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FLAGS IN EUCLIDEAN THREE-SPACE

Boris **Odehnal**

Institut für Diskrete Mathematik und Geometrie, Technische Universität Wien, Wiedner Hauptstraße 8-10/104/3, 1040 Wien, Austria

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Abstract: The present paper is devoted to the study of flags in Euclidean three-space. Coordinates of flags are defined and a point model in a projective space is discussed. We study special subsets of the manifold of flags and show the close relations to Euclidean kinematics and Non-Euclidean geometries.

1. Introduction

A flag $\mathcal{F}$ in a three-dimensional space is a triplet (P, G, E), where P is a point, G is a line, and E is a plane with $P \in G \subset E$. A possible way to introduce coordinates in the set of flags is the following: Consider the three-dimensional space to be a projective one. Then points in $\mathbb{P}^3$ can be identified with the one-dimensional subspaces of a vector space K^4, where K is any commutative field (if the space is assumed to be Pappian). The planes of $\mathbb{P}^3$ can be identified with the linear forms of K^4 or equivalently with the one-dimensional subspaces of the dual vector space $K^{4\star}$. The lines $G \subset \mathbb{P}^3$, represented by homogeneous

E-mail address: odehnal@geometrie.tuwien.ac.at

Plücker coordinates correspond to certain one-dimensional subspaces of $K^4 \wedge K^4$.

In this setting exactly those one-dimensional subspaces of the tensor product $V := K^4 \otimes (K^4 \wedge K^4) \otimes K^{4\star}$ correspond to flags in $\mathbb{P}^3 = \mathbb{P}^3(K^4)$ that represent triplets comprising points P, lines G, planes E, such that $P \in G \subset E$.

In [9] this model was used in order to study the flag variety of a projective three-space. There the flag variety appears as the intersection of the Segre variety $\mathcal{S}_{3,5,3} \subset \mathbb{P}^{95}$ (see [7] for the exact definition) with a 63-dimensional subspace of $\mathbb{P}^{95}$. Automorphisms of the flag variety and automorphisms of $\mathbb{P}^3$ turn out to determine each other mutually. Unfortunately, V has a very high dimension which is not useful in practical computations.

The flag variety associated with n-dimensional projective spaces is treated in [3], [4]. The close relation to representations of the group $\mathrm{PGL}(n+1, \mathbb{C})$ is discussed in [5], [6].

The contributions of the present paper are the following: We consider the above mentioned three-dimensional space to be a Euclidean one and introduce coordinates in the set of flags in Sec. 2. Afterwards we present a point model of the set of flags in Sec. 3. The equiform transformations of Euclidean $\mathbb{R}^3$ induce linear automorphisms (collineations) of a certain six-dimensional cone in the model space. Sec. 3 also deals with pencils of flags and other submanifolds of the flag manifold. As an application of coordinates of flags as defined in Sec. 2 we show how to characterize pairs of flags in Sec. 4. In this section we distinguish between flags only with respect to incidence relations of components. Sec. 5 is given in order to show that the manifold of flags in Euclidean space $\mathbb{R}^3$ admits an embedding as hyperquadric in a seven-dimensional projective space. Finally we conclude and discuss the results in Sec. 6.

2. Coordinates of flags

2.1. Points, lines, and planes

Consider Euclidean three-space $\mathbb{R}^3$. A point P can be represented by Cartesian coordinates $p = (p_1, p_2, p_3)^T$. Oriented planes E are given by plane coordinates $(e_0, e_1, e_2, e_3)^T = (e_0, e^T)$ with $\|e\| = 1$. An equation of E is then given by $e_0 + \langle e, x \rangle = 0$, where $\langle \cdot, \cdot \rangle$ is the standard scalar product and $x = (x_1, x_2, x_3)^T$.

A line G in Euclidean three-space can be described by normalized Plücker coordinates $(g, \overline{g})$, where g is a unit vector parallel to G. The vector $\overline{g}$ is called momentum vector and is defined by $\overline{g} := x \times g$, if X is any point on G. The momentum vector is independent of the choice of X on G. Obviously, the thus defined Plücker coordinates of G satisfy

$$(1) \qquad \langle g, \overline{g} \rangle = 0.$$

A line G in Euclidean space carries two oriented ones. The coordinate vectors $(g, \overline{g})$ and $(-g, -\overline{g})$ represent the different oriented lines in the same geometric object, i.e. the line G without orientations.

The coordinates $(g, \overline{g})$ of a non-oriented line G are homogeneous: the vector $(\lambda g, \lambda \overline{g})$ with $\lambda \in \mathbb{R} \setminus \{0\}$ is a coordinate vector of the same line G. Thus the six coordinates of G can be considered as coordinates of a point in a five-dimensional projective space $\mathbb{P}^5$.

The mapping $\gamma : \ G \mapsto (g, \overline{g})\mathbb{R} \in \mathbb{P}^5$ is not onto. Only the points whose coordinates satisfy Eq. (1) appear as images of lines. The quadratic hypersurface $\operatorname{im} \gamma = M_2^4$ is called Klein quadric or Plücker quadric.

Remark. Here and in the following, the superscript and the subscript denote the dimension and the algebraic degree, respectively. $\diamondsuit$

Remark. We dropped the norming condition $\|g\| = 1$ and thus we allowed $g = 0$. Without saying it we performed the projective closure of Euclidean three-space. The mapping γ is thus defined for lines at infinity as well. They are characterized by $g = 0$ and their γ-images constitute a plane in M_2^4. $\diamondsuit$

M_2^4 is a point model for the set of lines in projective three-space. The point model for the set of lines in Euclidean three-space is M_2^4 without the plane $g = 0$, see [18], [19], [20]. In order to describe the set of lines in an elliptic three-space one has to add appropriate norming conditions for Plücker coordinates to (1), see [1], [13], [15], [17].

2.2. Line elements

A flag $\mathcal{F}$ in Euclidean three-space $\mathbb{R}^3$ consists of a line element (P, G) and a plane E containing (P, G). In order to define coordinates for flags in $\mathbb{R}^3$ we use coordinates for line elements in $\mathbb{R}^3$ as defined in [14].

Let $(g, \overline{g})$ be normalized Plücker coordinates of the line G. Let further P be the point on G. Then the normalized Plücker coordinates of the line element (P, G) is the vector $(g, \overline{g}, \gamma) \in \mathbb{R}^7$, where $\gamma := \langle g, p \rangle$.

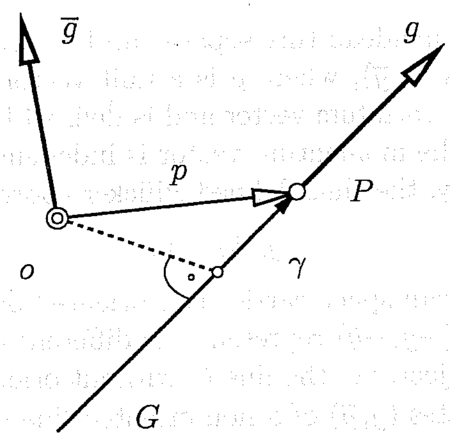

Figure 1. Coordinates for line elements

The real value γ has a geometric meaning (see Fig. 1): It is the signed distance of $P \in G$ to the pedal point of G taking into account the orientation of G.

A line element in Euclidean space $\mathbb{R}^3$ carries two oriented ones. The vectors $(g, \overline{g}, \gamma)$ and $(-g, -\overline{g}, -\gamma)$ coordinatize the two oriented line elements on G pointing in opposite directions, but both are coordinates of the non-oriented line element (P, G).

Remark. The concept of Plücker coordinates of lines can be extended to lines at infinity. This is not the case for coordinates of line elements. Though line elements (G_∞, G) with proper lines G and their ideal point G_∞ can be described by $(g, \overline{g}, \infty)$, we cannot assign coordinates to line elements located in the ideal plane in this way. $\Diamond$

Again we can drop the norming condition. It is obvious that $(\lambda g, \lambda\overline{g}, \lambda\gamma)$ describes the same line element in Euclidean space $\mathbb{R}^3$, if $\lambda \in \mathbb{R} \setminus \{0\}$ and $g \neq 0$. Obviously the thus defined coordinates of line elements are homogeneous. They allow an interpretation as coordinates of points in a six-dimensional projective space $\mathbb{P}^6$.

Eq. (1) represents a quadratic cone M_2^5 in $\mathbb{P}^6$ which at least includes the set of points representing the coordinate vectors of line elements in Euclidean $\mathbb{R}^3$.

Remark. The point model for the set of line elements in Euclidean three-space is a subset of the quadratic cone M_2^5. The three-dimensional projective subspace spanned by $V = (0,0,0;0,0,0;1)\mathbb{R}$ and $g_1 = g_2 = g_3 = 0$ does not belong to the point model. V is the vertex of M_2^5. $\Diamond$

In [14] this point model for the set of line elements was introduced in order to point out the close relation between equiform kinematics and the geometry of line elements. The geometry of line elements turned out to be useful for the recognition and reconstruction of surfaces that are invariant under one-parameter subgroups of the group of equiform motions, see [10].

2.3. Flags

In order to describe a flag $\mathcal{F} \subset \mathbb{P}^3$ we recall that a flag consists of a line element (P, G) and a plane E containing (P, G). The plane E is fixed by a unit normal vector $\widehat{g}$ (see Fig. 2). We define:

Definition 2.1. The vector $(g, \overline{g}, \widehat{g}, \gamma) \in \mathbb{R}^{10}$ with $g \neq 0$, $\widehat{g} \neq 0$ is the coördinate vector of a flag $\mathcal{F} = (P, G, E)$ in Euclidean space $\mathbb{R}^3$, where $(g, \overline{g}, \gamma)$ are the normalized Plücker coordinates of the line element (P, G) in $\mathbb{R}^3$ and $\widehat{g}$ is a unit vector with $\langle g, \widehat{g} \rangle = 0$.

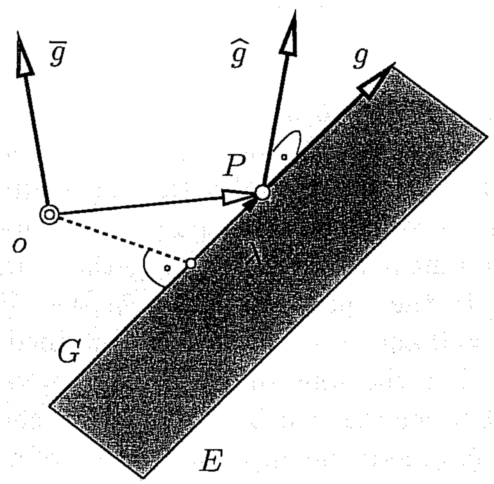

Figure 2. Coordinates of flags

Remark. We exclude $g = 0$, as this does not define a line in Euclidean space. Lines with Plücker coordinates $(0, \overline{g})$ are lines in the ideal plane of the projective closure of $\mathbb{R}^3$. The plane component would not be defined if $\widehat{g} = 0$. Therefore there is the four-dimensional subspace W: $g_1 = g_2 = g_3 = g_7 = g_8 = g_9 = 0$ of $\mathbb{R}^{10}$ whose points do not correspond to flags in $\mathbb{R}^3$. $\diamond$

We observe the following phenomena concerning orientations: Both the line element $(P, G) \subset \mathcal{F}$ and the plane $E \in \mathcal{F}$ can be oriented in two different ways. The orientations of G and E do not depend on each other. Thus the flag $\mathcal{F} = (P, G, E)$ carries four different oriented flags all of them being in the same orbit with respect to SO_3.

So the coordinate vectors $(g,\overline{g},\widehat{g},\gamma)$, $(g,\overline{g},-\widehat{g},\gamma)$, $(-g,-\overline{g},\widehat{g},-\gamma)$, and $(-g,-\overline{g},-\widehat{g},-\gamma)$ describe the four different oriented flags belonging to only one geometric object: the flag F without any orientation.

The coordinates $(g,\overline{g},\widehat{g},\gamma)$ of a flag F as defined in Def. 2.1 determine $\mathcal{F}$ uniquely. The point, the line, and the plane can be extracted from $(g,\overline{g},\widehat{g},\gamma)$:

Lemma 2.1. *Each vector* $(g,\overline{g},\widehat{g},\gamma)\in\mathbb{R}^{10}$ *with* $\|g\|=\|\widehat{g}\|=1$ *and* $\langle g,\overline{g}\rangle=\langle g,\widehat{g}\rangle=0$ *is the coordinate vector of a flag* $\mathcal{F}=(P,G,E)$ *in Euclidean three-space* $\mathbb{R}^3$.

The point P, *the line* G, *and the plane* E *can be found according to*

$$P=g\times\overline{g}+\gamma g, \tag{2}$$

$$G=(g,\overline{g}), \tag{3}$$

$$E=(-\det(g,\overline{g},\widehat{g}),\widehat{g}). \tag{4}$$

Proof. Given the vector $(g,\overline{g},\widehat{g},\gamma)\in\mathbb{R}^{10}$ with the required properties it is obvious that the line component of $\mathcal{F}$ has coordinates $(g,\overline{g})$. By the definition of line element coordinates, especially by the definition of γ, we find the coordinates of the point component $P\in\mathcal{F}$ as $p= = g\times\overline{g}+\gamma g$. The unit normal vector $\widehat{g}$ together with the point $P\in E$ and $G\subset E$ leads to the equation $-\det(g,\overline{g},\widehat{g})+\langle\widehat{g},x\rangle=0$ of E. By assumption $\langle g,\widehat{g}\rangle=0$ and so the line G is contained in the plane E. ◊

Remark. Note that the four differently oriented flags attached to $\mathcal{F}$ mentioned in Lemma 2.1 can be obtained by choosing appropriate orientations of g, $\overline{g}$, $\widehat{g}$, and the appropriate sign of γ. ◊

3. A point model for the set of flags

The coordinates $(g,\overline{g},\widehat{g},\gamma)$ of a flag $\mathcal{F}$ are homogeneous, which means that the vector $(\lambda g,\lambda\overline{g},\lambda\widehat{g},\lambda\gamma)$ with $\lambda\in\mathbb{R}\setminus\{0\}$ also is a coordinate vector of the flag $\mathcal{F}$. This can easily be seen using Eqs. (2), (3), and (4) from Lemma 2.1. They lead to the same point, line, and plane that define the flag.

In the previous section we required that the vectors g and $\widehat{g}$ are unit vectors. In the following when we deal with homogeneous flag coordinates we always assume that g and $\widehat{g}$ are of equal length, i.e. $\langle g,g\rangle=\langle\widehat{g},\widehat{g}\rangle$.

Now we can interpret the homogeneous coordinates $(g,\overline{g},\widehat{g},\gamma)=$

$= (g_1, \dots, g_{10})$ of a flag $\mathcal{F}$ as coordinates of points in a nine-dimensional projective space $\mathbb{P}^9$ with base points $B_1, \dots, B_{10}$. In the analytical model $\mathbb{R}^{10}$, B_i are the one-dimensional subspaces spanned by the canonical basis vectors.

Remark. We observe a strange kind of homogeneity of coordinates of a flag. If $\mathcal{F} = (P, G, E)$ is represented by $(g, \overline{g}, \widehat{g}, \gamma)$ then the line element (P, G) with coordinates $(g, \overline{g}, \gamma)$ remains unchanged, if we multiply its coordinate vector by $\lambda \in \mathbb{R} \setminus \{0\}$. The normal vector $\widehat{g}$ defines the plane E even it is not a unit vector. So the vector $(\lambda g, \lambda \overline{g}, \mu \widehat{g}, \lambda \gamma)$ with $\mu \in \mathbb{R} \setminus \{0\}$ is the coordinate vector of the same flag $\mathcal{F}$. $\diamond$

Now we can state:

Theorem 3.1. *The point model of the set of flags in Euclidean three-space is contained in the six-dimensional algebraic variety $M^6 \subset \mathbb{P}^9$ given by the equations*

$$\langle g, g \rangle - \langle \widehat{g}, \widehat{g} \rangle = 0 = g_1^2 + g_2^2 + g_3^2 - g_7^2 - g_8^2 - g_9^2, \tag{5}$$

$$\langle g, \overline{g} \rangle = 0 = g_1 g_4 + g_2 g_5 + g_3 g_6, \tag{6}$$

$$\langle g, \widehat{g} \rangle = 0 = g_1 g_7 + g_2 g_8 + g_3 g_9. \tag{7}$$

Remark. Before starting the proof we note that the points in the subspace $V: \ g = \widehat{g} = 0$ do not correspond to flags in $\mathbb{R}^3$. V is a four-dimensional subspace of $\mathbb{R}^{10}$ and defines a three-dimensional projective subspace of the model space $\mathbb{P}^9$. Points in this subspace do not correspond to flags in $\mathbb{R}^3$. $\diamond$

Proof. Given a vector $(g, \overline{g}, \widehat{g}, \gamma) = (g_1, \dots, g_{10}) \in \mathbb{R}^{10}$ whose entries satisfy (5), (6), (7) we can apply Lemma 2.1 in order to find the point, the line, and the plane defining the flag $\mathcal{F}$.

According to Def. 2.1 we can assign coordinates to a flag such that they satisfy (5),(6), and (7). $\diamond$

We are able to give even a rational parametrization of the manifold M^6. This enables us to give a very low upper bound for the algebraic degree of M^6 considered as an algebraic variety:

Theorem 3.2. *The manifold M^6 defined by Eqs.* (5), (6), *and* (7) *admits a rational parametrization and is at most of algebraic degree* 5. *M^6 is a cone with* 0*-dimensional vertex B_{10}.*

Proof. In order to give a parametrization we let $P = (u_4, u_5, u_6)^T$ be the point of a flag. We let further G be parallel to $g = (2u_1, 2u_2, 1 - u_1^2 - u_2^2)^T / M$, where $M = 1 + u_1^2 + u_2^2$. Obviously $\|g\| = 1$ and $\overline{g} = p \times g$.

Since g is an isotherm parametrization of the Euclidean unit sphere S^2, we have $\|g_{,1}\| = \|g_{,2}\|$ and $\{g, g_{,1}, g_{,2}\}$ form an orthogonal frame. Here $_{,j}$ denotes the partial derivative with respect to u_j.

The normal vector $\widehat{g}$ of the plane component E has to satisfy $\langle g, \widehat{g}\rangle = \langle g, g\rangle - \langle \widehat{g}, \widehat{g}\rangle = 0$. So we let $\widehat{g} = \cos\phi\; g_{,1}\|g_{,1}\|^{-1} + \sin\phi\; g_{,2}\|g_{,2}\|^{-1}$. Substituting $\cos\phi = (1-u_3^2)/N$, $\sin\phi = 2u_3/N$, where $N = 1+u_3^2$, and using the abbreviation $Z = 1-u_1^2-u_2^2$, we obtain the rational parametrization

$$
\begin{aligned}
&MN\mathcal{F}(u_1,u_2,u_3,u_4,u_5,u_6) = \\
&= \big[2u_1N, 2u_2N, NZ; N(u_5Z-2u_2u_6), N(2u_1u_6-u_4Z), \\
&\qquad 2N(u_2u_4-u_1u_5); \\
&\quad (1-u_3^2)(M-2u_1^2)-4u_1u_2u_3, 2u_3(M-2u_2^2)-2(1-u_3^2)u_1u_2, \\
&\quad -2u_1(1-u_3^2)-4u_2u_3; N(2u_1u_4+2u_2u_5+u_6Z)\big]^T.
\end{aligned} \tag{8}
$$

Since M, N, and Z are quadratic polynomials in the parameters u_i, the degree of the denominator MN of all coordinate functions equals four. The numerators of the fourth, fifth, and tenth coordinate function in (8) are polynomials of degree five.

Since x_{10} does show up in any of the equations defining M^6 it can be chosen independently and so M^6 is a cone with generators passing through B_{10}. $\Diamond$

Remark. The manifold M^6 is the intersection of three quadratic hypersurfaces in $\mathbb{P}^6$. According to the theorem by Bézout one could expect that the algebraic degree of M^6 equals eight.

On the other hand, there are four different vectors $(g, \overline{g}, \widehat{g}, \gamma)$, $(-g, -\overline{g}, \widehat{g}, -\gamma)$, $(-g, -\overline{g}, -\widehat{g}, -\gamma)$, and $(g, \overline{g}, -\widehat{g}, \gamma)$ corresponding to the four different orientations of a flag $\mathcal{F}$. So we could expect that the degree of M^6 is four. $\Diamond$

Remark. Eqs. (6) and (7) are the equations of two quadratic cones Γ_1 and Γ_2, respectively. The vertices of Γ_1 and Γ_2 are the three-dimensional projective subspaces $V_1 = [B_7, B_8, B_9, B_{10}]$ and $V_2 = [B_4, B_5, B_6, B_{10}]$, respectively. Here and in the following $[X_1, \ldots, X_n]$ denotes the projective subspace spanned by points $X_1, \ldots, X_n$. With (6) and (7) we see that $V_1 \subset \Gamma_2$ and $V_2 \subset \Gamma_1$.

The five-dimensional projective subspaces $W_1 : \widehat{x} = 0,\ x_{10} = 0$ $(x_7 = x_8 = x_9 - x_{10} = 0)$ and $W_2 : \overline{x} = 0,\ x_{10} = 0$ $(x_1 = x_2 = x_2 = x_{10} = 0)$ contain four-dimensional base-quadrics $Q_1 := \Gamma_1 \cap W_1$ and

$Q_2 := \Gamma_2 \cap W_2$ both being projectively equivalent to the Klein quadric.

Since the Klein quadric carries two three-parameter families of planes, the cones Γ_i carry two three-parameter families of six-dimensional projective subspaces comprising the set of generators.

The intersection of Γ_1 and Γ_2 splits into three-space $I = [B_1, B_2, B_3, B_{10}]$ of multiplicity two and a remaining quadratic surface. This can easily be verified by intersecting $\Gamma_1 \cap \Gamma_2$ with an arbitrary line, which neither is a part of Γ_1 nor of Γ_2.

Eq. (5) is the equation of quadratic cone Δ that is projectively equivalent to both Γ_1 and Γ_2. It shares the vertex V_2 with Γ_2. A base quadric of Δ is given by (5) together with equations $x_4 = x_5 = x_6 = x_{10} = 0$. ◊

3.1. The group of equiform motions in $\mathbb{R}^3$

In the following we want to describe the automorphism of the variety M^6 induced by the equiform motions in $\mathbb{R}^3$.

An equiform motion $\mu : \mathbb{R}^3 \to \mathbb{R}^3$ transforms points x in Euclidean space according to

$$x' = \alpha A x + a, \tag{9}$$

where $\alpha \in \mathbb{R} \setminus \{0\}$, $A \in \mathrm{SO}_3$ and $a \in \mathbb{R}^3$.

Lemma 3.1. *The seven parameter group of linear automorphisms (automorphic collineations) of M^6 induced by an equiform motion $\mu : \mathbb{R}^3 \to \mathbb{R}^3$ given by (9) transforms homogeneous flag coordinates $(g, \overline{g}, \widehat{g}, \gamma)$ according to*

$$\begin{bmatrix} g' \\ \overline{g}' \\ \widehat{g}' \\ \gamma' \end{bmatrix} = \alpha \begin{bmatrix} A & 0 & 0 & 0 \\ A^{\times} A & A & 0 & 0 \\ 0 & 0 & A & 0 \\ a^T A & 0^T & 0^T & \alpha \end{bmatrix} \begin{bmatrix} g \\ \overline{g} \\ \widehat{g} \\ \gamma \end{bmatrix}, \tag{10}$$

where $A^{\times}$ is the skew-symmetric matrix of the linear mapping $x \mapsto a \times x$.

Remark. Eq. (10) is nothing but a different representation of the group of equiform motions in Euclidean space $\mathbb{R}^3$. ◊

Proof. Applying an equiform motion to a flag $F = (P, G, E)$, the coordinates $(g, \overline{g}, \widehat{g}, \gamma)$ change according to $g' = \alpha A g$, $\widehat{g}' = \alpha A \widehat{g}$, $\gamma' = \langle p', g' \rangle = \langle \alpha A (g \times \overline{g} + \gamma g) + a, \alpha A g \rangle = \alpha a^T A g + \alpha^2 \gamma$, and $\overline{g}' = \alpha A \overline{g} + \alpha A^{\times} A g$. Using block matrix notation we find (10). ◊

Remark. The group of Euclidean motions is a subgroup of the group of equiform motions. Inserting $\alpha = 1$ into (10) we obtain the subgroup of automorphic collineations of M^6 induced by Euclidean motions. ◊

3.2. Flags sharing two components: pencils of flags

Now we are going to study certain submanifolds of the set of flags in $\mathbb{R}^3$ and ask for the corresponding point sets in the manifold M^6. According to [9] we define:

Definition 3.1. A *pencil* of flags is the set of flags sharing exactly two components.

Obviously there are three different types of pencils: the flags of a pencil differ in the point, line, or plane component. The respective pencils of flags will be denoted by $\mathcal{F}_{G,E}$, $\mathcal{F}_{P,E}$, or $\mathcal{F}_{P,G}$, where the subscripts point to the fixed elements (see Fig. 3).

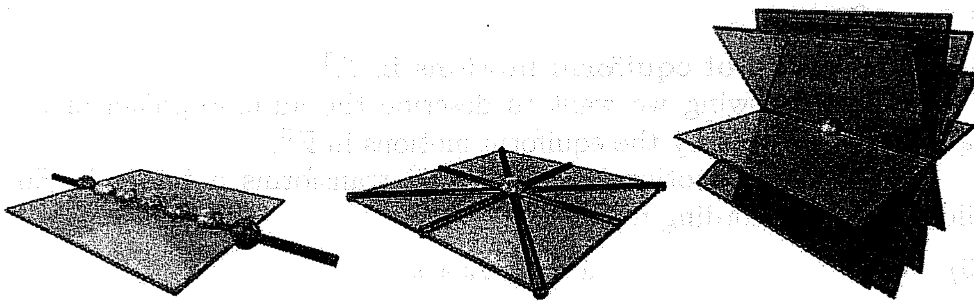

Figure 3. Pencils of flags: $\mathcal{F}_{G,E}$, $\mathcal{F}_{P,E}$, $\mathcal{F}_{P,G}$

From Th. 3.2 we know that M^6 is a cone and thus it contains at least one-dimensional subspaces (lines) of $\mathbb{P}^9$. The following result is elementary to verify:

Theorem 3.3. *The pencils of flags correspond to lines in* M^6.

Proof. It means no restriction to show Th. 3.3 for special pencils of flags. A Euclidean motion can be used to map the special pencil to any pencil of the same type.

We can assume that $\mathcal{F}_{G,E}$ is given by $G = (1,0,0;0,0,0)$ and $\widehat{g} = (0,0,1)$. Since $P \in G$ we have $P = (t_1/t_0, 0, 0)$ and $\mathcal{F}_{G,E} = = (t_0, 0, 0; 0, 0, 0; 0, 0, t_0; t_1)\mathbb{R}$, which is a parametrization of a line in M^6 depending on the homogeneous parameter $t_1 : t_0 \neq 0 : 0$. Note that it is an affine line, i.e. the point $\mathcal{F}_{G,E}(0:1) = B_{10}$ is missing, since it does not correspond to a flag in $\mathbb{R}^3$.

Let now $\mathcal{F}_{P,E}$ be given by $P = (0,0,0)$ and $E = (0,0,0,1)$. Then $g = (t_0/t_1, 1, 0)$ and we find $\mathcal{F}_{P,E} = (t_0, t_1, 0; 0, 0, 0; 0, 0, t_0; t_1)\mathbb{R}$, which is a line parametrized by a the homogeneous parameter $t_0 : t_1 \neq \neq 0 : 0$. Like in the previous case a point is missing. $\mathcal{F}(0:1) = = (0,1,0;0,0,0;0,0,0;1)\mathbb{R}$ does not define a flag in $\mathbb{R}^3$.

Finally we let $\mathcal{F}_{P,G}$ contain $P = (0,0,0)$ and $G = (1,0,0;0,0,0)$. With $\widehat{g} = (0, t_0/t_1, 1)$ we arrive at $\mathcal{F}_{P,G} = (t_0,0,0;0,0,0;0,t_1,t_0;0)\mathbb{R}$. As in the two above mentioned cases, this is an affine line in M^6. Like in the previously mentioned cases the point $\mathcal{F}_{P,G}(0:1) = B_8$ does not correspond to a flag in $\mathbb{R}^3$. $\Diamond$

3.3. Flags sharing one component: bundles of flags

Fixing exactly one component of a flag $\mathcal{F}$ we find three different sets of flags, not all of them corresponding to projective subspaces in M^6. With $\mathcal{F}_P$, $\mathcal{F}_G$, and $\mathcal{F}_E$ we denote the set of flags sharing the point P, the line G, and the plane E, respectively (see Fig. 4). As in the previous cases the subscripts indicate the shared component.

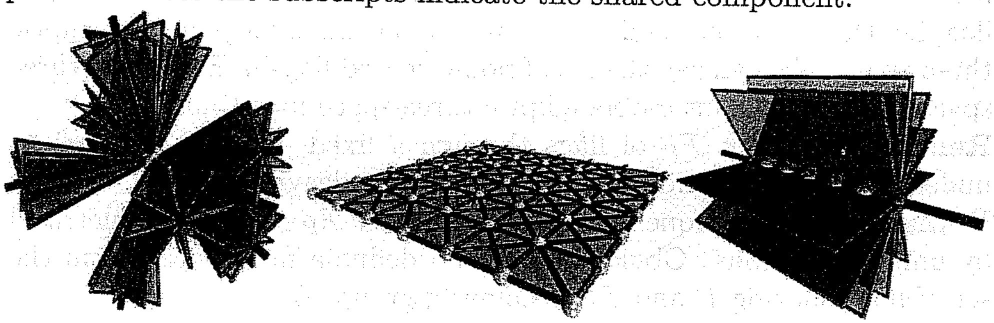

Figure 4. Flags sharing only one component: $\mathcal{F}_P$, $\mathcal{F}_E$, $\mathcal{F}_G$

3.3.1. Flags through a fixed point

Without loss of generality we can assume that the fixed point P coincides with the origin of the coordinate system. Thus we can parametrize $\mathcal{F}_P$ by $(g, 0, \widehat{g}, 0)$. Obviously this manifold of flags is three-dimensional and we have the following result:

Theorem 3.4. *There is a one-to-one correspondence between the set of oriented flags in Euclidean three-space sharing a fixed point and the group* SO_3. *The set of oriented flags through a fixed point forms a three-dimensional elliptic space.*

Proof. It means no restriction to assume that the fixed point is the origin $P = (0,0,0)^T$ of the underlying Cartesian coordinate system. Assume that $g = (-c_a s_b, -s_a s_b, c_b)^T$ and $\widehat{g} = (-c_a c_b s_c - s_a c_c, -s_a c_b s_c + c_a c_c, -s_b s_c)^T$, respectively, where $s_x = \sin x$ and $c_x = \cos x$. Note that this orients the flag. We attach a Cartesian coordinate system with each flag $\mathcal{F}_P$ through P: We let g and $\widehat{g}$ be the third and second basis vector, respectively. In order to form a right handed basis the first vector equals $\widehat{g} \times g$.

Therefore the matrix $A = [(\widehat{g} \times g)^T, \widehat{g}^T, g^T] \in \mathrm{SO}_3$ describes the uniquely determined rotation about the origin that moves the oriented flag $\mathcal{F}_P^0 = (0,0,1;0,0,0;0,1,0;0)$ to the oriented flag $\mathcal{F}_P$. The values a, b, c thus can be interpreted as the Euler angles of this rotation, see [2], [11], [12].

It is well known (see e.g. [1], [8], [13]) that the set of Euclidean rotations about a fixed point forms a three-dimensional elliptic space. ◊

As outlined earlier in this paper a flag in Euclidean space carries four differently oriented ones. Thus there are four different rotations about P transforming a certain oriented proto-flag $\mathcal{F}_P^0$ to a non-oriented flag $\mathcal{F}_P$. These rotations differ in 180°-rotations of the flags about their line component or the plane's normal. Therefore the geometric object flag, i.e. the non-oriented flag, belongs to four different points in elliptic three-space. We can say the set of non-oriented flags in Euclidean three space with fixed point covers elliptic three-space four times.

Remark. The set $\mathcal{F}_P$ of flags through a fixed point P is invariant under the three-parametric group of rotations leaving P fixed. ◊

Remark. As a consequence of Th. 3.4 the set $\mathcal{F}_P$ can be parametrized by unit quaternions. Obviously one can define a multiplication in the set of flags sharing P and $\mathcal{F}_P$ becomes a group. ◊

3.3.2. Flags with a common plane

The dual counter part (at least from the projective geometric point of view) of the set of flags $\mathcal{F}_P$ through a given point P is the set of flags $\mathcal{F}_E$ with a fixed plane E. It is easily shown that the following theorem holds:

Theorem 3.5. *There is a one-to-one correspondence between the set of oriented flags with a fixed plane E and the set of Euclidean motions in the plane E. The set of oriented flags with fixed plane component is a three-dimensional quasi-elliptic space.*

Proof. Without loss of generality we can assume E to be the plane $x_3 = 0 \subset \mathbb{R}^3$. The flags with common plane E only differ in their line elements and can thus be identified with them. So there exists a one-to-one correspondence between oriented line elements in E and oriented flags sharing E.

Let $\mathcal{F}_O = (O, X, E)$ with O being the origin and X being the first axis of the Cartesian coordinate system. Let $\mathcal{F}_P = (P, G, E)$ be the flag with line element (P, G). Then there exists a unique Euclidean motion $\mu: E \to E$ with $\mathcal{F}_O \mapsto \mathcal{F}_P$. It is well known, see [8], that the set of Euclidean motions of E forms a three-dimensional quasi-elliptic space. ◊

The set of non-oriented flags with a common plane covers the quasi-elliptic three-space twice, since both the line elements and the plane allow two different orientations without changing the geometric objects.

Remark. The set of flags with common plane is invariant under the three-parametric group of Euclidean motions in E. ◊

3.3.3. Flags with a common line

Finally we pay attention to the self-dual configuration (at least from the projective geometric point of view) of flags $\mathcal{F}_G$ sharing a line G. Obviously $\mathcal{F}_G$ is a two-dimensional manifold of flags and we have:

Theorem 3.6. *The set $\mathcal{F}_G$ of flags in Euclidean three-space sharing a line G corresponds to a plane in M^6.*

Proof. Without loss of generality we can assume $G = (1,0,0;0,0,0)$. We let $P = (1, t_1/t_0, 0, 0)$ and $E = (0,0,1,t_2/t_0)$. So the homogeneous coordinate vector of all flags in $\mathcal{F}_G$ is given by $\mathbb{P}^2 = (t_0, 0, 0; 0, 0, 0; 0, t_0, t_1; t_2)\mathbb{R}$, which obviously is a plane in M^6.

We can not allow $t_0 = 0$ otherwise we loose the line component in $\mathcal{F}_G$. Thus a line $\mathbb{P}^1$ in the image plane $\mathbb{P}^2$ of $\mathcal{F}_G$ is missing. Since the vector $\widehat{g}$ is prohibited to vanish, the point $t_0 : t_1 = 0 : 0$ on $\mathbb{P}^1$ is also not part of the image of $\mathcal{F}_G$. Consequently the image $\mathcal{F}_G$ is a plane minus a line element. ◊

4. Characterization of pairs of flags

In this section we show how to characterize pairs $(\mathcal{F}_1, \mathcal{F}_2)$ of flags $\mathcal{F}_1 = (P, G, E)$ and $\mathcal{F}_2 = (Q, H, F)$ can be characterized by means of their coordinates. The coordinatization of flags presented in [9] does not benefit this.

We discuss pairs of flags with respect to the incidence of points, lines, and plane components. We do not deal with orthogonality and parallelity of components in order to avoid lengthy discussions.

Again we use normalized coordinate vectors, i.e. $\|g\| = \|\widehat{g}\| = = \|h\| = \|\widehat{h}\| = 1$.

We do not allow point components to coincide with the origin of the coordinate system, and we also do not allow lines and planes to pass through the origin. Otherwise the number of subcases would grow rapidly while the number of different pairs of flags would not. This can be avoided by choosing appropriate coordinate systems.

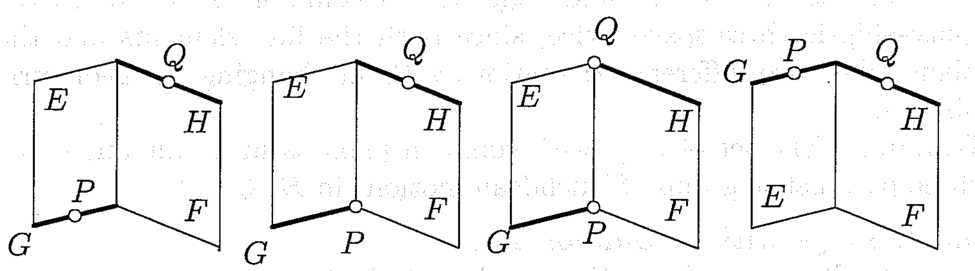

Figure 5. Cases 1–4: 1. $\mathcal{F}_1 \neq \mathcal{F}_2$, 2. $P \in F$, 3. $P \in F$, $Q \in E$, 4. $G \cap H \neq \emptyset$

Case 1. The flags $\mathcal{F}_1$ and $\mathcal{F}_2$ differ in each component and there is no remarkable incidence relation except the trivial ones $P \in G \subset E$ and $Q \in H \subset F$. There is nothing to characterize.
Case 2. $P \in F$ is obviously characterized by $-\det(h, \overline{h}, \widehat{h}) + \langle p, \widehat{h}\rangle = 0$. Since $p = g \times \overline{g} + \gamma g$, we have

(11) $$\langle g \times \overline{g} - h \times h + \gamma g, \widehat{h}\rangle = 0.$$

Case 3. In case of $P \in F$ and $Q \in E$ Eq. (11) is fulfilled. According to Case 2 we have the additional relation

(12) $$\langle h \times \overline{h} - g \times g + \eta h, \widehat{g}\rangle = 0.$$

Case 4. The pair of flags with $G \cap H \neq \emptyset$ is easily characterized by characterizing intersecting lines G and H, respectively. This is done with the well-known formula (see [15], [17])

(13) $$\langle g, \overline{h}\rangle + \langle \overline{g}, h\rangle = 0.$$

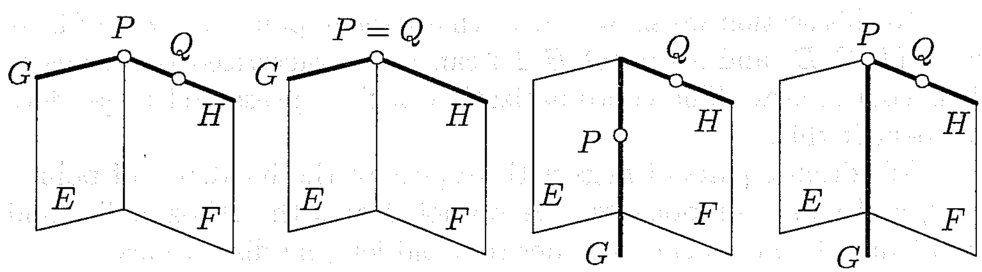

Figure 6. Cases 5–8: 5. $P \in H$, 6. $P = Q$, 7. $P \in F$, $G \cap H \neq \emptyset$, 8. $E \cap F = G, P \in H$

Case 5. Consider the case where G and H intersect in P. From the previous case we have the first condition (13) for the coordinates of

$\mathcal{F}_1$ and $\mathcal{F}_2$, respectively. Since $P \in H$ we have $p = g \times \overline{g} + \gamma g = \overline{g} \times \overline{h}/\langle h, \overline{g}\rangle$. Here we used a formula for the intersection point of intersecting lines G and H. This formula can be found in [15]. (It is only valid in those cases where G does not pass through the origin of the coordinate system.) Multiplying the latter equation by g we find the remaining characterization as

$$\gamma\langle h, \overline{g}\rangle = \det(g, \overline{g}, \overline{h}). \tag{14}$$

Case 6. Now we consider the case $G \cap H = P = Q$. Obviously (13) is valid since the intersection point of G and H exists. With Case 5 we have (14) and

$$\eta\langle \overline{h}, g\rangle = \det(h, \overline{h}, \overline{g}). \tag{15}$$

Case 7. Let now $E \cap F = G$. Consequently $G \cap H \neq \emptyset$ and thus (13) is fulfilled. Since G is contained in F we have

$$\langle g, \widehat{h}\rangle = 0 \text{ and } \det(h, \overline{h}, \widehat{h})g + h \times \overline{g} = 0, \tag{16}$$

because there is no unique intersection point of F and G.

Case 8. Now we consider $E \cap F = G$ and $G \cap H = P$. Since $G \cap H \neq \emptyset$ (13) is valid. As indicated in Case 5 we additionally have (14). Since G is contained in F, the point P is too. So (11) is valid.

The condition for $G \subset F$ is given by (16) and the condition for $H \subset E$ reads

$$\langle h, \widehat{g}\rangle = 0 \text{ and } \det(g, \overline{g}, \widehat{g}) + \widehat{g} \times h = 0. \tag{17}$$

The Plücker coordinate vector of $E \cap F$ and the coordinate vector $(g, \overline{g})$ of G are a multiple of each other. This can easily be verified with the above relations.

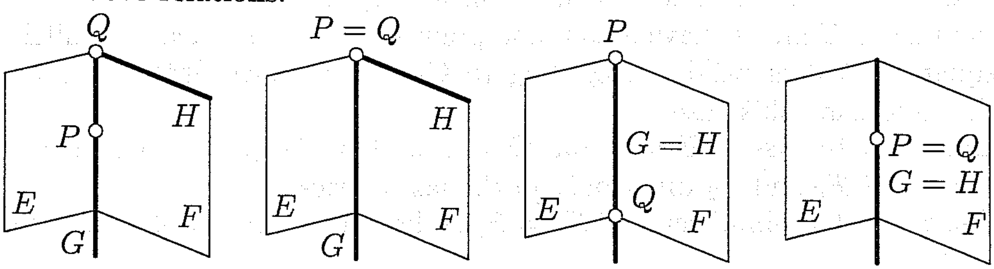

Figure 7. Cases 9–12: 9. $Q \in G = E \cap F$, 10. $E \cap F = G, P = Q$, 11. $G = H$, 12. $G = H, P = Q$

Case 9. Now we consider $E \cap F = G$ and $G \cap H = Q$. Obviously (13) and (15) are valid. For $G \subset F$ and $H \subset E$ the respective conditions (16) and (17) are fulfilled.

Case 10. Now we let $E \cap F = G$ and $P = Q$. This case is characterized by (13), (16), (14), and (15) since $P = Q$.
Case 11. The case $G = H$ is easily characterized by $(g, \overline{g}) = (h, \overline{h})$.
Case 12. If $G = H$ and $P = Q$ we have

$$(g, \overline{g}, \gamma) = (h, \overline{h}, \eta). \tag{18}$$

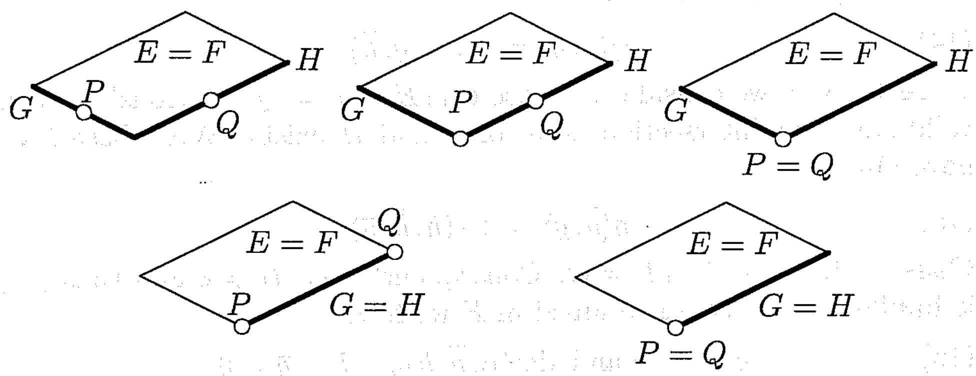

Figure 8. Cases 13–17: 13. $E = F$, 14. $G \cap H = P$, 15. $P = Q$, 16. $G = H$, 17. $\mathcal{F}_1 = \mathcal{F}_2$

Case 13. The case $E = F$ of coinciding planes appears if and only if

$$(-\det(g, \overline{g}, \widehat{h}), \widehat{g}) = (-\det(h, \overline{h}, \widehat{h}), \widehat{h}). \tag{19}$$

In this case the lines G and H have a common point or are at least parallel. So (13) is also valid.
Case 14. If now $G \cap H = P$ we have (14) besides the equations of the previous case, i.e. (13) and (19).
Case 15. If $E = F$ and $P = Q$ we have (19). As outlined in Case 13 the planes E and F have a common point or they are at least parallel, equation (13) is valid. According to Case 5 we have (14) and (15) characterizing this case.
Case 16. In case of $E = F$ and $G = H$ we have (18). The coordinate vectors of $\mathcal{F}_1$ and $\mathcal{F}_2$ differ only in the last entries.
Case 17. The simple case of $\mathcal{F}_1 = \mathcal{F}_2$ is detected immediately, because we have

$$(g, \overline{g}, \widehat{g}, \gamma) = (h, \overline{h}, \widehat{h}, \eta).$$

Figures 5, 6, 7, and 8 are given in order to help the reader. These are only incidence tables.
Remark. The above discussion could easily be extended. Pairs of flags can not only be characterized with respect to incidence relations.

They could also be characterized by with respect to orthogonality and parallelity of components. ◊

Remark. In a similar manner to the definitions in [9] we can call two flags $\mathcal{F}_1$, and $\mathcal{F}_2$ *related*, if they differ in exactly two components. We call two flags *adjacent*, if they differ in exactly one component.

With the discussions above we can give analytic characterizations of related and adjacent flags. The related ones appear in Cases 6, 10, 11, 13, 14. Adjacent pairs of flags are described in Cases 12, 15, 16. ◊

5. Flags and the group of Euclidean motions

Consider a flag $\mathcal{F} = (P, G, E)$ in Euclidean three-space $\mathbb{R}^3$. The coordinates $(g, \overline{g}, \widehat{g}, \gamma)$ of $\mathcal{F}$ may be taken with respect to the coordinate system $\Sigma_0 := \{(0,0,0)^T; e_1, e_2, e_3\}$ with e_i being the canonical basis of $\mathbb{R}^3$. In the following the vectors g and $\widehat{g}$ may be unit vectors.

Now we attach a Cartesian coordinate system Σ with $\mathcal{F}$. We let P be the origin of Σ, the first axis points in the direction of G, and the second axis shall coincide with the plane's normal. Thus the third axis is uniquely determined if we want Σ to be right handed and $\mathcal{F}$ becomes oriented.

Obviously $\mathcal{F}$ can be uniquely represented by the position of Σ relatively to Σ_0. It is well known (see e.g. [2], [11], [12]) that there exists a uniquely determined Euclidean motion $\mu : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ that transforms Σ_0 to Σ. Thus we can say:

Theorem 5.1. *There is a one-to-one correspondence between the set of oriented flags in Euclidean three-space $\mathbb{R}^3$ and the set of Euclidean motions in $\mathbb{R}^3$. The group of Euclidean motions considered as a differentiable manifold is a point model for the set of flags in $\mathbb{R}^3$.*

Again we recall that the geometric object flag carries four orientations which only differ by certain rotations. So there are four Euclidean motions moving a certain proto-flag to a given non-oriented flag.

Remark. The matrix A, the vector a, and α from (9) determining the equiform (indeed Euclidean) motion moving Σ_0 to Σ can be expressed in terms of the coordinates $(g, \overline{g}, \widehat{g}, \gamma)$ of the flag $\mathcal{F}$ associated with Σ: $\alpha = 1$, $A = [(\widehat{g} \times g)^T, \widehat{g}^T, g^T]$ and $a = g \times \overline{g} + \gamma g$. ◊

Furthermore it is well known that Euclidean motions can be represented by normed biquaternions (see e.g. [2], [11], [12]). The identification of Euclidean motions with normed biquaternions performs a mapping of Euclidean motions to points of the well known Study quadric S_2^6 [16], [17], which is a point model of the set of Euclidean motions. With Th. 5.1 we have:

Theorem 5.2. *There is a one-to-one correspondence between the set of oriented flags in Euclidean three-space and the points of the Study quadric. The Study quadric S_2^6 can serve as a point model of the set of flags in Euclidean $\mathbb{R}^3$.*

The four different orientations of a flag lead to four different Euclidean motions and thus to four different points in the study quadric all of them belonging to the same geometric object: the naked flag without orientations.

Remark. By identification of flags with Euclidean motions we solve the embedding problem of the set of flags in Euclidean three-space $\mathbb{R}^3$. Obviously we find a hypersurface (quadric) $S_2^6 \subset \mathbb{P}^7$ being a point model for the set of flags in $\mathbb{R}^3$. $\Diamond$

Remark. The coordinatization of the set of flags by means of biquaternions is not very useful in practical applications and computations. Th. 5.2 only answers the question whether it is possible to embed the flag manifold of Euclidean $\mathbb{R}^3$ into a low dimensional space or not, and how low can this dimension be. $\Diamond$

6. Conclusion and future research

We defined coordinates of flags in Euclidean space $\mathbb{R}^3$ by means of Plücker coordinates of lines and line elements. These coordinates can be used to characterize pairs of flags and decide whether these are related or adjacent. The characterization can be extended to parallel and even orthogonal elements. This extension seems to be not very complicated but long winded and uninteresting to the author.

The homogeneity of coordinates of flags leads to the interpretation of coordinates for points in a projective space. The manifold M^6 is defined by the obvious constraints to the coordinates of a flag. This model has some advantages. One is the very low dimension of the embedding space compared to the model used in [9]. The other advantage is the possibility to characterize flags which (from the incidence geometric point of view) seems to be useful. Even the group of equiform transformation admits a simple description in the presented model (cf. Lemma 3.1).

We will not keep in secret that the presented model has disadvantages. The coordinates of flags defined in Def. 2.1 can not be extended to flags containing points, lines, or planes at infinity. Thus the projective closure as performed for the Klein model of line space and the resulting closure of the Klein quadric by adding a plane to it, does nor work for M^6. The subspaces which are missing here can not be added to M^6 such that the points therein represent flags in projectively extended Euclidean three-space.

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