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rg-COMPACT SPACES AND *rg*-CONNECTED SPACES

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Abstract: The aim of this paper is to study rg-continuous map and rg-irresolute map and to introduce and study rg-compact spaces and rg-connected spaces.

1. Introduction

In 1993, N. Palaniappan and K. Chandrasekhara [3], introduced the concept of regular generalized closed (briefly, rg-closed) sets and regular generalized open (briefly, rg-open) sets in a topological space. They also defined regular generalized continuous (briefly, rg-continuous) map and regular generalized irresolute (briefly, rg-irresolute) map between topological spaces and studied some of their properties.

The purpose of this paper is to study these mappings and to define and study the concept of regular generalized compact spaces and regular generalized connected spaces.

Throughout this paper, spaces X, Y and Z mean topological spaces $(X,\tau), (Y,\sigma)$ and (Z,γ), respectively. For a subset A of X, the closure, the interior and the complement of A are denoted by $\mathrm{cl}(A), \mathrm{int}(A)$ and A^C, respectively.

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2. Definitions and basic properties

Definition 2.1. (1) A set A is said to be regular open (briefly, r-open) (resp. regular closed (briefly, r-closed)) if $A = \text{int}(\text{cl}(A))$ (resp. $A = = \text{cl}(\text{int}(A))$). The family of r-open (resp. r-closed) sets of a space X is denoted by $RO(X)$ (resp. $RC(X)$) [4].

(2) A set A is said to be semi-open (briefly, s-open) (resp. semi-closed (briefly, s-closed)), if $A \subset \text{cl}(\text{int}(A))$ (resp. $\text{int}(\text{cl}(A)) \subset A$) [2].

(3) A set A is said to be rg-closed if $\text{cl}(A) \subset U$ whenever $A \subset \subset U$, where U is r-open. It is said to be rg-open if A^C is rg-closed (equivalently $F \subset \text{int}(A)$ whenever $F \subset A$ and F is r-closed) [3].

Remark 2.2. In [3], Palaniappan and Chandrasekhara proved that the union of two rg-closed sets is rg-closed, however the intersection of two rg-closed sets is not rg-closed as seen in the following example.

Example 2.3. Let $X = \{a, b, c, d\}, \tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, X\}$. Take $A = \{a, c\}, B = \{a, b\}$,then A, B are rg-closed but $A \cap B = \{a\}$ is not rg-closed since $\{a\} \subset \{a\}$ which is r-open and $\text{cl}\{a\} = \{a, c, d\} \not\subseteq \{a\}$.

Definition 2.4. The intersection of all rg-closed sets containing a set A is called the regular generalized closure of A and is denoted by rg-$\text{cl}(A)$. If A is rg-closed, then rg-$\text{cl}(A) = A$. The converse is not true, since the intersection of rg-closed sets need not be rg-closed.

Lemma 2.5. *If $A \subset X$, then $A \subset rg\text{-cl}(A) \subset \text{cl}(A)$.*

Proof. Obvious, since a closed set is rg-closed. ◊

Lemma 2.6. *If $A \subset B$, then $rg\text{-cl}(A) \subset rg\text{-cl}(B)$.*

Proof. Obvious. ◊

Theorem 2.7. *If A is rg-closed and $A \subset B \subset \text{cl}(A)$, then B is rg-closed.*

Proof. Let A be rg-closed and $A \subset B$. Let U be r-open set containing B, then $A \subset U$, so $\text{cl}(A) \subset U$But since $B \subset \text{cl}(A)$, so $\text{cl}(B) \subset \text{cl}(A) \subset \subset U$. Hence, B is rg-closed. ◊

3. Regular generalized continuous map

Definition 3.1. A map $f : X \to Y$ is called regular generalized continuous (briefly, rg-continuous) if $f^{-1}(V)$ is rg-closed in X for every closed set V of Y. f is called regular-continuous (briefly, r-continuous) if $f^{-1}(V)$ is r-closed in X whenever V is closed in Y ([3]).

Theorem 3.2. *Let $f : X \to Y$ be a map, then the following statements are equivalent:*

(1) *f is rg-continuous.*

(2) *The inverse image of each open set V in Y is rg-open in X.*

Proof. Obvious. ◊

Theorem 3.3. *Let $f : X \to Y$ be rg-continuous map, then $f(rg\text{-cl}(A)) \subset \subset \text{cl}(f(A))$ for every $A \subset X$.*

Proof. Let $A \subset X$ and f be rg-continuous map. Since $\text{cl}(f(A))$ is a closed subset in Y, therefore $f^{-1}(\text{cl}(f(A)))$ is rg-closed in X, but $A \subset \subset f^{-1}(f(A)) \subset f^{-1}(\text{cl}(f(A)))$. Hence $f^{-1}(\text{cl}(f(A)))$ is rg-closed subset of X containing A, therefore $rg\text{-cl}(A) \subset f^{-1}(\text{cl}(f(A))$, so $f(rg\text{-cl}(A)) \subset \subset \text{cl}(f(A))$. ◊

Theorem 3.4. *Let $f : X \to Y$ be a map, then the following statements are equivalent:*

(1) *For each $x \in X$ and each open set V in Y with $f(x) \in V$, there exists an rg-open set U in X such that $x \in U, f(U) \subset V$.*

(2) *For every subset A of X, $f(rg\text{-cl}(A)) \subset \text{cl}(f(A))$.*

(3) *For every subset B of Y, $rg\text{-cl}(f^{-1}(B)) \subset f^{-1}(\text{cl}(B))$.*

Proof. (1)⇒(2). Let $A \subset X$, $x \in A$ and $y = f(x) \in f(rg\text{-cl}(A))$. Let V be an open set containing y. From (1), there exists an rg-open set U in X such that $x \in U, f(U) \subset V$. Since $x \in rg\text{-cl}(A)$, therefore $U \cap A \neq \phi$. Hence, $f(U) \cap f(A) \neq \phi$. So $V \cap f(A) \neq \phi$ and $y \in \text{cl}(f(A))$. Therefore (2) holds.

(2)⇒(1). Let $x \in X, V$ be an open set containing $f(x)$. Let $A = = f^{-1}(V^C)$, then $x \notin A$. From (2), $f(rg\text{-cl}(A)) \subset \text{cl}(f(A)) \subset V^C$. So $rg\text{-cl}(A) \subset f^{-1}(\text{cl}(f(A))) \subset f^{-1}(V^C) = A$. Therefore $A = rg\text{-cl}(A)$. Then $x \notin rg\text{-cl}(A)$ and so there exists an rg-open set U containing x such that $U \cap A = \phi$. Hence, $f(U) \subset f(A^C) \subset V$.

(2)⇒(3). Let $B \subset Y$, put $A = f^{-1}(B)$. From (2), we have $f(rg\text{-cl}(f^{-1}(B))) \subset \text{cl}(f(f^{-1}(B))) \subset \text{cl}(B)$. Hence $rg\text{-cl}(f^{-1}(B)) \subset \subset f^{-1}(\text{cl}(B))$.

(3)⇒(2). Let $A \subset X$ and $B = f(A)$. From (3), $rg\text{-cl}(A) \subset \subset rg\text{-cl}(f^{-1}(B)) \subset f^{-1}(\text{cl}(f(A)))$. So $f(rg\text{-cl}(A)) \subset \text{cl}(f(A))$. This completes the proof. ◊

Remark 3.5. It is obvious that if f is continuous, then it is rg-continuous but the converse is not true as seen in the following example:

Example 3.6. Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, X, \{a, b\}\}$, $\sigma = = \{\phi, Y, \{a, b\}, \{c\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be defined by $f(a) = c$,

$f(b) = b$, $f(c) = a$. Then f is rg-continuous but it is not continuous since $f^{-1}(\{c\}) = \{a\}$ is not closed in X.

Remark 3.7. It is proved in [3] that the composition of two rg-continuous maps need not be rg-continuous.

Theorem 3.8. *Let $f : X \to Y$ be rg-continuous and $g : Y \to Z$ be continuous, then $g \circ f : X \to Z$ is rg-continuous.*

Proof. Let V be closed set in Z, then $g^{-1}(V)$ is closed in Y. But since f is rg-continuous then $f^{-1}(g^{-1}(V))$ is rg-closed in X. Therefore, $g \circ f$ is rg-continuous. ◊

Definition 3.9. A map $f : X \to Y$ is called rg-irresolute if $f^{-1}(V)$ is rg-closed in X for every rg-closed set V of Y [3]. It is called irresolute if $f^{-1}(V)$ is s-open in Y for every s-open set V of Y ([1]).

Theorem 3.10. *A map $f : X \to Y$ is rg-irresolute if for every rg-open set A of Y, $f^{-1}(A)$ is rg-open set in X.*

Proof. Obvious. ◊

Remark 3.11. It is obvious that if $f : X \to Y$ is rg-irresolute, then it is rg-continuous but the converse is not true as seen in the following example:

Example 3.12. Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$, $\sigma = \{\phi, \{a\}, Y\}$. Define $f : (X, \tau) \to (Y, \sigma)$ by: $f(a) = b$, $f(b) = f(c) = c$, then f is rg-continuous but it is not rg-irresolute since $\{b\}$ is rg-closed in Y but $f^{-1}(\{b\}) = \{a\}$ is not rg-closed in X.

Theorem 3.13. *Let $f : X \to Y$, $G : Y \to Z$ be rg-irresolute mappings, then $g \circ f : X \to Z$ is rg-irresolute map.*

Proof. Let V be an rg-closed set in Z, then $g^{-1}(V)$ is an rg-closed set in Y. But since f is rg-irresolute, therefore $f^{-1}(g^{-1}(V))$ is an rg-closed set in X. So we have that $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is an rg-closed set in X. Therefore $g \circ f$ is rg-irresolute. ◊

Theorem 3.14. *If $f : X \to Y$ is rg-irresolute mapping, then for every $A \subset X$, $f(rg\text{-cl}(A)) \subset rg\text{-cl}(f(A))$.*

Proof. Let $A \subset X$, then rg-cl$(f(A))$ is rg-closed in Y, but f is rg-irresolute, so $f^{-1}(\text{-cl}(f(A)))$ is rg-closed in X. Also, $A \subset f^{-1}(f(A)) \subset f^{-1}(rg\text{-cl}(f(A)))$. So, $rg\text{-cl}(A) \subset rg\text{-cl}(f^{-1}(rg\text{-cl}(f(A))) = f^{-1}(rg\text{-cl}(f(A)))$. So $f(rg\text{-cl}(A)) \subset rg\text{-cl}(f(A))$. ◊

Remark 3.15. Irresolute maps and rg-irresolute maps are independent of each other as shown in the following example.

Example 3.16. (1) Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$, $\sigma = \{\phi, \{a\}, \{b, c\}, Y\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity map, then f is irresolute but not rg-irresolute since $f^{-1}(\{a\}) = \{a\}$ is not rg-closed in X.

(2) Let $X = \{a,b,c,d\}$, $\tau = \{\phi, \{c,d\}, X\}, Y = \{p,q\}$, $\sigma = \{\phi, \{p\}, Y\}$. Let $f : (X,\tau) \to (Y,\sigma)$ defined by $f(a) = f(b) = f(d) = p$, $f(c) = q$. Then f is rg-irresolute but it is not irresolute since $\{p\}$ is s-closed in Y but $f^{-1}(\{p\}) = \{a,b,d\}$ is not s-closed in X.

4. *rg*-compact spaces

Definition 4.1. (1) A collection $\{A_\alpha : \alpha \in \nabla\}$ of rg-open sets in a topological space X is called rg-open cover of a subset B of X if $B \subset \cup\{A_\alpha : \alpha \in \nabla\}$ holds.

(2) A topological space X is called regular generalized compact (briefly, rg-compact) if every rg-open cover of X has a finite subcover.

(3) A subset B of X is called rg-compact relative to X if for every collection $\{A_\alpha : \alpha \in \nabla\}$ of rg-open subsets of X such that $B \subset \cup\{A_\alpha : \alpha \in \nabla\}$, there exist a finite subset $\nabla_\circ$ of ∇ such that $B \subset \cup\{A_\alpha : \alpha \in \nabla_\circ\}$.

(4) A subset B of X is said to be rg-compact if B is rg-compact as a subspace of X.

Theorem 4.2. *Every rg-closed subset of rg-compact space X is rg-compact relative to X.*

Proof. Let A be rg-closed subset of X, then A^C is rg-open. Let $O = \{G_\alpha : \alpha \in \nabla\}$ be a cover of A by rg-open subsets of X. Then $W = O \cup A^C$ is an rg-open cover of X, i.e., $X = (\cup\{G_\alpha : \alpha \in \nabla\}) \cup A^C$. By hypothesis, X is rg-compact. Hence W has a finite subcover of X say $(G_1 \cup G_2 \cup \cdots \cup G_n) \cup A^C$. But A and A^C are disjoint, hence $A \subset G_1 \cup G_2 \cup \cdots \cup G_n$. So O contains a finite subcover for A, therefore A is rg-compact relative to X. $\Diamond$

Theorem 4.3. *Let $f : X \to Y$ be a map:*

(1) *If X is rg-compact and f is rg-continuous bijective, then Y is compact.*

(2) *If f is rg-irresolute, and B is rg-compact relative to X, then $f(B)$ is rg-compact relative to Y.*

(3) *If X is compact and f is continuouse surjective, then Y is rg-compact.*

Proof. (1) Let $f : X \to Y$ be an rg-continuous bijective map, and X be an rg-compact space. Let $\{A_\alpha : \alpha \in \nabla\}$ be open cover for Y, then

$\{f^{-1}(A_\alpha) : \alpha \in \nabla\}$ is an rg-open cover of X. Since X is rg-compact, it has a finite subcover say $\{f^{-1}(A_1), f^{-1}(A_2), ldots, f^{-1}(A_n)\}$ but f is surjective, so $\{A_1, A_2, \ldots, A_n\}$ is a finite subcover of Y. Therefore, Y is compact.

(2) Let $B \subset X$ be rg-compact relative to X, $\{A_\alpha : \alpha \in \nabla\}$ be any collection of rg-open subsets of Y such that $f(B) \subset \cup\{A_\alpha : \alpha \in \nabla\}$. Then $B \subset \cup\{f^{-1}(A_\alpha) : \alpha \in \nabla\}$. By hypothesis, there exist a finite subset $\nabla_\circ$ of ∇ such that $B \subset \cup\{f^{-1}(A_\alpha) : \alpha \in \nabla_\circ\}$. Therefore, we have $f(B) \subset \cup\{A_\alpha : \alpha \in \nabla_\circ\}$ which shows that $f(B)$ is rg-compact relative to Y.

(3) Let $A = \{A_\alpha : \alpha \in \nabla\}$ be an rg-open cover of Y. Since f is continuouse, therefore $f^{-1}(A_\alpha)$ is open in X. The collection $W = \{f^{-1}(A_\alpha) : \alpha \in \nabla\}$ is an open cover of X. Since X is compact, W has a finite subset say $\{f^{-1}(A_1), f^{-1}(A_2), \ldots, f^{-1}(A_n)\}$ which cover X. Since $X = f(A_1) \cup f(A_2) \cup \cdots \cup f(A_n)$ and f is surjective, therefore $Y = f(X) = f(f^{-1}(A_1) \cup f^{-1}(A_2) \cup \cdots \cup f^{-1}(A_n)) =$ $= f(f^{-1}(A_1)) \cup f(f^{-1}(A_2)) \cup \cdots \cup f(f^{-1}(A_n)) \subset A_1 \cup A_2 \cup \cdots \cup A_n$. Thus $\{A_1, A_2, \ldots, A_n\}$ is a finite rg-open subcover of Y, and Y is rg-compact. $\Diamond$

5. *rg*-connected spaces

Definition 5.1. A space X is said to be regular generalized connected (briefly, rg-connected) if it can not be written as a disjoint union of two non empty rg-open sets, otherwise it said to be rg-disconnected. A subset of X is said to be rg-connected if it is rg-connected as a subspace of X.

Theorem 5.2. *For a space X, the following are equivalent:*

(1) *X is rg-connected.*

(2) *X and ϕ are the only subsets of X which are both rg-open and rg-closed.*

(3) *Each rg-continuous map of X into some discrete space Y with at least two points is a constant map.*

Proof. (1)$\Rightarrow$(2). Let X be rg-connected, and A be an rg-open and rg-closed subset of X. So A^C is both an rg-open and an rg-closed subset of X. Since X is the disjoint union of the rg-open sets A and A^C, one of these must be empty, that is, $A = \phi$, or $A = X$.

(2)$\Rightarrow$(1). Let X be rg-disconnected, i.e., $X = A \cup B$ where A and B are disjoint non-empty rg-open subsets of X. Then A is both

rg-open and rg-closed. By assumption $A = \phi$ or $A = X$, therefore X is rg-connected.

(2)⇒(3). Let $f : X \to Y$ be rg-continuous map from X into the discrete space Y with at least two points, then $\{f^{-1}(y) : y \in Y\}$ is a covered of X by rg-open and rg-closed sets. By assumption, $f^{-1}(y) = = \phi$ or X for each $y \in Y$. If $f^{-1}(y) = \phi$ for all $y \in Y$, then f is not a map. So there exist s exactly one point $y \in Y$ such that $f^{-1}(y) \neq \phi$ and hence $f^{-1}(y) = X$. This shows that f is a constant map.

(3)⇒(2). Let $O \neq \phi$ be both an rg-open and an rg-closed subset of X. Let $f : X \to Y$ be rg-continuous map defined by $f(O) = \{y\}$ and $f(O^c) = \{w\}$ for some distinct points y and w in Y. By assumption f is constant, therefore $0 = X$. ◊

Remark 5.3. It is obvious that every rg-connected space is connected but the converse is not true as seen in the following example:

Example 5.4. Let $X = \{a, b, c\}, \tau = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$. X is connected but not rg-connected since $\{a, c\}, \{b\}$ are disjoint rg-open sets and $X = \{a, c\} \cup \{b\}$.

Theorem 5.5. *Let $f : X \to Y$ be a map:*

(1) *If X is rg-connected and f is rg-continuous surjective, then Y is connected.*

(2) *If X is rg-connected and f is rg-irresolute surjective, then Y is rg-connected.*

(3) *If X is connected and f is continuouse surjective, then Y is rg-connected.*

Proof. (1) Let Y be disconnected, then $Y = A \cup B$, where A and B are disjoint non-empty open subsets of Y. Since f is rg-continuous surjective, therefore $X = f^{-1}(A) \cup f^{-1}(B)$ where $f^{-1}(A), f^{-1}(B)$ are disjoint non-empty rg-open subsets of X. This contradicts the fact that X is rg-connected. Hence, Y is connected.

(2) Suppose that Y is not rg-connected. Put $Y = A \cup B$, where A, B are disjoint non-empty rg-open subsets of Y. Since f is rg-irresolute surjective, therefore $X = f^{-1}(A) \cup f^{-1}(B)$, where $f^{-1}(A)$, $f^{-1}(B)$ are disjoint non-empty rg-open subsets of X. So X is not rg-connected, a contradiction.

(3) Suppose that Y is not rg-connected. Let $Y = A \cup B$, where A, B are disjoint non-empty rg-open subsets of Y, so they are open. Since f is continuose surjective, $f^{-1}(A), f^{-1}(B)$ are disjoint open subsets of X and $X = f^{-1}(A) \cup f^{-1}(B)$ contradict the fact that X is connected, therefore Y is rg-connected. ◊

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