cdot x_3}{x_2}\right)}1/p^3,\ \asymp\exp\left(\frac{-2x_1\cdot x_3}{x_2}\right)\cdot\frac{x_2}{x_1x_3},$$

and by an easy computation

$$O\left(\frac{\log\log 1/\rho_x}{\left(\log\frac{1}{\rho_x}\right)\left(\log\log\log\frac{1}{\rho_x}\right)}\right)=O\left(\frac{x_2^2}{x_1x_3^2}\right).$$

From Lemma 1 we obtain

**Theorem 2.** *Let* $\psi(n)=\log\sigma_{-1,r+1}(n)$. *Let* $H_r(u)$ *be the distribution function the characteristic function* $\varphi_r(t)$ *of which is defined by*

$$\varphi_r(t)=\prod_p\left(1-\frac{1}{p}\right)\left(1+\sum_{k=1}^{\infty}\frac{e^{i\tau\psi p^k}}{p^k}\right).$$

*Then*

$$\frac{1}{x}\#\{n \le x \mid \psi(n) < u\} - H_r(u) = O\left(\frac{x_2^2}{x_1 \cdot x_3^2}\right).$$

**Remark.** If $-1 < a < 0$, then it is not known, whether the condition (1) in Lemma 1 holds for $\psi(n) = \log \sigma_{-a,r+1}(n)$ or not. Naturally, the limit distribution exists, since the conditions of the Erdős–Wintner theorem [4] are satisfied.

According to the Erdős–Wintner theorem an additive arithmetical function $g(n)$ has the limit distribution $F$ if and only if the series

$$\sum_{|g(p)|<1} \frac{g(p)}{p}, \quad \sum_{|g(p)|<1} \frac{g^2(p)}{p}, \quad \sum_{|g(p)|\ge 1} \frac{1}{p}$$

are convergent, Furthermore,

$$\varphi_F(t) = \prod_p \left(1 - \frac{1}{p}\right)\left(1 + \sum_{k=1}^{\infty} p^{-k} e(tg(p^k))\right),$$

($\varphi_F(t)$ is the characteristic function corresponding to $F$).

$F$ can be interpreted as the distribution function of the random variable $\eta = \sum \zeta_p$, where $\zeta_p$ are independent random variables with the purely discrete distribution, and

$$\varphi_{\zeta_p}(t) = \left(1 - \frac{1}{p}\right)\left(1 + \sum_{k=1}^{\infty} p^{-k} e(tg(p^k))\right).$$

P. Levy [7] proved: If $\sum \zeta_p = \eta$ is a convergent sum, then $F(= F_\eta)$ is continuous (everywhere) if and only if

$$\sum P(\zeta_p \ne 0) = \infty. \tag{3.2}$$

If (3.2) holds, then $F_\eta$ is of pure type, either absolutely continuous or singular (Lukács [8]).

For some distribution function $F$ let

$$Q_F(h) := \sup_x (F(x+h) - F(x)), \tag{3.3}$$

the concentration function of $F$. It was proved that

$$\frac{1}{(\log t)} \ll Q_F(1/t) \ll \frac{1}{(\log t)} \quad (t > 2) \tag{3.4}$$

holds for the following additive function $g(n)$:

a. $g(n) = \log \dfrac{\varphi(n)}{n}$ (Tjan [10]),

b. $g(n)=\log\frac{\sigma(n)}{n}$ (Erdős [2]),

c. if $g$ is strongly additive and

$$\sum_{p>t^A}\frac{|g(p)|}{p}<1/t,\quad |g(p_1)-g(p_2)|>\frac{1}{t},$$

if $p_1\neq p_2<t^\delta$, ($p_1,p_2$ run over the primes) hold with suitable positive constants $A$ and $\delta$ for every large $t$ (Erdős and Kátai [3]).

Easy to see that the assertion remains valid if the "strongly additiveness" is changed to additiveness.

The last conditions are clearly satisfied for $g(n)=\log\sigma_{-a,r+1}(n)$, thus the following assertion is true, since $g(n)=\frac{1}{p^a}+O\left(\frac{1}{p^{a+1}}\right)$.

**Theorem 3.** *Let $F$ be the limit distribution function of* $\log\sigma_{-a,r+1}(n)$, *and $Q_F$ be defined by* (3.3). *Then* (3.4) *holds true.*

A similar theorem can be proved for $\log\sigma_{-a,r+1}(P(n))$, $\log\sigma_{-a,r+1}(P(p))$, where $P$ is an integer valued polynomial, and $p$ runs over $\mathcal{P}$.

These follow from a theorem of Indlekofer and Kátai [6].

**4.** Let $A(n)=\sigma_{1,r+1}(n)$ and $A_k(n)$ be the $k$ fold iterate of $A(n)$. We can estimate $\omega(A_k(n))$.

**Theorem 4.** *Let $k,r$ be fixed positive integers. Then*

$$\lim_{x\to\infty}\sup_{z\in\mathbb{R}}\left|x^{-1}\#\left\{n\le x\;\middle|\;\frac{\omega(A_k(n))-a_k\cdot x_2^{k+1}}{b_k\cdot x_2^{k+1/2}}<z\right\}-\Phi(z)\right|=0,$$

*where*

$$a_k=\frac{1}{(k+1)!},\quad b_k=\frac{1}{k!\sqrt{2k+1}}.$$

The assertion with $\omega(\varphi_k(n))$ instead of $\omega(A_k(n))$ is proved in the paper of Bassily, Kátai and Wijsmuller [1]. Th. 4 can be proved on the same way. We omit the proof.

**5.** Assume that $0<a\le 1$,

$$A_{a,r+1}(x)=\sum_{n\le x}\sigma_{-a,r+1}(n). \tag{5.1}$$

Since

$$A_{a,1}(x) = \sum_{d\le x} \frac{1}{d^a}\left[\frac{x}{d}\right] = x\left(\zeta(1+a) - \frac{x^{-a}}{a}\right) + O\left(x^{1-a}\right) =$$
$$= \zeta(1+a)x + O\left(x^{1-a}\right)$$

and

$$A_{a,r+1}(x) = \sum_{d\le x} \frac{1}{d^{ra}} A_{a,r}\left(\frac{x}{d}\right),$$

by induction on $r$, we can deduce that

$$A_{a,r+1}(x) = \zeta(1+a)\cdots\zeta(1+ra)x + O\left(x^{1-a}\right).$$

The error term can be reduced, by using some more complicated method. We hope to return to this question in our forthcoming paper.

We consider only the case $a=1$. It is known that

$$A_{1,1}(x) = \sum_{n\le x} \sigma_{-1}(n) = \zeta(2)x - \frac{1}{2}\log x + \Delta(x),$$

where

$$\Delta(x) \ll (\log x)^{1/3}.$$

From the obvious identity

$$A_{a,r}(x) = \sum_{d\le x} \frac{1}{d^{ar}} A\left(\frac{x}{d}\right)$$

we obtain that

$$A_{1,2}(x) = \sum_{d\le x} \frac{1}{d^2}\left(\zeta(2)\frac{x}{d} - \frac{1}{2}\log\frac{x}{d}\right) + O\left(\sum_{d\le x}\frac{1}{d^2}\left(\log\frac{x}{d}\right)^{1/3}\right) =$$
$$= \zeta(2)x\left(\sum_{d\le x^3}\frac{1}{d^3}\right) - \frac{1}{2}(\log x)\sum_{d\le x}\frac{1}{d^2} + O\left((\log x)^{1/3}\right) =$$
$$= \zeta(2)x\cdot\zeta(3) - \frac{1}{2}(\log x)\zeta(2) + O\left((\log x)^{1/3}\right).$$

We can prove by induction that

$$A_{1,t}(x) = \zeta(2)\ldots\zeta(t+1)x - \frac{1}{2}\zeta(2)\ldots\zeta(t)\log x + O\left((\log x)^{1/3}\right), \tag{5.2}$$

$(t = 1, 2, \ldots)$.

This is clear:

$$A_{1,t+1}(x)=\sum_{d\le x}\frac{1}{d^{t+1}}\cdot A_{1,t}(x)=$$

$$=x\cdot\zeta(2)\dots\zeta(t+1)\sum_{d\le x}\frac{1}{d^{t+1}}-\frac{1}{2}\zeta(2)\dots\zeta(t)\cdot\sum_{d\le x}\frac{1}{d^{t+1}}\log\frac{x}{d}+$$

$$+O\left((\log x)^{1/3}\right)=$$

$$=x\zeta(2)\dots\zeta(t+1)\zeta(t+2)-\frac{1}{2}\zeta(2)\dots\xi(t+1)\log x+O\big((\log x)^{1/3}\big).$$

Thus the assertion (5.2) holds for every fixed $t$.

## References


[1] BASSILY, N. L., KÁTAI, I., WIJSMULLER, M.: Number of prime divisors of $\varphi_k(n)$, where $\varphi_k$ is the $k$-fold iterate of $\varphi$, *Journal of Number Theory* **65**/2 (1997), 226–239.

[2] ERDŐS, P.: On the distribution of numbers of the form $\sigma(n)|n$ and on some related questions, *Pacific J. Math.* **52** (1974), 59–65.

[3] ERDŐS, P., KÁTAI, I.: On the concentration of additive functions, *Acta Sci. Math.* **91** (1978), 295–305.

[4] ERDŐS, P., WINTNER, A.: Additive arithmetical functions and statistical independence, *Amer. Journal Math.* **61** (1939), 713–721.

[5] FAJNLEIB, A. S.: Generalization the Erseen inequality and its application in probabilistic in number theory (in Russian), *Izvestya Akademii Nauk, ser. matem.* **32** (4) (1968), 859–879.

[6] INDLEKOFER, K.-H., KÁTAI, I.: On the modulus of continuity of the distribution of some arithmetical functions, New Trends in Probability and Statistics Vol. 2. Analytic and Probabilistic Methods in Number Theory, VSP/TEV 1992, Vilnius, pp. 223–234.

[7] LEVY, P.: Sur les séries dont les terms son des variables eventuelles indépendentes, *Studia Math.* **3** (1931), 119–155.

[8] LUKÁCS, E.: Characteristic Functions, Griffin, London, 1960.

[9] POSTNIKOV, A. G.: Introduction to the analytic number theory, Nauka, Moskva, 1971 (in Russian).

[10] TJAN, M. M.: On the question of the distribution of values of the Euler function $\varphi(n)$, *Liet. Matem. Rink.* **6** (1966), 105–119 (in Russian).