GENERALIZED PRIMARY RINGS AND IDEALS

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Abstract: The concept of a primary ideal has been extended in several ways to noncommutative ring theory. Generically these are called *generalized primary conditions* (right and left). This paper continues the authors' investigation of generalized primary ideals and rings. Conditions are given for the intersection of generalized primary ideals to be generalized primary. Ascending chain conditions on ideals are useful in this context. The Kuratowski–Moore minimal principle is used to establish the existence of minimal generalized primary ideals. Consequences of having semicentral idempotents are developed. Various set inclusion relations and permutation identities are used to establish conditions for one-sided generalized primary conditions to be two-sided. Examples are given to illustrate and delimit the theory developed.

1. Introduction

The concept of primary ideal in the context of abstract commutative rings with unity was introduced by Emmy Noether in her seminar paper of 1921, [11]. There she used this idea to obtain a decomposition of ideals in terms of finite intersections of primary ideals. (See [13,

Ch. 4].) Various authors have extended these ideas to a noncommutative ring setting, e.g., see [1], [6], [8]. In [9] we began the examination of several such generalizations of the primary ideal concept from the viewpoint of the structure theory of rings. This paper continues that investigation. Here R will always denote a nonzero ring (associative, not necessarily being commutative or having unity).

An ideal I of R is said to be a *generalized right primary (g.r.p.)* ideal if whenever A and B are ideals of R such that $AB \subseteq I$, then either $A \subseteq I$ or $B^n \subseteq I$, for some n. This concept was introduced by Chatters and Hajarnavis [6], who used the terminology "right primary ideal." We say R is a *g.r.p. ring* if the zero ideal is a g.r.p. ideal.

Similarly, I is a *principal generalized right primary (p.g.r.p.)* ideal if whenever A and B are principal ideals of R such that $AB \subseteq I$, then either $A \subseteq I$ or $B^n \subseteq I$, for some n. If the zero ideal is p.g.r.p., then we say R is a *p.g.r.p. ring*. Analogously define *generalized left primary (g.l.p.)* ideals and rings, and *principal generalized left primary (p.g.l.p.)* ideals and rings. Generally we refer to these four properties as *generalized primary conditions*, when taken individually or in batches.

It is immediate that g.r.p. (g.l.p.) implies p.g.r.p. (p.g.l.p.). Examples are given showing the converse is not true – even for commutative rings. Examples are also given to illustrate that the conditions indeed are one-sided. In Sec. 4 various set inclusion relations or permutation identities are discussed which yield various one-sided generalized primary conditions imply other sided conditions of the same type. In Sec. 3 we continue the development of properties of generalized primary ideals began in [9]. Exemplary of the results in that section are those concerning the intersection of generalized primary ideals and those on semicentral idempotents. Examples are given throughout the paper to illustrate and delimit the theory.

2. Basic results

We use the following notation. (Here X is a nonempty subset of R.)

(i) $\langle X \rangle_R$ is the ideal of R generated by X, and $\langle \{b\} \rangle_R = \langle b \rangle_R$; if no ambiguity will arise, use $\langle \ \rangle$ in place of $\langle \ \rangle_R$;

(ii) $\mathbf{r}(X) = \{r \in R : Xr = 0\}$ and $\mathbf{l}(X) = \{r \in R : rX = 0\}$;

(iii) $I \lhd R$ is used for "I is an ideal of R";

(iv) If $T, I \lhd R$, then $(T : I)_r = \{r \in R : Tr \subseteq I\}$ and $(T : I)_l =$

$= \{r \in R : rT \subseteq I\}$;

(v) $\mathcal{I}(R)$ is the set of all ideals in R and $(\mathcal{I}(R), \cdot)$ is the associated multiplicative semigroup;

(vi) $\mathbb{N}$ and $\mathbb{Z}$ are the set of natural numbers and the set of rational integers, respectively.

Note that if $(\mathcal{I}(R), \cdot)$ has a left or right identity, then R itself must be that element. However, in such a situation R need not be a two-sided identity for $(\mathcal{I}(R), \cdot)$, as the following example illustrates.

2.1. Example. Let S be the two element semigroup with each element a left identity and let T be the semigroup ring $\mathbb{Z}_2[S]$. Then T is a left identity for $(\mathcal{I}(T), \cdot)$, but not a right identity. A routine calculation shows that that T is g.l.p. but neither g.r.p. nor p.g.r.p.

For our purposes the condition of $(\mathcal{I}(R), \cdot)$ having a left or right identity can often be used in place of stronger hypothesis of the ring R having a left or right unity.

In [9] we gave a lengthy list of conditions which are equivalent to an ideal being g.r.p. (respectively: g.l.p., p.g.r.p., p.g.l.p.). The following such conditions will be useful in the sequel.

2.2. Lemma. *Let $I \lhd R$.*

(i) *I is a p.g.r.p. ideal of R if and only if when $A, B \lhd R$, B finitely generated, and $AB \subseteq I$, then $A \subseteq I$ or $B^n \subseteq I$, for some n.*

(ii) *I is a p.g.l.p. ideal of R if and only if when $A, B \lhd R$, A finitely generated, and $AB \subseteq I$, then $B \subseteq I$ or $A^n \subseteq I$, for some n.*

Proof. (i) Let I be a p.g.r.p. ideal. Consider $A, B \lhd R$ such that $AB \subseteq I$ and $B = \langle b_1 \rangle + \cdots + \langle b_n \rangle$. So if $a \in A$, $a \neq 0$, then $\langle a \rangle \langle b_j \rangle \subseteq I$ and hence some power of $\langle b_j \rangle$ is contained in I. So some power of B is contained in I. The required converse is immediate.

(ii) Proceed similarly as in (i). $\lozenge$

Observe that in each of (i) and (ii) of Lemma 2.2 one obtains an equivalent condition if A is restricted to being finitely generated.

Note that if R is either nilpotent or prime, then R is both g.r.p. and g.l.p. Also, if R is semiprime and either g.r.p. or g.l.p., then R is prime. A commutative nil ring is always both p.g.r.p. and p.g.l.p. However, a commutative nil ring need not be g.r.p. (nor g.l.p.), as the next example illustrates.

2.3. Example. Let p be a prime in a unique factorization domain D. Use $\overline{p} = p + \langle p^n \rangle$ in $D / \langle p^n \rangle$ and let W_n be the ideal of $D / \langle p^n \rangle$ generated by $\overline{p}$. Then W_n is nilpotent of index exactly n. Let $W = \sum_2^\infty \oplus W_n$, $A = W_2$, and $B = \sum_3^\infty \oplus W_n$. Since $AB = BA = 0$ and B is not

nilpotent, the ring W is neither g.r.p. nor g.l.p. Since W is nil and commutative it is p.g.l.p. and p.g.r.p.

2.4. Example. Nil rings are not necessarily p.g.r.p. or p.g.l.p. Let A and B be nil simple rings that are not nilpotent. (Examples of such rings were given by Smoktunowicz, [14].) Then $S = A \oplus B$ is a nil ring which is neither p.g.r.p. nor p.g.l.p., since $\langle a \rangle \langle b \rangle = 0$ for any nonzero $a \in A, b \in B$.

3. Ideals

In this section some further properties of generalized primary ideals and of ideals in a generalized primary ring are developed. Many of these results are motivated by analogous results for primary ideals in the setting of commutative rings with unity, as found in [13, Ch. 4].

3.1. Proposition. *Let $I \lhd R$. If R is a g.r.p. (g.l.p.) ring, then I is a g.r.p. (g.l.p.) ring.*

Proof. Consider A, B ideals of the ring I such that $AB = 0$ and $A \neq 0$. Let $X = \langle B \rangle_R$. By the Andrunakievic Lemma [7, p. 107], $X^3 \subseteq B$. Thus $AX^3 = 0$. Now consider $\langle A \rangle_R I X^3 = (A + AR + RA + RAR)IX^3 \subseteq AX^3 + AX^3 + RAX^3 + RAX^3 = 0$. Since R is g.r.p. we have $\langle A \rangle_R I = 0$ or X^3 is nilpotent and thus $B \subseteq X$ is nilpotent. So consider the case where $\langle A \rangle_R I = 0$. Since $A \neq 0$, $\langle A \rangle_R I = 0$ implies $B \subseteq I$ is nilpotent. Thus I is g.r.p. ◊

Recall that a subring S of a ring R is a subideal of R if there is a sequence $S = I_0 \lhd I_1 \lhd \cdots \lhd I_n = R$. The smallest such n is called the level of the subideal. A subideal of level one is an ideal of R.

3.2. Corollary. *Let I be a subideal of R. If R is a g.r.p. (g.l.p.) ring, then I is a g.r.p. (g.l.p.) ring.*

Proof. This can be established by a routine induction argument on the level of the subideal. ◊

3.3. Corollary. *If T is an ideal of R and I is a g.r.p. (g.l.p.) ideal of R, then $I \cap T$ is a g.r.p. (g.l.p.) ideal of the ring T.*

Proof. Since $(T + I)/I$ is an ideal of the g.r.p. ring R/I, by Prop. 3.1 we have $(T+I)/I$ is a g.r.p. ring, and hence so is $T/(T \cap I)$. Thus $T \cap I$ is a g.r.p. ideal in the ring T. ◊

Observe that the following are equivalent for a ring R:
(i) the sum of any set of nilpotent ideals is nilpotent;
(ii) the sum of any set of nilpotent principal ideals is nilpotent.

Under either of these conditions a p.g.r.p. (p.g.l.p.) ring is g.r.p. (g.l.p.).

A routine calculation establishes that if R has a.c.c. on nilpotent ideals, then the sum of any set of nilpotent ideals of R is nilpotent. We will make use of this in the following arguments.

For I any ideal of a ring R, we will use $\sqrt{I}$ for the sum of all ideals T of R such that $T^m \subseteq I$, for some $m \in \mathbb{N}$. Equivalently, $\sqrt{I}$ is the inverse image, under the natural homomorphism $\eta : R \to R/I$, of the sum of all the nilpotent ideals in R/I. Observe that under condition (i) above, $(\sqrt{I})^n \subseteq I$, for some $n \in \mathbb{N}$.

3.4. Proposition. *Let I be a p.g.r.p. (p.g.l.p.) ideal of R. If R/I has a.c.c. on nilpotent ideals, then $\sqrt{I}$ is a prime ideals of R, and it is the minimal semiprime ideal containing I.*

Proof. Let $A, B \triangleleft R$ such that $AB \subseteq \sqrt{I}$ and $A \not\subseteq \sqrt{I}$. The latter implies $A^n \not\subseteq I$, for each $n \in \mathbb{N}$. However, there exists $m \in \mathbb{N}$ such that $(AB)^m \subseteq I$, as a consequence of the chain condition given. Without loss of generality, take m to be minimal such. If $m = 1$, then $AB \subseteq \subseteq I$ and $B^k \subseteq I$, for some k; so $B \subseteq \sqrt{I}$. So take $m > 1$. Then $[(AB)^{m-1}A]B \subseteq I$ and hence either some power of B is contained in I, or $(AB)^{m-1}A \subseteq I$. So $(AB)^{m-1} \subseteq I$, a contradiction to the minimality of m. Thus $\sqrt{I}$ is a prime ideal. If S is any semiprime ideal of R with $I \subseteq S$, then $(\sqrt{I})^n \subseteq S$, for some $n \in \mathbb{N}$, and hence $\sqrt{I} \subseteq S$. ◊

3.5. Corollary. *Let R have a.c.c. on ideals. For any p.g.r.p. (p.g.l.p.) ideal I of R, $\sqrt{I}$ is a prime ideal of R.*

3.6. Proposition. *Let $I_1, \dots, I_n$ be g.r.p. (g.l.p.) ideals of R. If R has a.c.c. on ideals and $\sqrt{I_j} = \sqrt{I_1}$, $j = 2, \dots, n$, then $\cap_{i=1}^n I_i$ is a g.r.p. (g.l.p.) ideal of R.*

Proof. Let $n = 2$ and let $A, B \triangleleft R$ with $AB \subseteq I_1 \cap I_2$, and $A \not\subseteq I_1 \cap I_2$. Consider the case where $A \not\subseteq I_1$. Since $AB \subseteq I_1$, we have $B^m \subseteq I_1$, for some $m \in \mathbb{N}$. Thus $B \subseteq \sqrt{I_1}$ and hence $B \subseteq \sqrt{I_2}$. Using this and that R/I_2 has a.c.c. on ideals we obtain $B^k \subseteq I_2$, for some $k \in \mathbb{N}$. Thus some power of B is contained in $I_1 \cap I_2$. Proceed similarly for $A \not\subseteq I_2$.

A routine induction argument establishes the desired result for general n. ◊

Note that the hypothesis that R has a.c.c. on ideals could be replaced by: R/I_j has a.c.c. on nilpotent ideals for each $j = 1, \dots, n$.

Also observe that the equality of the $\sqrt{I_j}$ terms cannot be eliminated from the hypothesis of Prop. 3.6. For example, in the ring $\mathbb{Z}$ we have $\langle 2 \rangle$ and $\langle 3 \rangle$ are g.r.p. ideals, but $\langle 2 \rangle \cap \langle 3 \rangle = \langle 6 \rangle$ is not.

3.7. Lemma. *If $I \lhd R$ and $b \in \sqrt{I}$, then there exist $n \in \mathbb{N}$ such that $(\langle b \rangle)^n \subseteq I$.*

Proof. There exist ideals $I_1, \ldots, I_m$ of R such that $b \in (I_1 + \ldots + I_m)$. The image of each I_j is nilpotent in $\overline{R} = R/I$, so the image of $I_1 + \ldots + I_m$ in $\overline{R}$ is nilpotent. Consequently there exists $n \in \mathbb{N}$ such that $(I_1 + \ldots + I_m)^n \subseteq I$. Hence $(\langle b \rangle)^n \subseteq I$. ◊

3.8. Proposition. *Let $(\mathcal{I}(R), \cdot)$ have a right (left) identity. If I is an ideal of R such that $\sqrt{I}$ is a maximal ideal of R, then I is a p.g.r.p. (p.g.l.p.) ideal of R.*

Proof. Let $a, b \in R$ such that $\langle a \rangle \langle b \rangle \subseteq I$ with $\langle a \rangle \nsubseteq I$. Since $I \subseteq \sqrt{I}$ and since $\sqrt{I}$ is also a prime ideal of R, we have $\langle a \rangle \subseteq \sqrt{I}$ or $\langle b \rangle \subseteq \sqrt{I}$. If $\langle b \rangle \subseteq \sqrt{I}$, then by Lemma 3.7, $(\langle b \rangle)^n \subseteq I$, for some n. Assume $\langle b \rangle \nsubseteq \sqrt{I}$. Then $\langle a \rangle \subseteq \sqrt{I}$ and hence $(\langle a \rangle)^m \subseteq I$, for some m. Without loss of generality take m to be minimal. Note that $m > 1$. Since $\sqrt{I}$ is a maximal ideal, we have $\langle b \rangle + \sqrt{I} = R$; so $\langle a \rangle \langle b \rangle + \langle a \rangle \sqrt{I} = \langle a \rangle R = \langle a \rangle$. Then $\langle a \rangle \subseteq I + \langle a \rangle \sqrt{I}$ and hence $(\langle a \rangle)^{m-1} \subseteq (\langle a \rangle)^{m-1} I + (\langle a \rangle)^m \sqrt{I} \subseteq I$, a contradiction to the minimality of m. ◊

Recall that for any proper ideal I of a ring R there is a unique smallest prime ideal containing I. Attempting to establish an analogous result for g.r.p. ideals runs into the difficulty of possibly having an unbounded increasing sequence of exponents arise for the ideals involved. This can be avoided by invoking an additional condition.

3.9. Definition. If there exists $n \in \mathbb{N}$ such that for each g.r.p. ideal I of R we have $(\sqrt{I})^n \subseteq I$, then R is said to be *uniformly g.r.p. bounded.* The smallest such n for which this holds is called the *uniform g.r.p. bound* for R.

In the proof of the next theorem we make use of a set theoretic result which is equivalent to the Axiom of Choice in Zermelo–Fraenkel set theory.

3.10. Kuratowski–Moore minimal principle ([10, p. 223]). *Let $\mathcal{S}$ be a nonempty family of sets. If for each chain $\mathcal{C}$ in $(\mathcal{S}, \supseteq)$, the set $\cap \mathcal{C}$ is in $\mathcal{S}$, then $(\mathcal{S}, \supseteq)$ has a minimal term.*

In establishing the next corollary it is useful to note that if $I \lhd R$, then R/I is a p.g.r.p. (p.g.l.p.) ring if and only if I is a p.g.r.p. (p.g.l.p.) ideal of R.

3.11. Proposition. *Let I be a proper ideal of R and let Ω be the set of all g.r.p. ideals which contain I. If R has uniform g.r.p. bound n, then $(\Omega, \supseteq)$ contains a minimal term. In particular, the set of all g.r.p.*

ideals of R contains a minimal term.

Proof. Let C be a chain in $(\Omega, \supseteq)$ and let $V = \cap C$. So V is an ideal of R which contains I. To show V is a g.r.p. ideal, let $A, B \lhd R$ with $AB \subseteq V$ and $A \nsubseteq V$. So $AB \subseteq X$, for each $X \in C$, and there exists $Y \in C$ so that $A \nsubseteq X$, for each $X \in C$ with $X \subseteq Y$. Then $B^n \subseteq X$, for each such X, and hence $B^n \subseteq V$. So $V \in \Omega$. By the Kuratowski–Moore Minimal Principle, $(\Omega, \supseteq)$ has a minimal term. Using $I = 0$ we get the desired minimal g.r.p. ideal of R. $\Diamond$

Similarly one can introduce the concept of *uniform g.l.p. bound* and get a g.l.p. version of Prop. 3.11.

We next consider the consequences of having a semicentral idempotent in a generalized primary ring. Recall that an idempotent $e \in R$ is left (right) semicentral if $exe = xe$, (respectively, $exe = ex$), for each $x \in R$, [2]. Note that e is left (right) semicentral if and only if $eRe = Re$, (respectively, $eRe = eR$). We make use of the following result.

3.12. Lemma. ([5, Lemma 1.1].) *Let R be a ring with unity and let e be a left semicentral idempotent of R. Then eR, $R(1-e)$, and $eR(1-e)$ are ideals of R, and $1 - e$ is a right semicentral idempotent in R.*

An analogous result holds for right semicentral idempotents.

3.13. Proposition. *Let R be a ring with unity and let e be either a left or right semicentral idempotent in R. If R is p.g.r.p. (p.g.l.p.), then e is either 0 or 1.*

Proof. Take e to be left semicentral. Assume $e \neq 0$. From $eR(1 - e) \cdot eR = 0$ and R p.g.r.p. we have $eR(1 - e) = 0$, and hence $ex = exe$, for each $x \in R$. Hence e is central and consequently $eR \cdot (1 - e)R = 0 = (1 - e)R \cdot eR$, which with R g.r.p. gives the desired result. Proceed similarly if e is right semicentral. $\Diamond$

3.14. Corollary. *Let R be a ring with unity and let I be a p.g.r.p. (p.g.l.p.) ideal of R. If $\overline{R} = R/I$ has a left or right semicentral idempotent $\overline{e} = e + I$, then either* (i) *$e \in I$ and hence $eR + Re \subseteq I$, or* (ii) *$\mathbf{r}(e) + \mathbf{l}(e) \subseteq I$.*

Proof. If $\overline{e} = 0$, then $e \in I$ and hence $eR + Re \subseteq I$. If $\overline{e}$ is the unity element in $\overline{R}$, then $\overline{ex} = \overline{x} = \overline{xe}$, for each $\overline{x} = x + I$, and hence $ex - x$ and $xe - x$ are in I, yielding $\mathbf{r}(e) + \mathbf{l}(e) \subseteq I$. $\Diamond$

3.15. Corollary. *Let R be a ring with unity. If every maximal ideal of R is generated by either a left or a right semicentral idempotent, then every proper p.g.r.p. (p.g.l.p.) ideal of R is a maximal ideal.*

Proof. Let I be a proper p.g.r.p. ideal of R. Then I is contained in a maximal ideal $M = \langle e \rangle$, where e is a left or a right semicentral

idempotent. If $e \in I$, then $I = M$. Otherwise R/I contains a nonzero left or right semicentral idempotent, $e + I$, and hence $R = \mathbf{r}(e) + eR \subseteq$ $\subseteq M$, a contradiction. $\Diamond$

4. Set inclusion relations and permutation identities

In this section we introduce various set inclusion relations and permutation identities. By imposing these conditions the assumption of various one-sided generalized primary conditions on rings or ideals will imply the other sided condition of the same holds, e.g., g.r.p. implies g.l.p.

4.1. Set inclusion relations

In each of the following if R does not have unity then assume the arithmetic is carried out in a suitable extension ring which has unity. This takes care of the situation that arises for zero exponents.

4.1.1. If $X, Y \lhd R$, then there exist $i, j, k, l \geq 0$ such that $XR^iYR^j \subseteq$ $\subseteq YR^kXR^l$.

4.1.2. If $X, Y \lhd R$, then there exist $i, j, k, l \geq 0$ such that $R^jXR^iY \subseteq$ $\subseteq R^lYR^kX$.

4.1.3. If $X, Y \lhd R$, then there exist $i, j, k, l, m, n \geq 0$ such that $R^iXR^jYR^k \subseteq R^lYR^mXR^n$.

4.1.4. If X and Y are finitely generated ideals of R, then there exist $m, n \geq 0$ such that $XR^mY \subseteq YR^nX$.

4.1.5. If X and Y are finitely generated ideals of R, then there exist $i, j, k, l \geq 0$ such that $R^iXR^jY \subseteq R^kYR^lX$.

4.1.6. If X and Y are finitely generated ideals of R, then there exist $i, j, k, l \geq 0$ such that $XR^iYR^j \subseteq YR^kXR^l$.

Note that in each of the above relations the exponents may depend on the choice of X and Y.

Observe that 4.1.1 implies 4.1.3 and 4.1.6, while 4.1.2 implies 4.1.3 and 4.1.5. Furthermore, 4.1.4 implies both 4.1.5 and 4.1.6. Also note that if R has unity then each of the six set inclusion relations above yields $XY \subseteq YX$. Since X and Y are arbitrary this then yields $XY =$ $= YX$.

Of course all commutative rings satisfy each of the above set inclusion relations. However, there are many noncommutative rings that satisfy one or more of these. Some such rings arise from permutation identities, which we discuss next.

4.2. Definition. A semigroup S is said to satisfy a *permutation identity* if there exists a permutation π on $1, \ldots, n$, $n > 1$, such that $s_1 \cdots s_n = s_{\pi(1)} \cdots s_{\pi(n)}$ for each $s_1, \ldots, s_n \in S$. A ring R is said to be a *permutation identity ring* if the multiplicative semigroup $(R, \cdot)$ satisfies a permutation identity. For background on permutation identity rings see [4] and on semigroups satisfying a permutation identity see [12].

Observe that if $(R, \cdot)$ satisfies a permutation identity π, then $(\mathcal{I}(R), \cdot)$ also satisfies the same identity. However it is possible for $(\mathcal{I}(R), \cdot)$ to satisfy a permutation identity and $(R, \cdot)$ not satisfy any permutation identity. For example, if R is a noncommutative simple ring with unity, then $(R, \cdot)$ satisfies no permutation identity, but $(\mathcal{I}(R), \cdot)$, being commutative, satisfies the identity $XY = YX$.

4.3. Example. Let π be a permutation on $1, \ldots, n$, with $n > 2$ and $\pi(1) \neq 1$. If $(\mathcal{I}(R), \cdot)$ satisfies the permutation identity given by π, then R will satisfy 4.1.1. In particular this occurs if $(\mathcal{I}(R), \cdot)$ is left permutable, i.e., $xyz = yxz$, for each $x, y, z \in \mathcal{I}(R)$. So left permutable rings satisfy 4.1.1. Similarly, if π is a permutation with $\pi(n) \neq n$, and $(\mathcal{I}(R), \cdot)$ satisfies the permutation identity given by π, then R will satisfy 4.1.2. In particular this occurs if $(\mathcal{I}(R), \cdot)$ is right permutable, i.e., $xyz = xzy$, for each $x, y, z \in \mathcal{I}(R)$. Examples of such rings and methods of constructing further examples can be found in [3], [4].

4.4. Example. Let S be the two element semigroup with both elements being right identities. Then the semigroup ring $T = \mathbb{Z}_2[S]$ is right permutable, hence satisfies 4.1.1. But it does not satisfy 4.1.2. This ring is g.r.p., but not g.l.p. nor p.g.l.p.

4.5. Example. Putcha and Yaqub have shown that if S is a semigroup satisfying a permutation identity, then there exist $m \in \mathbb{N}$ such that $xaby = xbay$, for each $a, b \in S, x, y \in S^m$, [12]. Thus every ring R for which $(\mathcal{I}(R), \cdot)$ satisfies a permutation identity must satisfy 4.1.3 with $j = m = 0$. In particular, every permutation identity ring satisfies 4.1.3 with $j = m = 0$.

We now use set inclusion relations to establish connections between generalized primary conditions, one-sided yielding two-sided.

4.6. Proposition. *If R satisfies 4.1.1 (4.1.2) and R is g.r.p. (g.l.p.), then R is g.l.p. (g.r.p.).*

Proof. Let R satisfy 4.1.1 and be g.r.p. Consider $A, B \triangleleft R$ such that $AB = 0$, with $A \neq 0$, $B \neq 0$. Then there exist $i, j, k, l \geq 0$ such that $BR^i A R^j \subseteq AR^k BR^l \subseteq AB = 0$. Since $B \neq 0$, we have $R^i A R^j$ is nilpotent, and hence A is nilpotent. Thus R is g.l.p.

Proceed similarly if R satisfies 4.1.2 and is g.l.p. ◊

4.7. Corollary. *Let I be a g.r.p. (g.l.p.) ideal of R such that $\overline{R} = R/I$ satisfies 4.1.1 (4.1.2). Then I is a g.l.p. (g.r.p.) ideal of R.*

4.8. Corollary. *Let R satisfy 4.1.1 (4.1.2). Then every g.r.p. (g.l.p.) ideal of R is a g.l.p. (g.r.p.) ideal of R.*

Proof. The set inclusion relations 4.1.1 and 4.1.2 are inherited by homomorphic images. ◊

4.9. Proposition. *Let R satisfy 4.1.3.*

(i) *If $\mathbf{r}(X) = 0$ and R is g.r.p., then R is g.l.p.*

(ii) *If $\mathbf{l}(R) = 0$ and R is g.l.p., then R is g.r.p.*

Proof. (i) Let $A, B \lhd R$ such that $AB = 0$, with $A \neq 0, B \neq 0$. Then there exist $i, j, k, l, m, n \geq 0$ such that $R^i B R^j A R^k \subseteq R^l A R^m B R^n \subseteq \subseteq AB = 0$. So either $R^i B = 0$ or $R^j A R^k$ is nilpotent. The latter yields A is nilpotent, and $R^i B = 0$ cannot occur because $\mathbf{r}(R) = 0$, and $B \neq 0$.

(ii) Proceed similarly as in (i). ◊

4.10. Corollary. *Let R satisfy 4.1.3.*

(i) *If I is a g.r.p. ideal of R and $(R : I)_r \subseteq I$, then I is a g.l.p. ideal of R.*

(ii) *If I is a g.l.p. ideal of R and $(R : I)_l \subseteq I$, then I is a g.l.p. ideal of R.*

Proof. (i) The condition $(R : I)_r \subseteq I$ implies $\mathbf{r}(R/I) = 0$. This and the inheritance from R of 4.1.3 in R/I, yields the desired result using Prop. 4.9(i).

(ii) Proceed similarly using Prop. 4.9(ii). ◊

4.11. Proposition. *Let R satisfy 4.1.4.*

(i) *If $\mathbf{l}(R) = 0$ and R is p.g.r.p., then R is p.g.l.p.*

(ii) *If $\mathbf{r}(R) = 0$ and R is p.g.l.p., then R is p.g.r.p.*

Proof. (i) Consider $A, B \lhd R$ such that $AB = 0$, $A \neq 0$, $B \neq 0$, and B is finitely generated. Then there exist $m, n \geq 0$ such that $BR^m A \subseteq \subseteq AR^n B \subseteq AB = 0$. So either A is nilpotent or $BR^m = 0$. The latter, together with $\mathbf{l}(R) = 0$, implies $B = 0$, a contradiction.

(ii) Proceed similarly as in (i). ◊

4.12. Corollary. *Let $I \lhd R$ such that R/I satisfies 4.1.4.*

(i) *If $(R : I)_r \subseteq I$ and I is a p.g.r.p. ideal, then I is a p.g.l.p. ideal.*

(ii) *If $(R : I)_l \subseteq I$ and I is a p.g.l.p. ideal, then I is a p.g.r.p. ideal.*

Observe that R/I may satisfy 4.1.4 without R itself satisfying 4.1.4 since an ideal X/I of R/I can be finitely generated, yet the ideal X in R not be.

4.13. Proposition. *Let* $\mathbf{l}(R) = 0 = \mathbf{r}(R)$.

(i) *If* R *satisfies* 4.1.5 *and* R *is p.g.r.p., then* R *is p.g.l.p.*

(ii) *If* R *satisfies* 4.1.6 *and* R *is p.g.l.p., then* R *is p.g.r.p.*

Proof. (i) Let A, B be finitely generated ideals of R such that $AB = 0$, $A \neq 0$, $B \neq 0$. Then there exist $i, j, k, l \geq 0$ such that $R^i B R^j A \subseteq \subseteq R^k A R^l B \subseteq AB = 0$. Using that R is p.g.r.p. we have either A is nilpotent or $R^i B R^j = 0$. The latter and $\mathbf{l}(R) = 0 = \mathbf{r}(R)$ yields $B = 0$, a contradiction.

(ii) Proceed similarly as in (i). $\Diamond$

4.14. Corollary. *Let* $I \lhd R$ *and assume that* $(R : I)_r \cup (R : I)_l \subseteq I$.

(i) *If* R/I *satisfies* 4.1.5 *and* I *is a p.g.r.p. ideal, then* I *is a p.g.l.p. ideal.*

(ii) *If* R/I *satisfies* 4.1.6 *and* I *is a p.g.l.p. ideal, then* I *is a p.g.r.p. ideal.*

It is worth noting that the set inclusion relations given in 4.1 can also be considered as conditions imposed on the ordered semigroup $(\mathcal{I}(R), \cdot, \subseteq)$. Furthur connections between this ordered semigroup and the structure of the underlying ring R will be given in a subsequent paper by the second author and R. Tucci.

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