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SQUARE-FREE VALUES OF THE CARMICHAEL FUNCTION

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Abstract: Asymptotic expansion for the mean-value of square-free values of the Carmichael function is given.

§ 1. Introduction

This paper contains a short remark to the result of F. Pappalardi, F. Saidak and J. E. Shparlinski in [1].

Let $\lambda(n)$ be the Carmichael function, defined for prime power p^ν as follows:

$$\lambda(p^\nu) = \begin{cases} p^{\nu-1}(p-1) & \text{if } \ p \geq 3 \text{ or } \nu \leq 2, \\ 2^{\nu-2} & \text{if } \ p = 2 \text{ and } \nu \geq 3, \end{cases} \tag{1.1}$$

furthermore, for $n \geq 2$,

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$$\lambda(n) = LCM\left[\lambda(p_1^{\nu_1}), \dots, \lambda(p_s^{\nu_s})\right], \tag{1.2}$$

where $n = p_1^{\alpha_1} \dots p_s^{\nu_s}$ is the prime factorization of n. Furthermore let $\lambda(1) = 1$.

Let $\mathcal{M}$ be the set of square-free numbers, p, q with or without suffixes denote prime numbers. Let $x_1 = \log x$, $x_2 = \log x_1, \dots$.

Let $L(x)$ be the numbers of those $n \le x$ for which $\lambda(n) \in \mathcal{M}$. In [1], by using the well-known theorem of E. Wirsing [2] it was proved that

$$L(x) = (\kappa + o(1)) \frac{x}{x_1^{1-\alpha}},$$

where κ is a positive constant, and

$$\alpha := \prod_p \left(1 - \frac{1}{p(p-1)}\right) = 0.37395\dots . \tag{1.3}$$

By using a result of B. V. Levin and A. S. Fainleib [3] (Lemma 1) we can deduce almost immediately the following

Theorem. *Let N be an arbitrary positive integer. There exists suitable real numbers $E_0, E_1, \dots,$ such that*

$$L(x) = x \sum_{l=0}^{N} E_l \, x_1^{-l-1-\alpha} + O\left(x \cdot x_1^{-N-2+\alpha}\right), \tag{1.4}$$

$E_0 = \kappa$.

The coefficients E_ν can be computed by using Lemma 1.

Lemma 1. *Let $f(n)$ be a multiplicative function, $\lambda_f(n)$ be defined from the equation*

$$f(n) \log n = \sum_{d|n} f(d) \lambda_f\left(\frac{n}{d}\right). \tag{1.5}$$

Assume that

$$\sum_{n \le x} \frac{\lambda_f(n)}{n} = \tau x_1 + B + h(x), \tag{1.6}$$

$h(x) = O\left(x_1^{-N-1}\right)$,

$$\prod_{p \le x} \left(1 + \sum_{r=1}^{\infty} \frac{|f(p^r)|}{p^r}\right) = O(x_1^A), \tag{1.7}$$

and

(1.8)
$$\lim_{p\to\infty}\sum_{r=1}^{\infty}\frac{|f(p^r)|}{p^r}=0.$$

Then

(1.9)
$$(m(x) :=) \sum_{n\le x} f(n) = \sum_{1\le\nu<\operatorname{Re}\tau+N+1-A} \tau(\tau-1)\dots(\tau-\nu+1)c_\nu x_1^{\tau-\nu} +$$
$$+ O\left(x\cdot x_1^{A-N-1-\varepsilon}\right),$$

where

(1.10)
$$c_\nu = \sum_{\lambda+\mu=\nu-1}(-1)^\lambda a_\mu,$$

and

(1.11)
$$\nu a_\nu = -\sum_{\lambda+\mu=\nu-1} a_\lambda b_\mu,\ \ b_0 = B,$$
$$b_\mu = \frac{(-1)^{\mu-1}}{(\mu-1)!}\int_1^\infty \frac{h(u)(\log u)^{\mu-1}}{u}du,$$

$(\mu\ge 1)$.

§ 2. Proof of the theorem

Let f be a multiplicative function, $f(2)=f(2^2)=f(2^3)=1$, $f(2^j)=0$ if $j\ge 4$.

If p is an odd prime, then let

$$f(p^l)=0\ (l=1,2,\dots)\quad \text{if } p-1\notin\mathcal{M}$$

and

$$1=f(p)=f(p^2),\ f(p^l)=0\ (l=3,4,\dots)\quad \text{if } p-1\in\mathcal{M}.$$

In [1] it was proved that $\lambda(n)\in\mathcal{M}$ if and only if $f(n)=1$. Thus $L(x)=\sum_{n\le x} f(n)$.

We shall count $\lambda_f(n)$.

We have

(2.1)

$$F(s):=\sum \frac{f(n)}{n^s}=\left(1+\frac{1}{2^s}+\frac{1}{2^{2s}}+\frac{1}{2^{3s}}\right)\prod_{\substack{p-1\in\mathcal{M}\\ p>2}}\left(1+\frac{1}{p^s}+\frac{1}{p^{2s}}\right),$$

$$-\frac{F'}{F}(s)=\frac{-\frac{\log 2}{2^s}-\frac{2\log 2}{2^{2s}}-\frac{3\log 2}{2^{3s}}}{1+\frac{1}{2^s}+\frac{1}{2^{2s}}+\frac{1}{2^{3s}}}-\sum_{\substack{p-1\in\mathcal{M}\\ p>2}}\frac{\frac{\log p}{p^s}+\frac{2\log p}{p^{2s}}}{1+\frac{1}{p^s}+\frac{1}{p^{2s}}}=$$

$$=-\sum\frac{\Lambda_f(n)}{n^s}.$$

It is clear that $\Lambda_f(n)=0$, if n is not a prime, or prime power. Furthermore $\Lambda_f(p^\alpha)=0$ for every $\alpha\in\mathbb{N}$ if $p>2$, $p-1\notin\mathcal{M}$; $\Lambda_f(p)=$ $=\log p$, if $p-1\in\mathcal{M}$.

From (2.1) we can compute the values $\Lambda_f(p^\alpha)$ for every p and α. Let $p-1\in\mathcal{M}$. Then

$$\Lambda_f(p^\alpha)=\begin{cases}\log p & \text{if}\quad \alpha\equiv 1,2 \pmod 3,\\ -2\log p & \text{if}\quad \alpha\equiv 0 \pmod 3).\end{cases}$$

Furthermore

$$\Lambda_f(2^\alpha)=\begin{cases}\log 2 & \text{if}\quad \alpha\equiv 1,2,3 \pmod 4,\\ -3\log 2 & \text{if}\quad \alpha\equiv 0 \pmod 4).\end{cases}$$

Let us observe that the conditions of Lemma 1 are satisfied. (1.7) holds with $A=1$. The fulfilment of (1.8) is obvious.

Let $E(x)=\sum_{n\le x}\Lambda_f(n)$, $E_1(x)=\sum_{p-1\in\mathcal{M}}\log p$. Then $E(x)-$ $-E_1(x)=O(\sqrt{x})$. Furthermore

$$E_1(x)=\sum_{2<p\le x}(\log p)\sum_{d^2|p-1}\mu(d)=\sum_{d^2\le x}\mu(d)\vartheta(x,d^2,1), \tag{2.2}$$

where

$$\vartheta(x,k,l):=\sum_{\substack{q\le x\\ q\equiv l \pmod k}}\log q. \tag{2.3}$$

By using the Siegel–Walfisz theorem:

$$\vartheta(x,d^2,1)=\frac{x}{\varphi(d^2)}\left(1+O\left(e^{-c\sqrt{x_1}}\right)\right) \tag{2.4}$$

uniformly as $d^2\le x_1^M$, M is an arbitrary constant. Using this, and the trivial $\vartheta(x,d^2,1)\le x_1\cdot\dfrac{x}{d^2}$ estimation, we obtain immediately that

$$E_1(x) = x \sum_{d^2 \le x_1^M} \frac{\mu(d)}{\varphi(d^2)} + O\left(xe^{-c\sqrt{x_1}}\right) + O\Bigg(xx_1 \sum_{d \ge x_1^{M/2}} \frac{1}{d^2}\Bigg).$$

Since

$$\sum_{d^2 \le x_1^M} \frac{\mu(d)}{\varphi(d^2)} = \alpha - \sum_{d^2 \ge x_1^M} \frac{\mu(d)}{\varphi(d^2)} = \alpha + O\left(x_1^{-M/2}\right)$$

$$\sum_{d \ge x_1^{M/2}} 1/d^2 \ll x_1^{-M/2},$$

we obtain that

(2.4)
$$E_1(x) = \alpha x + O\left(x \cdot x_1^{1-M/2}\right), \quad \text{and so}$$
$$E(x) = \alpha x + O\left(x \cdot x_1^{1-M/2}\right).$$

By using partial integration, we obtain (1.6) immediately, and so we can apply Lemma 1.

Our theorem is straightforward. ◊

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