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SKEW-ORTHOGONAL POLYNOMIALS OF A DISCRETE VARIABLE

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Abstract: In this paper we present skew-orthogonal polynomials of a discrete variable. A skew-orthogonality property with respect to the powers $\{n^i\}_{i\geq 0}$ of a discrete variable n will play a very important role in our construction. This allows us to use the classic Charlier and Meixner polynomials.

1. Introduction

Classical orthogonal polynomials of a discrete variable have been extensively used in some problems of mathematics, physics, computational mathematics and engineering. The Hahn, Meixner, Kravchuk and Charlier polynomials are the principal examples of these special functions. It is very important to note that all these polynomials are solutions of some difference equations of hypergeometric type.

On the other hand the skew-orthogonal polynomials were introduced in 1970 by Dyson in his phenomenal work about correlations between the eigenvalues of a random matrix and later developed by Mehta (see [8]) and others. It should be observed that there is a deep connection between skew-orthogonal polynomials and the theory of integrable systems. This relation was noted by Adler, Horozov and Van Moerbeke in [1]. The first author has also presented a simple setting for the skew-orthogonal polynomials which emphasizes the appearance of non-uniqueness in the construction of these polynomials (see [6]).

To the best of our knowledge, skew-orthogonal functions of a discrete variable have not been studied. The principal objective of this paper is the construction of skew-orthogonal polynomials of a discrete variable.

Let F be an infinite dimensional real vector space. Suppose that over F is defined a bilinear form $\Omega(.,.)$ such that $\Omega(x,y) = -\Omega(y,x)$. In this case $\Omega(.,.)$ is called a skew-bilinear form.

Definition 1. Let $\Omega(.,.)$ be a skew-bilinear form over an infinite dimensional real vector space F. We say that the sequence $\{u_n\}_{n\geq 0} \subset F$ of vectors is *skew-orthogonal* if:

a) $\Omega(u_{2n}, u_{2m}) = 0 = \Omega(u_{2n+1}, u_{2m+1})$ for $n, m = 0, 1, \ldots,$

b) $\Omega(u_{2n}, u_{2m+1}) = \gamma_n \delta_{n,m}$ for $n, m = 0, 1, \ldots,$ where $\{\gamma_n\}_{n\geq 0}$ is a sequence of real non-zero numbers.

Throughout the paper we use the following remark which is evident but will be very important for us.

Remark 2. Let $\{e_0, e_1, \ldots\}$ be a linearly independent system of vectors belonging to an infinite dimensional real vector space F over which is defined a skew-bilinear form $\Omega(.,.)$. Let us assume that $\{\gamma_n\}_{n\geq 0}$ is a sequence of real non-zero numbers. Then, if the system $\{u_n\}_{n\geq 0}$ defined in the form $u_0 = e_0$ and for any n, $u_n = c_0^n e_0 + \cdots c_{n-1}^n e_{n-1} + e_n$ where $c_i^n \in \mathbb{R}$ for $i = 0, \ldots, n$ satisfies the following relations

$$\Omega(u_{2n}, u_{2n+1}) = \gamma_n,$$
$$\Omega(u_{2n}, e_m) = 0, \quad \text{for} \quad m = 0, \ldots, 2n-1,$$
$$\Omega(u_{2n+1}, e_m) = 0, \quad \text{for} \quad m = 0, \ldots, 2n-1,$$

then the system $\{u_n\}_{n\geq 0}$ is skew-orthogonal.

We would like to recall that other definitions of orthogonality have been used formerly, for example, Birkhoff [5] and James [7] (with respect to semi-inner-products) but this will be omitted in this paper.

Recall that on $\mathbb{Z}$ we have two important operators, namely the forward difference operator Δ for which $\Delta\varphi(n) = \varphi(n+1) - \varphi(n)$, and the backward difference operator ∇ defined by means of $\nabla\varphi(n) = \varphi(n) - \varphi(n-1)$.

Next we give a brief introduction to the Charlier and Meixner polynomials.

Recall that the classic monic Charlier polynomials $C_k(n;a)$ are defined for any $n \in \mathbb{Z}_+$, of the following form, the k-th polynomial is

$$C_k(n;a) = \sum_{i=0}^{k} \frac{k!}{i!\,(k-i)!} p_{n,i}\,(-a)^{k-i},$$

where $p_{n,i} = n(n-1)\cdots(n-i+1) =$ and $a \in (0,+\infty)$. The following properties of the Charlier polynomials can be found in [4], for any $n \in \mathbb{Z}_+$ and $k \geq 1$ we have

(c_1) $\triangle C_k(n;a) = kC_{k-1}(n;a)$,

(c_2) $nC_k(n;a) = C_{k+1}(n;a) + (k+a)C_k(n;a) + akC_{k-1}(n;a)$, where $C_{-1}(n;a) = 0$,

(c_3) $n \triangle \nabla C_k(n;a) + (a-n) \triangle C_k(n;a) + kC_k(n;a) = 0$.

Now we present the well known classic monic Meixner polynomials by means of their explicit representation as a hypergeometric function, that is, for $\gamma \in \mathbb{R}_+$ and $\beta \in (0,1)$ we have for the k-th polynomial ($k \geq 0$)

$$M_k^{(\gamma,\beta)}(n) = \left(\frac{\beta}{\beta-1}\right) \sum_{i=0}^{k} \binom{k}{i} (\gamma+i)_{k-i}(n-i+1)_i \left(1-\frac{1}{\beta}\right)^i,$$

where $(\gamma)_n$ is the usual Pochhammer symbol $(\gamma)_n = \gamma(\gamma+1)(\gamma+2)\cdots(\gamma+n-1)$ with $(\gamma)_0 = 1$. In [3] and [4] the following facts were presented.

(m_1) The Meixner polynomials satisfies a recurrence relation of the form

$$nM_k^{(\gamma,\beta)}(n) = M_{k+1}^{(\gamma,\beta)}(n) + \frac{k(1+\beta)+\beta\gamma}{1-\beta} M_k^{(\gamma,\beta)}(n) + \left(\frac{k\beta(k-1+\gamma)}{(\beta-1)^2}\right) M_{k-1}^{(\gamma,\beta)}(n),$$

for any $k \geq 0$, where $M_0^{(\gamma,\beta)}(n) = 1$ and $M_{-1}^{(\gamma,\beta)}(n) = 0$.

(m_2) $\left(\frac{n+\gamma}{k}\right) \triangle M_k^{(\gamma,\beta)}(n) = M_k^{(\gamma,\beta)}(n) + \left(\frac{\gamma+k-1}{1-\beta}\right) M_{k-1}^{(\gamma,\beta)}(n)$,

(m_3) $\frac{n}{k}\nabla M_k^{(\gamma,\beta)}(n) = M_k^{(\gamma,\beta)}(n) + \left(\frac{\beta}{\beta-1}\right)(1-\gamma-k)M_{k-1}^{(\gamma,\beta)}(n)$.

2. Skew-symmetric difference operators in $l_\rho^2(\mathbb{Z}_+)$

In this section we shall study some skew-bilinear forms which are defined by means of skew-symmetric linear difference operators in $l_\rho^2(\mathbb{Z}_+)$ where $\rho : \mathbb{Z}_+ \to \mathbb{R}$. It will be the first step in the construction of skew-orthogonal polynomials of a discrete variable.

Let ρ be a fixed real function of $\mathbb{Z}_+$ onto $\mathbb{R}$. Let $l_\rho^2(\mathbb{Z}_+)$ denote the collection of all real functions φ on $\mathbb{Z}_+$, such that, $\sum_{n=0}^{\infty} |\varphi(n)|^2 \rho(n)$ is finite. Then $l_\rho^2(\mathbb{Z}_+)$ is a real linear space with an inner product defined of the following form $(f,g) = \sum_{n=0}^{\infty} f(n) g(n) \rho(n)$. In what follows, we will assume that all the moments $\rho_k = \sum_{n\geq 0} n^k \rho(n)$ corresponding to ρ are finite.

Let $\rho(n) = e^{-a}\frac{a^n}{n!}$ be the usual Poisson distribution where a is a positive real number. It is well known that

$$(C_i(n;a), C_j(n;a)) = a^i i! \delta_{ij}, \tag{1}$$

and also

$$(C_i(n;a), n^s) = 0 \quad \text{for } s = 0, 1, \dots, i-1. \tag{2}$$

We introduce a linear operator L_c of $l_\rho^2(\mathbb{Z}_+)$ onto itself defined by means of $L_c\varphi = \Delta\varphi + \Lambda\varphi$, for any φ of $l_\rho^2(\mathbb{Z}_+)$, where $\Lambda\varphi(n) = \varphi(n) - \frac{n}{a}\varphi(n-1)$.

Next, we are going to show that the difference operator L_c is skew-symmetric.

Lemma 3. *The operator L_c is a skew-symmetric operator, that is,*

$$(L_c f, g) = -(L_c g, f).$$

Proof. Let f, $g \in l_\rho^2(\mathbb{Z}_+)$ then we compute $(L_c f, g)$ we get

$$(L_c f, g) = (\Delta f + \Lambda f, g) =$$

$$= e^{-a}\left(\sum_{n=0}^{\infty} \frac{a^n}{n!} f(n+1) g(n) - \sum_{n=1}^{\infty} \frac{a^{n-1}}{(n-1)!} f(n-1) g(n)\right),$$

on the other hand

$$-(f, L_c g) = -(f(n), g(n+1) - g(n)) - \left(f(n), g(n) - \frac{n}{a} g(n-1)\right) =$$

$$= -e^{-a}\sum_{n=0}^{\infty} \frac{a^n}{n!} f(n) g(n+1) + e^{-a}\sum_{n=1}^{\infty} \frac{a^{n-1}}{(n-1)!} f(n) g(n-1),$$

one can then replace $n+1$ by m in the first term and $n-1$ by m in the second. These changes lead to us to the following equality

$$-(f, L_c g) = -e^{-a}\sum_{m=1}^{\infty}\frac{a^{m-1}}{(m-1)!}f(m-1)\,g(m) + e^{-a}\sum_{m=0}^{\infty}\frac{a^m}{m!}f(m+1)\,g(m),$$

therefore $(L_c f, g) = -(f, L_c g)$. $\Diamond$

By means of this lemma it is now easy to see that $\Omega_c(f,g) = (L_c f, g)$ is a skew-bilinear form.

Take now the negative binomial distribution $\varrho(n) = \frac{\beta^n}{n!}(\gamma)_n$ where as before $(\gamma)_n$ is the Pochhammer symbol. Here $\gamma \in \mathbb{R}_+$ and $\beta \in (0,1)$. Then the following results are well known

$$\left(M_k^{(\gamma,\beta)}(n), n^j\right) = 0 \quad \text{for } j = 0, 1, \dots, k-1, \tag{3}$$

and

$$\left(M_k^{(\gamma,\beta)}(n), M_l^{(\gamma,\beta)}(n)\right) = d(k,l)\,\delta_{kl}, \tag{4}$$

where $d(k,l) > 0$ for any k and l (see [9]).

In this moment on $l_\rho^2(\mathbb{Z}_+)$ we will define a difference operator L_m which has the form $L_m\varphi(n) = \frac{n}{\beta}\varphi(n-1) - (n+\gamma)\varphi(n+1)$ for all $\varphi \in l_\rho^2(\mathbb{Z}_+)$. The operator L_m has the property of to be skew-symmetric as we state in the following lemma.

Lemma 4. *The difference operator L_m is skew-symmetric.*

Proof. To see this, note that for any f, $g \in l_\rho^2(\mathbb{Z}_+)$

$$(L_m f, g) = \left(\frac{n}{\beta}f(n-1) - (n+\gamma)f(n+1), g(n)\right) =$$

$$= \sum_{n=1}^{\infty}\frac{n}{\beta}f(n-1)\,g(n)\frac{\beta^n}{n!}(\gamma)_n - \sum_{n=0}^{\infty}(n+\gamma)f(n+1)\,g(n)\frac{\beta^n}{n!}(\gamma)_n,$$

thus

$$(L_m f, g) = \sum_{n=1}^{\infty}f(n-1)g(n)\frac{\beta^{n-1}}{(n-1)!}(\gamma)_n - \sum_{n=0}^{\infty}f(n+1)g(n)\frac{\beta^n}{n!}(\gamma)_{n+1},$$

now

$$(f, L_m g) = \left(f(n), \frac{n}{\beta}g(n-1) - (n+\gamma)\,g(n+1)\right) =$$

$$= \sum_{n=1}^{\infty}g(n-1)\,f(n)\frac{\beta^{n-1}}{(n-1)!}(\gamma)_n - \sum_{n=0}^{\infty}g(n+1)\,f(n)\frac{\beta^n}{n!}(\gamma)_{n+1},$$

doing $m = n-1$ in the first sum and $m = n+1$ in the second sum, then we obtain

$$(f, L_m g) = \sum_{m=0}^{\infty} g(m) f(n+1) \frac{\beta^m}{m!} (\gamma)_{m+1} -$$
$$- \sum_{n=0}^{\infty} g(m) f(m-1) \frac{\beta^{m-1}}{(m-1)!} (\gamma)_m = -(L_m f, g). \quad \diamond$$

This lemma allows us to assert that $\Omega_m(f,g) = (L_m f, g)$ is a skew-bilinear form.

Here concludes the first step in our construction.

3. Skew-orthogonal polynomials associated to the Charlier polynomials

The objective of this section is to build skew-orthogonal polynomials in $l^2_\rho(\mathbb{Z}_+)$ when $\rho(n) = e^{-a}\frac{a^n}{n!}$, starting from the discrete orthogonal Charlier polynomials and the skew-bilinear form Ω_c. We begin with the following lemma

Lemma 5. *For all $s \in \mathbb{N}$, we have*

$$nC_s(n-1;a) = C_{s+1}(n;a) + aC_s(n;a). \tag{5}$$

Proof. Firstly we note that

$$\nabla\Delta C_s(n;a) = C_s(n+1;a) - 2C_s(n;a) + C_s(n-1;a),$$

that is

$$C_s(n-1;a) = \nabla\Delta C_s(n;a) - C_s(n+1;a) + 2C_s(n;a) =$$
$$= \nabla\Delta C_s(n;a) - \Delta C_s(n;a) + C_s(n;a),$$

hence

$$nC_s(n-1;a) = n\nabla\Delta C_s(n;a) - n\Delta C_s(n;a) + nC_s(n;a). \tag{6}$$

Using now the property (c_3) we can replace the term $n\nabla\Delta C_s(n;a)$ in (6) and we obtain a new equation

$$nC_s(n-1;a) = -((a-n)\triangle C_s(n;a) + sC_s(n;a)) -$$
$$-n\Delta C_s(n;a) + nC_s(n;a) = -a \triangle C_s(n;a) - sC_s(n;a) + nC_s(n;a),$$

and from (c_2) it shows that

$$nC_s(n-1;a) = -a \triangle C_s(n;a) - sC_s(n;a) +$$
$$+ C_{s+1}(n;a) + (s+a)C_s(n;a) + asC_{s-1}(n;a),$$

finally, it follows of (c_1) that

$$nC_s(n-1;a) = -asC_{s-1}(n;a) - sC_s(n;a) + \\ + C_{s+1}(n;a) + (s+a)C_s(n;a) + asC_{s-1}(n;a),$$

thus

$$nC_s(n-1;a) = C_{s+1}(n;a) + aC_s(n;a),$$

therefore, the lemma is proved. $\diamondsuit$

The principal result of this section is the following assertion.

Theorem 6. *Let $\Omega_c(f,g) = \langle L_c f, g\rangle$ for all $f,g \in L^2_\rho$, where $\rho(n) = e^{-a}\frac{a^n}{n!}$. Then a system of discrete skew-orthogonal polynomials $\{q_s(n;a)\}_{s\geq 0}$ with respect to Ω_c, can be constructed with the help of the classic monic Charlier polynomials if we take*

$$q_{2m}(n;a) = 2^m a^m m! \sum_{k=0}^{m} \frac{1}{2^k a^k k!} C_{2k}(n;a), \tag{7}$$
$$q_{2m+1}(n;a) = C_{2m+1}(n;a),$$

and where the sequence $\{\gamma_m\}_{m\geq 0}$ is composed of the following numbers

$$\gamma_m = -\frac{1}{a}\|c_{2m+1}(n;a)\|^2_{L^2_\rho}. \tag{8}$$

Proof. For this end let us lean on the only concrete method we have at hand, Rem. 2. We write $q_{2m}(n)$ and $q_{2m+1}(n)$ as a linear combination of the Charlier polynomials, that is

$$q_{2m}(n;a) = C_{2m}(n;a) + \sum_{k=0}^{2m-1} b_k^{2m} C_k(n;a),$$

$$q_{2m+1}(n;a) = C_{2m+1}(n;a) + \sum_{k=0}^{2m} b_k^{2m+1} C_k(n;a),$$

where b_k^{2m} and b_k^{2m+1} are constant which will be determined below. For brevity, from now on, we write $q_s(n)$.instead of $q_s(n;a)$.

It is clear that $q_0 = 1$ and $q_1(n) = C_1(n;a) + b_0^1$. We must have $\Omega_c(q_0,q_1) = \left(L_c 1, C_1(n;a) + b_0^1\right) = \gamma_0$. Since $(L_c 1)(n) = 1 - \frac{n}{a}$ and $(1, C_1(n;a)) = 0$, it then follows that $\gamma_0 = -\left(\frac{n}{a}, C_1(n;a)\right) + \left(1, b_0^1\right) - \left(\frac{n}{a}, b_0^1\right) = -\left(\frac{n}{a}, C_1(n;a)\right) + b_0^1 - b_0^1 = -\left(\frac{n}{a}, C_1(n;a)\right)$.

Now

$$-\left(\frac{n}{a}, C_1(n;a)\right) = -\frac{1}{a}(n, C_1(n;a)) - \frac{1}{a}(a, C_1(n;a)) =$$

$$= -\frac{1}{a}\left\|C_1(n;a)\right\|^2_{L^2_\rho} = \gamma_0,$$

here we have used the fact that $C_1(n;a) = n - a$. Note that in fact b_0^1 is arbitrary. Hence we can take $b_0^1 = 0$ and $q_1 = C_1(n;a)$. Therefore $\Omega_c(q_0, q_1) = \gamma_0$ holds.

Now let us assume that $m \geq 1$. From Rem. 2 it follows that to compute b_k^{2m} when $k = 0, \ldots, 2m-1$, it is suitable to require that the equalities $\Omega_c(q_{2m}(n), n^i) = 0$ for $i = 0, \ldots, 2m-1$ are held. In this case this idea is feasible and Rem. 2 constitutes an implement to calculate the coefficients b_k^{2m}.

In fact

$$\Omega_c(q_{2m}(n), n^i) = \left(\Delta C_{2m}(n;a) + \sum_{k=0}^{2m-1} b_k^{2m} \, \Delta\, C_k(n;a), n^i\right) +$$
$$+\left(\Lambda C_{2m}(n;a) + \sum_{k=0}^{2m-1} b_k^{2m} \Lambda C_k(n;a), n^i\right).$$

By the property (c_1) of the Charlier polynomials and of the expression of Λ it follows that $\Omega_c(q_{2m}(n), n^i)$ can be written in the form

$$(9)\quad \Omega_c(q_{2m}(n), n^i) = \left(2mC_{2m-1}(n;a) + \sum_{k=0}^{2m-1} kb_k^{2m} C_{k-1}(n;a), n^i\right) +$$
$$+\left(C_{2m}(n;a) + \sum_{k=0}^{2m-1} b_k^{2m} C_k(n;a), n^i\right) -$$
$$-\left(\frac{n}{a} C_{2m}(n-1;a) + \frac{n}{a}\sum_{k=0}^{2m-1} b_k^{2m} C_k(n-1;a), n^i\right).$$

In the previous expression, in order to do all the polynomials depending only of n, but not simultaneously of n and $n-1$ we use Lemma 5 to write $C_s(n-1;a)$ by means of some other polynomials $C_l(n;a)$, where $l = 0, 1, \ldots, 2m$. Thus, from (5) and (9) it shows that $\Omega_c(q_{2m}(n), n^i)$ could be expressed as

$$\Omega_c(q_{2m}(n), n^i) = \left(2mC_{2m-1}(n;a) + \sum_{k=0}^{2m-1} kb_k^{2m} C_{k-1}(n;a), n^i\right) +$$

$$+\left(C_{2m}(n;a)+\sum_{k=0}^{2m-1} b_k^{2m} C_k(n;a), n^i\right)-$$
$$-\frac{1}{a}\left(C_{2m+1}(n;a)+aC_{2m}(n;a), n^i\right)-$$
$$-\frac{1}{a}\left(\sum_{k=0}^{2m-1} b_k^{2m}\left(C_{k+1}(n;a)+aC_k(n;a)\right), n^i\right).$$

Then, since $0 \le i \le 2m-1$ this would imply

$$\Omega_c(q_{2m}(n), n^i) = \left(2mC_{2m-1}(n;a)+\sum_{k=0}^{2m-1} k b_k^{2m} C_{k-1}(n;a), n^i\right)- \tag{10}$$
$$-\frac{1}{a}\left(\sum_{k=0}^{2m-1} b_k^{2m} C_{k+1}(n;a), n^i\right).$$

Next, each equation $\Omega_c(q_{2m}(n), n^i) = 0$ will be studied in detail using the orthogonality of the Charlier polynomials (2). We recall that $m \ge 1$, then for $i=0$ we deduce that $b_1^{2m}=0$. Let us suppose that $m>1$ and let s be such that $0<2s<2m-1$ then from (10)

$$\Omega_c(q_{2m}(n), n^{2s}) = \left((2s+1)\, b_{2s+1}^{2m} C_{2s}(n;a), n^{2s}\right)-$$
$$-\frac{1}{a}\left(b_{2s-1}^{2m} C_{2s}(n;a), n^{2s}\right),$$

hence, it is evident that the equations $\Omega_c(q_{2m}(n), n^{2s}) = 0$ imply that $b_{2s+1}^{2m} = \frac{1}{a(2s+1)} b_{2s-1}^{2m}$. Therefore $b_{2s+1}^{2m} = 0$ for $s = 0, 1, \ldots, m-1$.

Let $m>1$ and let s, such that, $0<2s+1<2m-1$ then taking $i = 2s+1$ in (10) we have

$$\Omega_c(q_{2m}(n), n^{2s+1}) = \left((2s+2)\, b_{2s+2}^{2m} C_{2s+1}(n;a), n^{2s+1}\right)-$$
$$-\frac{1}{a}\left(b_{2s}^{2m} C_{2s+1}(n;a), n^{2s+1}\right),$$

thus the equality $\Omega_c(q_{2m}(n), n^{2s+1}) = 0$ will be satisfied if $b_{2s}^{2m} = a(2s+2)\, b_{2s+2}^{2m}$ holds. On the other hand, it is easy to show that $b_{2m-2}^{2m} = 2am$ (when $m \ge 1$) which follows from the equation $\Omega_c(q_{2m}(n), n^{2m-1}) = 0$. Since all the odd coefficients are zero we have

$$q_{2m}(n) = C_{2m}(n;a) + 2amC_{2(m-1)}(n;a) +$$
$$+ 2^2a^2m(m-1)\, C_{2(m-2)}(n;a) + \cdots + 2^m a^m m! C_0(n;a) =$$

$$= 2^m a^m m! \sum_{i=0}^{m} \frac{1}{2^i a^i i!} C_{2i}(n;a).$$

Now, let us compute b_k^{2m+1}. As before, the equalities $\Omega_c(q_{2m}(n), n^i) = 0$ for $i = 0, 1, \ldots, 2m-1$ enables us to find these constants. A straightforward calculation tells us that for $i = 0, 1, \ldots, 2m-1$

$$\Omega_c(q_{2m+1}(n), n^i) = \left((2m+1)\, C_{2m}(n;a) + \sum_{k=0}^{2m} k b_k^{2m+1} C_{k-1}(n;a), n^i\right) - \tag{11}$$
$$- \frac{1}{a}\left(\sum_{k=0}^{2m} b_k^{2m+1} C_{k+1}(n;a), n^i\right).$$

Then for $i = 0$, it follows from (11) that $b_1^{2m+1} = 0$. On the other hand, if $m > 1$ and let $s \in \mathbb{N}$ be such that $0 < 2s < 2m-1$ one can evaluate $i = 2s$ in (11). In this case we obtain that

$$\Omega_c(q_{2m+1}(n), n^{2s}) = \left((2s+1)\, b_{2s+1}^{2m+1} C_{2s}(n;a), n^{2s}\right) -$$
$$- \frac{1}{a}\left(b_{2s-1}^{2m+1} C_{2s}(n;a), n^{2s}\right),$$

hence, in order to $\Omega_c(q_{2m+1}(n), n^{2s}) = 0$ is sufficient that $b_{2s+1}^{2m+1} = \frac{1}{a(2s+1)} b_{2s-1}^{2m+1}$. Here, we recall that $b_1^{2m+1} = 0$, so $b_{2s+1}^{2m+1} = 0$ for $s = 1, \ldots, m-1$.

Let m and s be chosen such that $m > 1$ and $0 < 2s+1 < 2m-1$. Taking in (11) $i = 2s+1$ we have

$$\Omega_c(q_{2m+1}(n), n^{2s+1}) = \left((2s+2)\, b_{2s+2}^{2m+1} C_{2s+1}(n;a), n^{2s+1}\right) -$$
$$- \frac{1}{a}\left(b_{2s}^{2m+1} C_{2s+1}(n;a), n^{2s+1}\right),$$

thus if $b_{2s}^{2m+1} = a2\,(s+1)\, b_{2(s+1)}^{2m+1}$ then $\Omega(q_{2m+1}(n), n^{2s+1}) = 0$ for $m > 1$ and $s = 0, 1, \ldots, m-2$. When $i = 2m-1$ turn out that

$$\Omega_c(q_{2m+1}(n), n^{2m-1}) = \left(2m b_{2m}^{2m+1} C_{2m-1}(n;a), n^{2m-1}\right) -$$
$$- \frac{1}{a}\left(b_{2m-2}^{2m+1} C_{2m-1}(n;a), n^{2m-1}\right).$$

It follows from this equality that $\Omega_c(q_{2m+1}(n), n^{2m-1})$ is zero if for instance $b_{2m-2}^{2m+1} = 2am b_{2m}^{2m+1}$. We choose $b_{2m}^{2m+1} = 0$. Under this selection we obtain $q_{2m+1}(n) = C_{2m+1}(n;a)$. It remains to see that $\gamma_m = \Omega_c(q_{2m}(n), q_{2m+1}(n))$. This will be done now. To see this, notice that

$$\Omega_c(q_{2m}(n), q_{2m+1}(n)) = (\Delta q_{2m}(n) + \Lambda q_{2m}(n), q_{2m+1}(n)),$$

hence

$$\Omega_c(q_{2m}, q_{2m+1}) =$$

$$= 2^m a^m m! \sum_{i=0}^{m} \frac{1}{2^i a^i i!} \left(2iC_{2i-1}(n;a) - \frac{1}{a} C_{2i+1}(n;a), C_{2m+1}(n;a) \right) =$$

$$= -\frac{1}{a} \| c_{2m+1}(n;a) \|^2_{L^2_\rho}. \quad \Diamond$$

4. Skew-orthogonal polynomials associated to the Meixner polynomials

This section is devoted to present skew-orthogonal polynomials of a discrete variable with respect to the skew-bilinear form $\Omega_m(f,g)$ in $l^2_\rho(\mathbb{Z}_+)$ when $\rho(n) = \frac{\beta^n}{n!}(\gamma)_n$ and under the following conditions $\gamma \in \mathbb{R}_+$ and $\beta \in (0,1)$. For this purpose it is then suitable to use the Meixner polynomials.

Let us start with a lemma.

Lemma 7. *Let* $I_k(n) = \frac{n}{\beta} M_k^{(\gamma,\beta)}(n-1) - (\gamma+n) M_k^{(\gamma,\beta)}(n+1)$ *for* k, $n \in \mathbb{N}$. *Then* $I_k(n)$ *can be expressed as*

$$I_k(n) = \left(\frac{1-\beta}{\beta}\right) M_{k+1}^{(\gamma,\beta)}(n) + \frac{k}{\beta-1}(k+\gamma-1) M_{k-1}^{(\gamma,\beta)}(n). \tag{12}$$

Proof. Since $I_k(n) = \frac{n}{\beta} M_k^{(\gamma,\beta)}(n) - \frac{n}{\beta} \nabla M_k^{(\gamma,\beta)}(n) - (\gamma+n) \Delta M_k^{(\gamma,\beta)}(n) - (\gamma+n) M_k^{(\gamma,\beta)}(n)$, then using now the properties (m_2) and (m_3) of the Meixner polynomials we have

$$I_k(n) = \frac{n}{\beta} M_k^{(\gamma,\beta)}(n) - \frac{k}{\beta} \left(M_k^{(\gamma,\beta)}(n) + \left(\frac{\beta}{\beta-1}\right)(1-\gamma-k) M_{k-1}^{(\gamma,\beta)}(n) \right) -$$
$$- k \left(M_k^{(\gamma,\beta)}(n) + \left(\frac{\gamma+k-1}{1-\beta}\right) M_{k-1}^{(\gamma,\beta)}(n) \right) - (\gamma+n) M_k^{(\gamma,\beta)}(n),$$

thus

$$I_k(n) = \left(\frac{1-\beta}{\beta}\right) n M_k^{(\gamma,\beta)}(n) - \frac{k(1+\beta)+\beta\gamma}{\beta} M_k^{(\gamma,\beta)}(n) + \\ + 2k \frac{k+\gamma-1}{\beta-1} M_{k-1}^{(\gamma,\beta)}(n). \tag{13}$$

Taking into account the recurrence relation (m_1), then from (13) it follows (12). ◊

Our last result is the following theorem.

Theorem 8. *Let $l^2_\rho(\mathbb{Z}_+)$ when $\rho(n)=\frac{\beta^n}{n!}(\gamma)_n$. With respect to $\Omega_m(f,g)$ there exists a system of discrete skew-orthogonal polynomials given as follows*

$$q_{2k}^{(\gamma,\beta)}(n) = \frac{2^k\beta^k k!\,(\gamma+1)_{(2k,2)}}{(\beta-1)^{2k}} \sum_{i=0}^{k} \frac{(\beta-1)^{2i}}{2^i\beta^i i!\,(\gamma+1)_{(2i,2)}} M_{2i}^{(\gamma,\beta)}(n),$$

and $q_{2k+1}^{(\gamma,\beta)}(n) = M_{2k+1}^{(\gamma,\beta)}(n)$ where $\gamma_k = \frac{1-\beta}{\beta}\|M_{2k+1}^{(\gamma,\beta)}(n)\|^2_{L^2_\rho}$ for any $k=0,1,\cdots$ and

$$(\gamma+1)_{(2s,2)} = (\gamma+1)(\gamma+1+2)\cdots(\gamma+1+2k-2).$$

Proof. In the proof of this theorem we use Lemma 7 and the orthogonality property of the Meixner polynomials (3). But it is very similar to the proof of Th. 6, therefore the proof of Th. 8 will be omitted. ◊

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