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A CHARACTERIZATION OF DAMPED AND UNDAMPED HARMONIC OSCILLATIONS BY A SUPERPOSITION PROPERTY II

Peter **Schöpf**

Institut für Mathematik, Karl-Franzens Universität Graz, Heinrichstraße 36, A–8010 Graz, Austria

Jens **Schwaiger**

Institut für Mathematik, Karl-Franzens Universität Graz, Heinrichstraße 36, A–8010 Graz, Austria

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Abstract: Let k_1, k_2 be two fields, $f : k_1 \to k_2$ and $a \in k_1, A \in k_2$. We call $f_{[a]} : k_1 \to k_2$, $f_{[a]}(x) := f(x+a)$ the a-translate and Af the A-scaled of f. A function $f : k_1 \to k_2$ has the superposition property, if for every $a \in k_1, A \in k_2$ the superposition $f + Af_{[a]}$ is again a scaled translate of f, i.e. there exist $b \in \in k_1, B \in k_2$ with $f + Af_{[a]} = Bf_{[b]}$. We characterize the functions with the superposition property which are continuous from $\mathbb{R}$ to $\mathbb{R}$ (resp. $\mathbb{R}$ to $\mathbb{C}$) or holomorphic from $\mathbb{C}$ to $\mathbb{C}$. The methods used hereby do not apply to functions from $\mathbb{C}$ to $\mathbb{R}$. This case is not treated in this paper and therefore an open problem.

In [4] the authors determined all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

E-mail addresses: peter.schoepf@uni-graz.at, jens.schwaiger@uni-graz.at

$$\forall_{a,A\in\mathbb{R}}\ \exists_{b,B\in\mathbb{R}}\ \forall_{x\in\mathbb{R}}\quad f(x)+Af(x+a)=Bf(x+b) \tag{1}$$

under the additional hypothesis that the number of zeroes of f is at most countable.

During the 41st ISFE Miklós Laczkovich presented a talk entitled Linear functional equations and Shapiro's conjecture ([1]). The second author, also a participant of this symposium, discussed with him possibilities to apply his results to solve (1). Indeed he found a way to do this. Interestingly enough it even turned out that "only" the ideas how to make his results applicable are of importance and not the results as such. The authors are indebted to Miklós Laczkovich for his important contribution to this paper.

In the following we will

1. prove that the main result of [4] remains true without the above hypothesis on the zeroes of f,

2. determine all continuous functions $g:\mathbb{R}\to\mathbb{C}$ such that

$$\forall_{a\in\mathbb{R},A\in\mathbb{C}}\ \exists_{b\in\mathbb{R},B\in\mathbb{C}}\ \forall_{x\in\mathbb{R}}\quad g(x)+Ag(x+a)=Bg(x+b) \tag{2}$$

and

3. all holomorphic functions $h:\mathbb{C}\to\mathbb{C}$ such that

$$\forall_{a,A\in\mathbb{C}}\ \exists_{b,B\in\mathbb{C}}\ \forall_{x\in\mathbb{C}}\quad h(x)+Ah(x+a)=Bh(x+b). \tag{3}$$

The basic (and deep) tools we need come from the famous paper [5].

In this paper Laurent Schwartz considers the (metrizable) topological vector spaces $C(\mathbb{R},\mathbb{C})$ and $\mathcal{O}(\mathbb{C})$ of continuous (holomorphic) mappings from $\mathbb{R}$ ($\mathbb{C}$) to $\mathbb{C}$, equipped with the topology of uniform convergence on compact subsets.

A subspace V of $C(\mathbb{R},\mathbb{C})$ is called translation invariant if it contains with f all the functions $x\mapsto f(x+t)$, where $t\in\mathbb{R}$. Likewise, a subspace V of $\mathcal{O}(\mathbb{C})$ is called translation invariant if it contains with h all the functions $z\mapsto h(z+t)$, where $t\in\mathbb{C}$.

A function E is called an exponential monomial if it is of the form $E(x)=x^j\exp(\lambda x)$ ($E(z)=z^j\exp(\lambda z)$) for some complex number λ and some $j\in\mathbb{N}_0$.

Then, Th. 5 from [5] together with some remarks from [5, p. 876] says the following.

Theorem 1 (Laurent Schwartz). *Any closed translation invariant subspace V of $C(\mathbb{R},\mathbb{C})$ different from $C(\mathbb{R},\mathbb{C})$ is the closure of the linear* *subspace of V generated by all exponential monomials contained in V.*

Especially, even if $V = C(\mathbb{R}, \mathbb{C})$, any closed translation invariant subspace $V \neq \{0\}$ of $C(\mathbb{R}, \mathbb{C})$ contains an exponential function E_λ, $E_\lambda(x) = \exp(\lambda x)$ for some $\lambda \in \mathbb{C}$ and all $x \in \mathbb{R}$.

For $\mathcal{O}(\mathbb{C})$ the situation is similar ([5, p. 926]).

Theorem 2 (Laurent Schwartz). *Any closed translation invariant subspace V of $\mathcal{O}(\mathbb{C})$ different from $\mathcal{O}(\mathbb{C})$ is the closure of the linear subspace of V generated by all exponential monomials contained in V.*

Especially, even if $V = \mathcal{O}(\mathbb{C})$, any closed translation invariant subspace $V \neq \{0\}$ of $\mathcal{O}(\mathbb{C})$ contains an exponential function E_λ, $E_\lambda(z) = \exp(\lambda z)$ for some $\lambda \in \mathbb{C}$ and all $z \in \mathbb{C}$.

Using the ideas of Laczkovich we now formulate and prove the following lemma.

Lemma 1. *Let $f \in C(\mathbb{R}, \mathbb{R})$ be a non zero solution of* (1), *let $b = b(a, A)$, $B = B(a, A)$ be suitable functions from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$ such that*

$$(4) \qquad f(x) + Af(x+a) = B(a, A) f\left(x + b(a, A)\right), \quad a, A, x \in \mathbb{R}.$$

Then there is some $\lambda \in \mathbb{C}$ such that $E_\lambda \in V$, where

$$V := \{g \in C(\mathbb{R}, \mathbb{C}) \,|\, g \text{ satisfies } (4)\}.$$

Proof. Obviously V as defined in the lemma is a closed translation invariant subspace of $C(\mathbb{R}, \mathbb{C})$. Since V contains $f \neq 0$, this subspace is different from $\{0\}$. Thus by Th. 1 there is some $\lambda \in \mathbb{C}$ such that $E_\lambda \in V$. $\diamondsuit$

The next lemma makes the ideas from [4] work.

Lemma 2. *Let $f \in C(\mathbb{R}, \mathbb{R})$ be a solution of* (1). *Then there are $A_0, a_0 \in \mathbb{R} \setminus \{0\}$, $A_0 < 0$ such that*

$$(5) \qquad f(x) + A_0 f(x + a_0) = 0, \quad x \in \mathbb{R}.$$

Proof. We obviously may assume that $f \neq 0$. Let $b = b(a, A)$ and $B = B(a, A)$ as in Lemma 1. Then, by this lemma there is some λ such that E_λ satisfies (4). Thus, with $\lambda = \mu + i\nu$, $\mu, \nu \in \mathbb{R}$ and after some calculations, we get

$$(6) \qquad 1 + A\exp(\lambda a) = B\exp(\lambda b), \quad a, A \in \mathbb{R},$$

and, on considering the real and imaginary part separately,

$$1 + A\exp(\mu a)\cos(\nu a) = B\exp(\mu b)\cos(\nu b), \quad a, A \in \mathbb{R} \tag{7}$$

$$A\exp(\mu a)\sin(\nu a) = B\exp(\mu b)\sin(\nu b), \quad a, A \in \mathbb{R}. \tag{8}$$

Obviously, even if $\nu = 0$, we may choose some $a_0 \in \mathbb{R} \setminus \{0\}$ such that $\cos(\nu a_0) = 1$. Then, putting $A_0 := -\exp(-\mu a_0) < 0$ equation (7) implies $B\cos(\nu b) = 0$. But $\sin(\nu a_0) = 0$ since $\cos(\nu a_0) = 1$. Accordingly by (8) we also have $B\sin(\nu b) = 0$. So

$$B^2 = B^2\cos^2(\nu b) + B^2\sin^2(\nu b) = 0 + 0 = 0.$$

Thus $B = B(a_0, A_0) = 0$, from which we get (5). $\Diamond$

So for any continuous solution $f : \mathbb{R} \to \mathbb{R}$ equation (5) holds for some $a_0 \neq 0, A_0 < 0$. Then the results from [4] may be applied to get the following theorem.

Theorem 3. *A function $f \in C(\mathbb{R}, \mathbb{R})$ satisfies* (1) *if and only if there are $a, b, c, d \in \mathbb{R}$ such that*

$$f(x) = \exp(cx + d)\cos(ax + b), \quad x \in \mathbb{R}. \tag{9}$$

Note that any function $f \neq 0$ of the form (9) may be written as the real part of some function $g : \mathbb{R} \to \mathbb{C}$ of the form $g(x) = \exp(px + q)$ ($p = c + ia, q = d + ib \in \mathbb{C}$). Obviously this g and $g = 0$ solve (2). Now we determine all continuous solutions of (2).

Theorem 4. *A function $g \in C(\mathbb{R}, \mathbb{C})$ satisfies* (2) *if and only if either $g = 0$ or $g(x) = \exp(px + q)$ for all $x \in \mathbb{R}$ with some $p, q \in \mathbb{C}$.*

Proof. As mentioned above these function indeed satisfy (2). So let $0 \neq g \in C(\mathbb{R}, \mathbb{C})$ be a solution of (2). Proceeding as in Lemma 1 we may find functions $b : \mathbb{R} \times \mathbb{C} \to \mathbb{R}$ and $B : \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ such that

$$g(x) + Ag(x + a) = B(a, A)g\left(x + b(a, A)\right), \quad x, a \in \mathbb{R}, A \in \mathbb{C}. \tag{10}$$

By the same ideas as in the proof of Lemma 1 we then may find an exponential function E_λ which satisfies (10). This implies

$$1 + A\exp(\lambda a) = B(a, A)\exp\left(\lambda b(a, A)\right), \quad a \in \mathbb{R}, A \in \mathbb{C}.$$

Putting $a_0 := 2\pi$ and $A_0 := -\exp(-\lambda a_0) \neq 0$ we find that $B = B(a_0, A_0) = 0$. Thus g satisfies

$$g(x) + A_0 g(x + a_0) = 0, \quad x \in \mathbb{R}.$$

Accordingly, following the procedures in [4], we may write g in the form

$$g(x) = \exp(\lambda x)h(x), \quad x \in \mathbb{R},$$

where $h : \mathbb{R} \to \mathbb{C}$ is a continuous function with period $a_0 = 2\pi$:

$$h(x + a_0) = h(x), \quad x \in \mathbb{R}.$$

By [4, Lemma 2] h is also a solution of (2). Thus the Fourier coefficients $c_k = c_k(h)$ of h,

$$c_k := c_k(h) := \frac{1}{2\pi} \int_0^{2\pi} h(t) \exp(-ikt)dt, \quad k \in \mathbb{Z},$$

satisfy

$$c_k \left(1 + A \exp(ika)\right) = c_k B \exp(ikb), \quad k \in \mathbb{Z}. \tag{11}$$

This immediately follows from multiplying (2) (with $t{=}x$) by $\exp(-ikt)$ and from integrating the resulting equation between 0 and 2π.

We want to show that at most one of the c_k's is different from zero. So, let $c_n \neq 0 \neq c_m$. Choosing $a \in \mathbb{R} \setminus 2\pi\mathbb{Q}$ and $A := -\exp(-ina)$ it follows from (11) that $B \exp(inb) = 0$. Thus $B \exp(imb) = 0$, too. Using $c_m \neq 0$ then implies $A = -\exp(-ima)$. Hence $\exp(ina) = \exp(ima)$ and $(n-m)a \in 2\pi\mathbb{Z}$, which for $a \notin 2\pi\mathbb{Q}$ is only possible when $n = m$.

Therefore for some $n \in \mathbb{Z}$ all Fourier coefficients of the continuous function $h - c_n(h)E_{in}$ vanish. Hence by [2, Th. 2.2, p. 261] $h - c_n(h)E_{in} = 0$ or

$$h(x) = c_n \exp(inx), \quad x \in \mathbb{R}$$

with some complex constant c_n.

Since $g \neq 0$ the function h also must be different from zero. Thus $c_n \neq 0$ and we get the desired form for g with $p = \lambda + in$ and $q = \log(c_n)$. ◊

Finally we consider (3) for $h \in \mathcal{O}(\mathbb{C})$.

Theorem 5. *A function $h \in \mathcal{O}(\mathbb{C})$ satisfies* (3) *if and only if either $h = 0$ or*

$$h(z) = \exp(pz + q), \quad z \in \mathbb{C} \tag{12}$$

with some $p, q \in \mathbb{C}$.

Proof. Obviously $h = 0$ and h with (12) satisfy (3). So, let $0 \neq h \in \mathcal{O}(\mathbb{C})$ satisfy (3). Then, after choosing functions $b, B : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ such that

(13) $$h(z) + Ah(z+a) = B(a, A)h\left(z + b(a, A)\right), \quad a, A, z \in \mathbb{C},$$

let

$$W := \{\ell \in \mathcal{O}(\mathbb{C}) \mid \ell \text{ satisfies } (13)\}.$$

Then $W \neq \{0\}$ since $0 \neq h \in W$. Moreover W is a translation invariant subspace of $\mathcal{O}(\mathbb{C})$ which is also closed with respect to the topology of uniform convergence on compact subsets of $\mathbb{C}$. Accordingly, by Th. 2, there is some complex number λ such that $E_\lambda \in W$. (13) for E_λ then implies

(14) $$1 + A\exp(\lambda a) = B\exp(\lambda b), \quad a, A \in \mathbb{C},$$

where $B = B(a, A)$ and $b = b(a, A)$.

Fixing $a_0 \neq 0$ and putting $A_0 := -\exp(-\lambda a_0)$ implies $B = 0$. Consequently h satisfies

(15) $$h(z) + A_0 h(z + a_0) = 0, \quad z \in \mathbb{C}.$$

Thus $h = \ell r$ with $\ell(z) := \exp(\lambda z)$ (and $\ell \in \mathcal{O}(\mathbb{C})$), where $r = h/\ell \in$ $\in \mathcal{O}(\mathbb{C})$ has period a_0 and satisfies (3).

By [3, Chap. 12.3, Th.] we have

(16) $$r(z) = \sum_{-\infty}^{\infty} c_k \exp\left(\frac{2k\pi i}{a_0} z\right), \quad z \in \mathbb{C},$$

where this Laurent series converges uniformly on compact subsets of $\mathbb{C}$ and where for arbitrary complex d the Fourier coefficients c_k are given by

(17) $$c_k = c_k(r) = \frac{1}{a_0} \int_{[d, d+a_0]} r(\zeta) \exp\left(-\frac{2\pi i}{a_0} k\zeta\right) d\zeta, \quad k \in \mathbb{Z}$$

($[d, d + a_0]$ denotes the path from d to $d + a_0$ along the straight line between d an $d + a_0$). It is easy to see that the Fourier coefficients of the translated function $r_{[a]}$, $r_{[a]} := r(z + a)$, are given by $c_k(r_{[a]}) =$ $= \exp\left(\frac{2k\pi i}{a_0} a\right) c_k(r)$.

Thus (3) for r implies

(18) $$c_k \left(1 + A\exp\left(\frac{2k\pi i}{a_0} a\right)\right) = c_k B \exp\left(\frac{2k\pi i}{a_0} a\right), \quad k \in \mathbb{Z}.$$

Since $\ell \neq 0$ we only have to show that $c_n \neq 0$ for exactly one $n \in \mathbb{Z}$.

Because of $r = h/\ell \neq 0$ there is some k, say $k = n$, such that $c_k \neq$ $\neq 0$. So, let $c_n \neq 0 \neq c_m$ and let $a \in \mathbb{C}$ be such that $a/a_0 \notin \mathbb{Q}$. (18) for $k = n$ and $A := -\exp\left(-\frac{2n\pi i}{a_0}a\right)$ implies $B = 0$. The same equation now for $k = m$ implies $1 + A\exp\left(\frac{2m\pi i}{a_0}a\right) = 0$. Accordingly $\exp\left(2(n-m)\pi i\frac{a}{a_0}\right) = 1$. Hence $2(n-m)\pi i\frac{a}{a_0} \in 2\pi i\mathbb{Z}$ which because of $\frac{a}{a_0} \notin \mathbb{Q}$ implies $n - m = 0$ or $n = m$. With $p := \lambda + i\frac{2\pi n}{a_0}$ and $q := \log(c_n)$ we finally get the desired result. $\Diamond$

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