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NUMERICAL SIMULATIONS FOR STOCHASTIC LATTICE EQUATIONS

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Abstract: The purpose of this paper is to study numerical approximations for the solution of stochastic lattice differential equations perturbed by additive fractional noise. An implicit Euler scheme is used and it is proved that the approximations converge almost surely to the solution of the considered equation. Computer simulations for these approximations are given.

1. Introduction

Let $\ell^2(\mathbb{Z})$ be the linear space of all families $(u_i)_{i\in\mathbb{Z}}$ of real numbers such that the family $(|u_i|^2)_{i\in\mathbb{Z}}$ is summable. It is well-known that the function $(\cdot,\cdot) : \ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z}) \to \mathbb{R}$, defined by

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$$(u,v) := \sum_{i\in\mathbb{Z}} u_i v_i$$

for all $u = (u_i)_{i\in\mathbb{Z}}, v = (v_i)_{i\in\mathbb{Z}}$ is a scalar product and that $\ell^2(\mathbb{Z})$ endowed with this scalar product is a Hilbert space. The corresponding norm we denote by $\|\cdot\|$.

In this paper, we study the numerical approximation of the solution for the following stochastic lattice differential equation

$$(1)\quad \frac{dU_i(t)}{dt} = (U_{i-1}(t) - 2U_i(t) + U_{i+1}(t)) - f(U_i(t)) + a_i \frac{d\beta_i^H(t)}{dt},\ i \in \mathbb{Z},$$

where $t \in [0,T]$, $U(t) = (U_i(t))_{i\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, f is a smooth function satisfying a dissipative condition, $(a_i)_{i\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, and $(\beta_i^H)_{i\in\mathbb{Z}}$ are independent fractional Brownian motions with Hurst index $H \in (0,1)$.

Stochastic lattice differential equations have many applications in models where the spatial structure has a discrete character and random influences are taken into account. In equation (1) we have an one-dimensional lattice with diffusive nearest neighbour interaction, a dissipative nonlinear reaction term and additive independent fractional noise at each node. The long time behaviour of the solution of equations of this type has been studied in [1].

In the present paper we give an implicit Euler scheme to approximate the solution of the considered equation and prove that the approximations converge in probability to the solution. We point out that our nonlinear reaction term f is assumed to be locally Lipschitz and the perturbation is a fractional noise (not only standard Brownian motion). Many results about numerical methods for stochastic differential equations with globally Lipschitz nonlinearities driven by standard Brownian motion can be found in the books [11] and [15]. The method in this article is closely related to implicit Euler schemes used for deterministic equations. We also use the properties of a fractional Brownian motion with values in a Hilbert space (as in [16]). For real-valued fractional Brownian motions we consider trigonometric series approximation as investigated in [9] and [14]. At the end of our paper we give computer simulations using Matlab programs.

Deterministic lattice differential equations are used to model systems as cellular neural networks ([6], [7]), propagation of pulses in myelinated axons ([2], [3]), in image processing and pattern recognition ([4], [5]), in chemical reaction theory ([13], [10]).

2. Stochastic lattice differential equation

We consider the stochastic lattice differential equation (1) over a probability space $(\Omega, \mathcal{F}, P)$.

For convenience, we will formulate system (1) as a stochastic differential equation in $\ell^2(\mathbb{Z})$. We denote by $e^i \in \ell^2(\mathbb{Z})$ $(i \in \mathbb{Z})$ the element having 1 at position i and all the other components 0.

Denote by A, A^* and $\mathcal{A}$ the operators from $\ell^2(\mathbb{Z})$ to $\ell^2(\mathbb{Z})$ defined for each $u = (u_i)_{i\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ as follows:

$$(Au)_i = u_{i+1} - u_i, \qquad (A^*u)_i = u_{i-1} - u_i,$$

and

$$(\mathcal{A}u)_i = -u_{i-1} + 2u_i - u_{i+1}$$

for each $i \in \mathbb{Z}$. Then it can be seen that $\mathcal{A} = AA^* = A^*A$ and that

$$(A^*u, v) = (u, Av) \quad \text{for all } u, v \in \ell^2(\mathbb{Z}).$$

Hence

$$(\mathcal{A}u, u) \geq 0 \quad \text{for all } u \in \ell^2(\mathbb{Z}). \tag{2}$$

$\mathcal{A}$ is a continuous linear operator. Its operator norm is denoted by $\|\mathcal{A}\|$.

The assumptions on the nonlinearity f occurring in the equation (1) are: Let $f \in C^1(\mathbb{R})$ be such that it satisfies

$$(f(x) - f(y))\,(x - y) \geq 0 \quad \text{for all } x, y \in \mathbb{R} \tag{3}$$

and the polynomial growth condition

$$|f(x)| \leq c_f |x|(1 + x^{2p}) \quad \text{for all } x \in \mathbb{R} \tag{4}$$

where p is a positive integer.

If $f(x) = \sum_{j=0}^{p} a_j x^{2j+1}$ with $a_j \geq 0$ for each $j = 0, \ldots, p$, then it is easy to verify that the conditions (3) and (4) are satisfied. This kind of nonlinearity was considered for deterministic equations in [4] and [8].

For each $u = (u_i)_{i\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, let $\tilde{f}(u) := (f(u_i))_{i\in\mathbb{Z}}$. For an arbitrary finite subset of indexes $Y \subset \mathbb{Z}$ we write

$$\sum_{i\in Y} |f(u_i)|^2 = \sum_{i\in Y} |f(u_i) - f(0)|^2 = \sum_{i\in Y} |f'(\xi_i)|^2 |u_i|^2, \tag{5}$$

with $\xi_i = \tau_i u_i$ for some $\tau_i \in (0, 1)$. Since $|\xi_i| \leq |u_i| \leq \|u\|$ and f is smooth, it follows that there exists a constant μ (depending on u) such that

$$\sum_{i \in Y} |f(u_i)|^2 \leq \mu \sum_{i \in \mathbb{Z}} |u_i|^2 = \mu \|u\|^2.$$

It follows that $\tilde{f}(u) \in \ell^2(\mathbb{Z})$ and $\tilde{f} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is the Nemytski operator associated with f. One can see that $\tilde{f}$ is locally Lipschitz from $\ell^2(\mathbb{Z})$ to $\ell^2(\mathbb{Z})$, more precisely, for each $R > 0$ there exists a constant l_R such that

(6)
$$\|\tilde{f}(x) - \tilde{f}(y)\| \leq l_R \|x - y\| \text{ for all } x, y \in \ell^2(\mathbb{Z}) \text{ with } \|x\| \leq R,\ \|y\| \leq R.$$

In the sequel, we identify $\tilde{f}$ with f.

A Gaussian random process $\left(\beta^H(t)\right)_{t \geq 0}$ is called one-dimensional fractional Brownian motion with Hurst index $H \in (0,1)$, if it has zero mean, continuous sample paths and the covariance function

$$E\left(\beta^H(s)\beta^H(t)\right) = \frac{1}{2}\left(t^{2H} + s^{2H} - |s-t|^{2H}\right), \quad s, t \in \mathbb{R}.$$

Note that if $H = \frac{1}{2}$, then the fractional Brownian motion is the ordinary standard Brownian motion.

Let $(\beta_i^H)_{i \in \mathbb{Z}}$ be independent one-dimensional fractional Brownian motions with Hurst index $H \in (0,1)$ and let $T > 0$. Then

$$B^H(t) := \sum_{i \in \mathbb{Z}} a_i \beta_i^H(t) e^i \quad \text{with} \quad (a_i)_{i \in \mathbb{Z}} \in \ell^2(\mathbb{Z}), \quad t \in [0,T], \tag{7}$$

is a fractional Brownian motion with values in $\ell^2(\mathbb{Z})$. Note that B^H is a.s. the sum of the family $(a_i \beta_i^H e^i)_{i \in \mathbb{Z}}$. B^H has a Hölder continuous version (because we can apply the Kolmogorov continuity criterion Th. 1.4.1 from [12] and use the properties of the moments of the increments of the one-dimensional fractional Brownian motion; see also [18], [16]): for all $\varepsilon > 0$ and $T > 0$, there exists a nonnegative random variable δ such that $E\delta^p < \infty$ for all $p \geq 1$ and

$$\|B^H(t) - B^H(s)\| \leq \delta |t-s|^{H-\varepsilon} \text{ for all } s, t \in [0,T].$$

In other words, the parameter H controls the regularity of the trajectories, which are almost surely Hölder continuous of order $\gamma = H - \varepsilon$. We will work with such a Hölder continuous version and denote it also by B^H.

The equation (1) with initial value $u_0 = (u_{0,i})_{i \in \mathbb{Z}} \in \ell^2(\mathbb{Z})$ can be rewritten as an integral equation in $\ell^2(\mathbb{Z})$:

$$(8)\quad U(t) = u_0 + \int_0^t \Big(-\mathcal{A}U(s) - f(U(s))\Big)ds + B^H(t), \quad t \in [0,T],\ \omega \in \Omega.$$

Theorem 2.1. *The equation (8) admits a unique solution $U \in \mathcal{L}^2(\Omega, \mathcal{C}([0,T],\ell^2))$. Moreover, there exists a constant $c > 0$ such that for all $\omega \in \Omega$ we have*

$$(9)\quad \sup_{t\in[0,T]} \|U(t)\|^2 \le c\Big(\|u_0\|^2 + \sup_{t\in[0,T]} \|B^H(t)\|^2 + \int_0^T \Big(\|B^H(s)\|^2 + \|B^H(s)\|^{4p+2}\Big)ds\Big).$$

Proof. Denote $z(t) = U(t) - B^H(t)$ for all $t \in [0,T]$. The equation (8) has a unique solution $U \in \mathcal{L}^2(\Omega, \mathcal{C}([0,T],\ell^2(\mathbb{Z})))$ for all $\omega \in \Omega$ if and only if for each $\omega \in \Omega$ the following equation

$$(10)\quad z(t) = u_0 + \int_0^t \Big(- \mathcal{A}z(s) - f(z(s) + B^H(s)) - \mathcal{A}B^H(s)\Big)ds,\ t \in [0,T],$$

has a unique solution $z \in \mathcal{L}^2(\Omega, \mathcal{C}([0,T],\ell^2(\mathbb{Z})))$. For each fixed $\omega \in \Omega$, the equation (10) is a deterministic equation. By standard argument (see [17]), the equation (10) has a unique local solution $z \in \mathcal{C}([0,T_{\max}),\ell^2(\mathbb{Z}))$, where $[0,T_{\max})$ is the maximal interval of existence of the solution of (10). We claim that this local solution is a global solution. Indeed, let $\omega \in \Omega$. From (10) it follows that for all $t \in [0,T_{\max})$ it holds

$$\|z(t)\|^2 = \|u_0\|^2 + 2\int_0^t \Big[- (\mathcal{A}z(s), z(s)) - (f(z(s) + B^H(s)) - f(B^H(s)), z(s)) + (-\mathcal{A}B^H(s) - f(B^H(s)), z(s))\Big] ds.$$

Hence

$$(11)\quad \|z(t)\|^2 \le \|u_0\|^2 + c_0 \int_0^t \Big(\|B^H(s)\|^2 + \|B^H(s)\|^{4p+2}\Big)ds,$$

for all $t \in [0,T_{\max})$, where c_0 is a positive constant depending on f

and $\mathcal{A}$. Then we obtain that $\|z\|$ is bounded by a continuous function, so there exists a global solution on the interval $[0,T]$. Using (11) it follows that for all $\omega \in \Omega$ we have

$$\sup_{t\in[0,T]} \|z(t)\|^2 \leq \|u_0\|^2 + c_0 \int_0^T \left(\|B^H(s)\|^2 + \|B^H(s)\|^{4p+2}\right) ds. \tag{12}$$

By taking expectation on both sides of this inequality it follows according to the properties of white noise that $z \in \mathcal{L}^2(\Omega, \mathcal{C}([0,T], \ell^2(\mathbb{Z})))$. Therefore equation (8) has a global solution $U \in \mathcal{L}^2(\Omega, \mathcal{C}([0,T], \ell^2(\mathbb{Z})))$. From (12) it follows that the inequality (9) holds, where $c = 2 + 2c_0$. ◊

3. Implicit Euler scheme

In this section we approximate (8) by using an implicit Euler scheme:

Let $N \in \mathbb{N}$ and let $\Delta = (t_0 = 0 < t_1 < \cdots < t_N = T)$ be a division of $[0,T]$. We denote $|\Delta| := \max_{k=0,1,\ldots,N-1} \tau_k$, where $\tau_k := t_{k+1} - t_k$ for each $k \in \{0,1,\ldots,N-1\}$. For each $\omega \in \Omega$ we put $U^0 := u_0$ and

$$U^{k+1} := U^k + \tau_k F(U^{k+1}) + B^H(t_{k+1}) - B^H(t_k), \tag{13}$$

for $k \in \{0,1,\ldots,N-1\}$, where $F : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is defined by $F = -\mathcal{A} - f$. The existence of $U^{k+1} \in \ell^2(\mathbb{Z})$ for each $k \in \{0,1,\ldots,N-1\}$ is assured by the well-known result of Browder for maximal monotone operators (in our case $\mathcal{A}$) perturbed by pseudo-monotone, demicontinuous, coercive operators (see [19], Th. 32.A, p. 867, and the remark on page 844).

Let $t \in [0,T)$. Then there exists a unique $k \in \{0,1,\ldots,N-1\}$ such that $t \in [t_k, t_{k+1})$. We define

$$U_\Delta(t) := U^k + (t - t_k)F(U^{k+1}) + B^H(t) - B^H(t_k) \tag{14}$$

and consider $U_\Delta(T) := U^N$. We observe that U_Δ is a continuous process on $[0,T]$. We set

$$\hat{U}_\Delta(t) = \sum_{k=0}^{N-1} U^{k+1} I_{[t_k,t_{k+1})}(t) \text{ for } t \in [0,T) \text{ and } \hat{U}_\Delta(T) = U^N, \tag{15}$$

where $I_{[t_k,t_{k+1})}$ is the indicator function of the interval $[t_k, t_{k+1})$. Then we have

$$U_\Delta(t) = u_0 + \int_0^t F(\hat{U}(s))ds + B^H(t) \text{ for each } t \in [0,T].$$

Lemma 3.1. *Let* $|\Delta| < \frac{1}{2}$. *There exists a random variable* $R > 0$ *independent of* Δ *such that for all* $\omega \in \Omega$ *the following inequalities hold:*

$$\sup_{t\in[0,T]} |\hat{U}_\Delta(t)| \leq R, \quad \sup_{t\in[0,T]} |U_\Delta(t)| \leq R \textit{ and } \sup_{t\in[0,T]} |U(t)| \leq R.$$

Proof. From (13) we get

$$U^{k+1} - B^H(t_{k+1}) = U^k - B^H(t_k) + \tau_k(-\mathcal{A}U^{k+1} - f(U^{k+1})) \tag{16}$$

for each $k \in \{0, 1, \ldots, N-1\}$. Put $Z_k := U^k - B^H(t_k)$ for $k \in$ $\in \{0, 1, \ldots, N\}$ and consider

$$G(x,t) := -\mathcal{A}x - f(x + B^H(t)) - \mathcal{A}B^H(t) = F(x + B^H(t)).$$

Then (16) can be written as follows:

$$Z_{k+1} = Z_k + \tau_k G(Z_{k+1}, t_{k+1}) \text{ for each } k \in \{0, 1, \ldots, N-1\}.$$

Now, fix any $k \in \{0, 1, \ldots, N-1\}$. By using (2) and (3) we obtain

$$\begin{aligned} \|Z_{k+1}\|^2 &= (Z_k, Z_{k+1}) + \tau_k(G(Z_{k+1}, t_{k+1}), Z_{k+1}) \leq \\ &\leq \frac{1}{2}\|Z_k\|^2 + \frac{1}{2}\|Z_{k+1}\|^2 + \frac{1}{2}\tau_k\|f(B^H(t_{k+1})) - \\ &\quad - \mathcal{A}B^H(t_{k+1})\|^2 + \frac{1}{2}\tau_k\|Z_{k+1}\|^2. \end{aligned}$$

Consequently, we have

$$\|Z_{k+1}\|^2 \leq \|Z_k\|^2 + \tau_k\|f(B^H(t_{k+1})) - \mathcal{A}B^H(t_{k+1})\|^2 + \tau_k\|Z_{k+1}\|^2,$$

hence

$$(1-\tau_k)\|Z_{k+1}\|^2 \leq \|Z_k\|^2 + \tau_k\|f(B^H(t_{k+1})) - \mathcal{A}B^H(t_{k+1})\|^2.$$

This inequality yields,

$$\|Z_{k+1}\|^2 \leq \|Z_k\|^2 + \frac{\tau_k}{1-\tau_k}\|Z_k\|^2 + \frac{\tau_k}{1-\tau_k}\|f(B^H(t_{k+1})) - \mathcal{A}B^H(t_{k+1})\|^2.$$

But $1 - |\Delta| \leq 1 - \tau_k$, so

$$\frac{1}{1-\tau_k} \leq \frac{1}{1-|\Delta|},$$

thus

(17)
$$\|Z_{k+1}\|^2 \le \|Z_k\|^2 + \frac{\tau_k}{1-|\Delta|}\Big(\|Z_k\|^2 + \|f(B^H(t_{k+1})) - \mathcal{A}B^H(t_{k+1})\|^2\Big).$$

Let $n \in \{1,\dots,N\}$. Summing up the inequalities obtained by putting $k := 0,\dots,n-1$ in (17), we get

(18)
$$\|Z_n\|^2 \le \|Z_0\|^2 + \frac{1}{1-|\Delta|}\sum_{k=0}^{n-1}\tau_k\Big(\|Z_k\|^2 + \|f(B^H(t_{k+1})) - \mathcal{A}B^H(t_{k+1})\|^2\Big).$$

We consider

$$\hat{Z}(t) := \sum_{k=0}^{N-1} I_{[t_k,t_{k+1})}(t) Z_k \text{ for } t \in [0,T) \text{ and } \hat{Z}(T) := Z_N,$$

$$\hat{B}(t) := \sum_{k=0}^{N-1} I_{[t_k,t_{k+1})}(t) B^H(t_{k+1}) \text{ for } t \in [0,T) \text{ and } \hat{B}(T) := B^H(T).$$

Then by (18) we obtain

$$\sup_{k\in\{0,1,\dots,n\}} \|Z_k\|^2 \le \|Z_0\|^2 + \frac{1}{1-|\Delta|}\int_0^{t_n}\Big(\|\hat{Z}(s)\|^2 + \|f(\hat{B}(s)) - \mathcal{A}\hat{B}(s)\|^2\Big)ds.$$

Let $t \in [0,T]$ be arbitrary. Then there exists a unique $n \in \{1,\dots,N-1\}$ such that $t_n \le t < t_{n+1}$ (we take $n = N$ if $t = T$). Then

$$\sup_{k\in\{0,1,\dots,n\}} \|Z_k\|^2 = \sup_{s\in[0,t]} \|\hat{Z}(s)\|^2 \le \|Z_0\|^2 + \frac{1}{1-|\Delta|}\int_0^{t}\Big(\|\hat{Z}(s)\|^2 + \|f(\hat{B}(s)) - \mathcal{A}\hat{B}(s)\|^2\Big)ds.$$

Using Gronwall's lemma we get

$$\sup_{s\in[0,t]} \|\hat{Z}(s)\|^2 \le \Big(\|Z_0\|^2 + \frac{1}{1-|\Delta|}\int_0^t \|f(\hat{B}(s)) - \mathcal{A}\hat{B}(s)\|^2 ds\Big)\, e^{\frac{1}{1-|\Delta|}t}.$$

Hence

$$\sup_{k\in\{0,1,\ldots,N\}} \|Z_k\|^2 = \sup_{s\in[0,T]} \|\hat{Z}(s)\|^2 \leq \tag{19}$$
$$\leq \left(\|u_0\|^2 + \frac{1}{1-|\Delta|}\int_0^T \|f(\hat{B}(s)) - \mathcal{A}\hat{B}(s)\|^2 ds\right) \mathrm{e}^{\frac{1}{1-|\Delta|}T}.$$

Next we prove that $\int_0^T \|f(\hat{B}(s)) - \mathcal{A}\hat{B}(s)\|^2 ds$ is bounded. We denote

$$M := \sup_{s\in[0,T]} \|B^H(s)\| < \infty.$$

Then we have

$$\int_0^T \|f(\hat{B}(s)) - \mathcal{A}\hat{B}(s)\|^2 ds \leq TM^2(c_f + ||\mathcal{A}|| + c_f M^{2p})^2,$$

because

$$\sup_{s\in[0,T]} \|\hat{B}(s)\| \leq M.$$

We assumed $|\Delta| < \frac{1}{2}$, then $\dfrac{1}{1-|\Delta|} < 2$. We consider the random variable

$$\rho^2 = \|u_0\|^2 + 2TM^2(c_f + ||\mathcal{A}|| + c_f M^{2p})^2 \mathrm{e}^{2T}.$$

From (19) we get

$$\sup_{k\in\{0,1,\ldots,N\}} \|Z_k\|^2 \leq \rho^2.$$

Since $U^k = Z_k + B^H(t_k)$, then

$$\sup_{k\in\{0,1,\ldots,N\}} \|U^k\|^2 \leq 2(\rho^2 + M^2).$$

With the notation $r^2 := 2(\rho^2 + M^2)$ we obtain

$$\sup_{k\in\{0,1,\ldots,N\}} \|U^k\|^2 = \sup_{t\in[0,T]} \|\hat{U}_\Delta(t)\|^2 \leq r^2.$$

Then by (14) we have

$$\|U_\Delta(t)\| \leq r + |\Delta|(c_f r + c_f r^{2p+1} + ||\mathcal{A}||r) + 2M$$

for all $t \in [0,T]$. Denote $\hat{r} := r + \frac{1}{2}(c_f r + c_f r^{2p+1} + ||\mathcal{A}||r) + 2M < \infty$. Then

$$\sup_{t\in[0,T]} \|\hat{U}_\Delta(t)\| \leq \hat{r}^2$$

and

$$\sup_{t\in[0,T]} \|U_\Delta(t)\| \le \hat{r}^2.$$

Finally we use the results from Th. 2.1 which estimate U and obtain the statement of this lemma. $\Diamond$

Lemma 3.2. *Using the notations* (14) *and* (15) *we have*

$$P\left(\lim_{|\Delta|\to 0}\int_0^T \|U_\Delta(t)-\hat{U}_\Delta(t)\|^2 dt\right)=1.$$

Proof. Let $\Delta = (t_0 = 0 < t_1 < \cdots < t_N = T)$ be a division of $[0,T]$ and let $t\in[0,T]$ be arbitrary. Then there exists a unique $k\in$ $\in \{0,1,\dots,N-1\}$ such that $t_k \le t < t_{k+1}$. Then $\hat{U}_\Delta(t) = U^{k+1}$ (we take $k = N-1$ if $t = T$) and by (14) we have

$$U_\Delta(t) = U^k + (t-t_k)F(U^{k+1}) + B^H(t) - B^H(t_k).$$

Then using (13) we have

$$\begin{aligned} U_\Delta(t)-\hat{U}_\Delta(t) &= U^k - U^{k+1} + (t-t_k)F(U^{k+1}) + B^H(t) - B^H(t_k) = \\ &= U^k - U^k - (t_{k+1}-t_k)F(U^{k+1}) - B^H(t_{k+1}) + \\ &\quad + B^H(t_k) + (t-t_k)F(U^{k+1}) + B^H(t) - B^H(t_k) = \\ &= (t-t_{k+1})F(U^{k+1}) + B^H(t) - \hat{B}(t), \end{aligned}$$

so

$$\|U_\Delta(t)-\hat{U}_\Delta(t)\|^2 \le 2|\Delta|R\Big(c_f + c_f R^{2p} + \|\mathcal{A}\|\Big) + 2\|B^H(t)-\hat{B}(t)\|^2.$$

This inequality yields

$$\begin{aligned} \int_0^T \|U_\Delta(t)-\hat{U}_\Delta(t)\|^2 dt \le 2|\Delta|RT\Big(c_f + c_f R^{2p} + \|\mathcal{A}\|\Big) + \\ + 2\int_0^T \|B^H(t)-\hat{B}(t)\|^2 dt. \end{aligned} \tag{20}$$

We prove that

$$\int_0^T \|B^H(t)-\hat{B}(t)\|^2 dt \to 0 \text{ as } |\Delta|\to 0.$$

B^H is Hölder continuous with exponent $\gamma < H$, then we get

$$\|B^H(t)-\hat{B}(t)\|^2 \le |t-t_{k+1}|^{2\gamma}\delta^2 \text{ for } t\in[t_k,t_{k+1}].$$

Then

$$\|B^H(t)-\hat{B}(t)\|^2 \le |\Delta|^{2\gamma}\delta^2,$$

so

$$\int_0^T \|B^H(t) - \hat{B}(t)\|^2 dt \leq |\Delta|^{2\gamma}\delta^2 T \to 0, \text{ when } |\Delta| \to 0.$$

Using this result in (20), follows the statement of this lemma. $\diamondsuit$

The main result of our paper is the following theorem.

Theorem 3.3. *The process U_Δ approximates the solution U of equation* (8) *in the following sense:*

$$P\left(\lim_{|\Delta|\to 0} \sup_{t\in[0,T]} \|U(t) - U_\Delta(t)\| = 0\right) = 1.$$

Proof. We have

$$U(t) - U_\Delta(t) = \int_0^t (F(U(s)) - F(\hat{U}_\Delta(s)))ds \quad \text{for each } t \in [0,T].$$

By using Lemma 3.1 and (6), it follows that

$$\|U(t) - U_\Delta(t)\|^2 = 2\int_0^t (F(U(s)) - F(\hat{U}_\Delta(s)), U(s) - U_\Delta(s))ds \leq$$

$$\leq 2\int_0^t (\|\mathcal{A}\| + l_R)\|U(s) - \hat{U}_\Delta(s)\| \cdot \|U(s) - U_\Delta(s)\|ds \leq$$

$$\leq \int_0^t 2(\|\mathcal{A}\| + l_R)\Big(\|U(s) - U_\Delta(s)\|^2 + \|U_\Delta(s) - \hat{U}_\Delta(s)\|^2\Big)ds.$$

According to Gronwall's lemma there exists a constant $C > 0$ such that

$$\sup_{t\in[0,T]} \|U(t) - U_\Delta(t)\|^2 \leq C\int_0^T \|U_\Delta(s) - \hat{U}_\Delta(s)\|^2 ds \cdot e^{CT},$$

To complete the proof we use Lemma 3.2. $\diamondsuit$

4. Simulations

We use Matlab programs to give simulations for the approximations U_Δ of the solution U of the infinite system (1) by considering a system of d equations. We take $d = 200$ components (using the notations from Sec. 3, the index i takes values in the set $\{-99, \ldots, 0, 1, \ldots, 100\}$), consider $f(x) = x^3$ for the reaction term, $T = 1$ and take an equidistant division of $[0, 1]$. For the one-dimensional fractional Brownian motion we use trigonometric series approximation, as in [9] and [14]. A simulated trajectory of an one-dimensional Brownian motion is given in Fig. 1.

We give a simulation of U_Δ at the endpoint $T = 1$ (in fact $U_\Delta = U^N$) by plotting the points (i, U_i^N) for $i = -99, \ldots, 0, 1, \ldots, 100$, in Fig. 2. The initial value is $u_0 = 0$.

Now we consider another initial value u_0, whose components u_{0i}, $i \in \{-99, \ldots, 0, 1, \ldots, 100\}$, are plotted in Fig. 3. The simulation of U_Δ at the endpoint $T = 1$ is shown in Fig. 4.

The evolution in time of the components with $i = 0$ and $i = 1$ of U_Δ is represented in Fig. 5. From this simulation we see that even two neighbour components of U_Δ have very different trajectories.

5. Figures

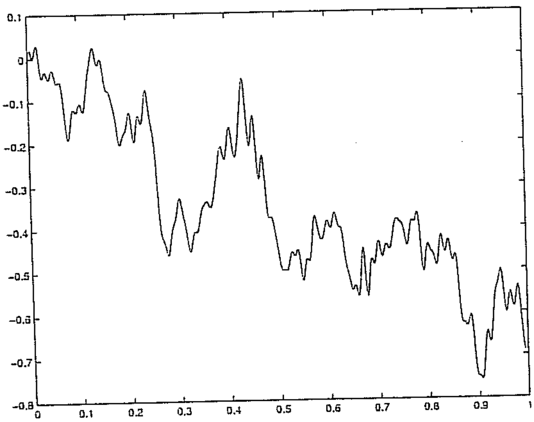

Fig. 1. Simulated trajectory of an one-dimensional fractional Brownian motion, with $H = 0.7$

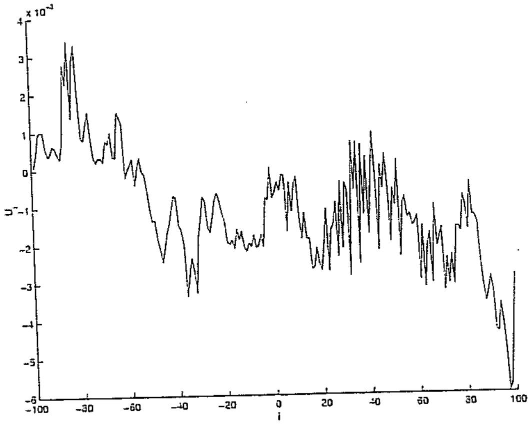

Fig. 2. Simulation of U_Δ at endpoint $T = 1$

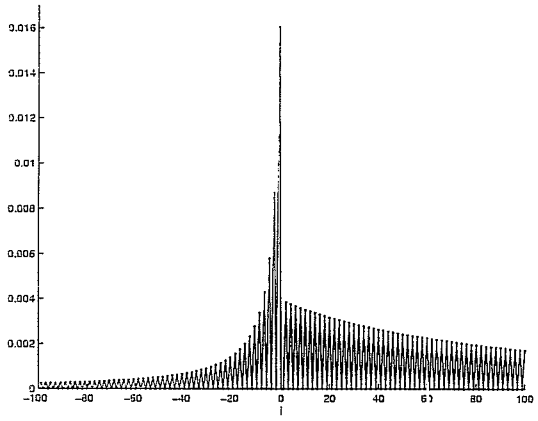

Fig. 3. Initial value u_0

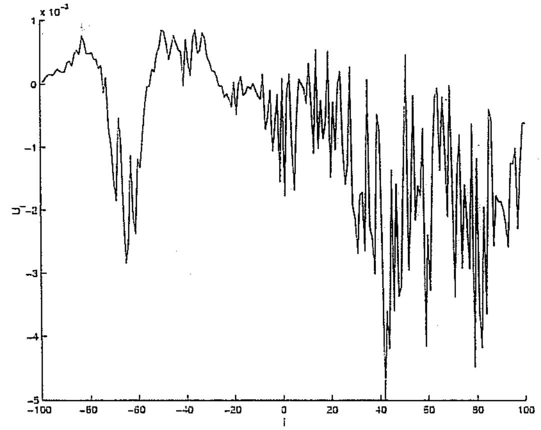

Fig. 4. Approximation U_Δ at the endpoint $T = 1$

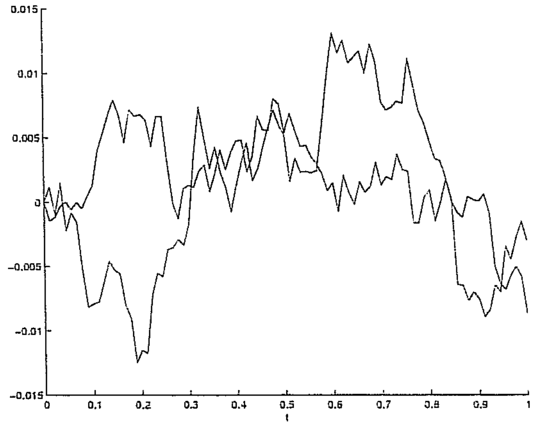

Fig. 5. Evolution in time of two components $U_{\Delta,i=0}$ and $U_{\Delta,i=1}$

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