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CENTRAL LIMIT THEOREMS FOR THE POTTS MODEL

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Abstract. We prove various q-dimensional Central Limit Theorems for the occurring of the colors in the q-state Potts model on $\mathbb{Z}^d$ at inverse temperature β, provided that β is sufficiently far from the critical point β_c. When $(d = 2)$ and $(q = 2$ or $q \geq 26)$, the theorems apply for each $\beta \neq \beta_c$. In the uniqueness region, a classical Gaussian limit is obtained. In the phase transition regime, the situation is more complex: when $(q \geq 3)$, the limit may be Gaussian or not, depending on the Gibbs measure which is considered. Particularly, we show that free boundary conditions lead to a non-Gaussian limit. Some particular properties of the Ising model are also discussed. The limits that are obtained are identified relatively to FK-percolation models.

1. Introduction

The aim of this study is to answer a natural question relative to Gibbs measures in the q-state Potts model: take a finite box in $\mathbb{Z}^d$ and consider the frequencies of each of the q states in the box. By the ergodic theorem, the vector of empirical frequencies obviously converges to a constant when the considered Gibbs measure is ergodic. Then, it is natural to ask whether we can have a central limit theorem with a standard renormalization.

Three decades after the seminal paper by Fortuin and Kasteleyn [FK72], it has become obvious that most problems related to the Potts model encounter the road of the Fortuin-Kasteleyn random cluster measure – see for example Häggström [Häg98] for a self-contained introduction to the relations between these models.

Roughly speaking, we can say that a realization of the q-state Potts model with free boundary conditions in a finite box is a random coloring of the vertices of a realization of a free random-cluster measure in the box Λ, with the constraint that connected components are mono-color. Actually, we can consider the Potts model as the restriction to its vertices of a measure on "colored graphs": there is

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randomness on the set of open bonds and also on the color of vertices, with the condition that connected components are mono-color.

Then, it is not difficult to guess that in the supercritical regime, the presence of an infinite cluster strongly modifies the fluctuation of the empiric repartitions. Thus, our first step is the study of the normal fluctuations of the density of the infinite cluster in large boxes. Since Fortuin-Kasteleyn random cluster measures enjoy the FKG property, there is a tool that is particularly appropriate to show central limit theorems; it is Newman's theorem [New80], which reduces the problem to proving the convergence of a series. Newman and Schulman noticed [NS81a, NS81b] that this could be useful for the study of the fluctuations of the density of the infinite cluster for some percolation models, but time passed without anybody using it for a precise model, not even for standard percolation. Thus we give in this paper a theorem that explains how to obtain a CLT for the density of the infinite clusters in the case of a percolation model satisfying FKG: it suffices to bound the probability of large finite clusters and the correlation of some local events.

This leads to a simple proof of the CLT in the case of Bernoulli percolation. The case of FK percolation appears to be more intricate, and there are values of the parameters for which one did not succeed in obtaining the desired estimates. Naturally, these gaps are reflected on the subsequent theorems relative to the Potts model.

The paper is organized as follows:

- The first part begins with the general CLT theorem for the density of the infinite cluster(s) announced above. Then, we prove that for each $q \geq 1$, there exists $p_r(q) < 1$ such that for $p > p_r(q)$, the number of points in large boxes $(\Lambda_n)_{n\geq 1}$ which belongs to the infinite cluster has a normal central limit behaviour under the random cluster measure $\phi_{p,q}$:
$$\frac{|\Lambda_n \cap C_\infty| - \phi_{p,q}(0 \in C_\infty)|\Lambda_n|}{|\Lambda_n|^{1/2}} \Longrightarrow \mathcal{N}(0, \sigma^2_{p,q}),$$
where C_∞ is the infinite cluster for FK percolation and
$$\sigma^2_{p,q} = \sum_{k\in\mathbb{Z}^d} \big(\phi_{p,q}(0 \leftrightarrow \infty \text{ and } k \leftrightarrow \infty) - \phi_{p,q}(0 \in C_\infty)^2\big).$$
The result is much better on the square lattice: when $q = 1, 2$ or $q \geq 26$, $p_r(q)$ is equal to the classical critical value $p_c(q)$, so the result applies in the whole supercritical zone.
- In the second part, we prove a q-dimensional central limit theorem for the fluctuation of the empiric repartitions of colors in a coloring model, that is a model where the connected components of a random graph (not necessarily satisfying the FKG inequalities) are painted independently, provided that the fluctuations of the number of points in large boxes which belong to an infinite cluster satisfy a central limit theorem.
- In the third part, we combine these results to obtain q-dimensional central limit theorems for the fluctuation of the empiric repartitions of the q colors in the q-state Potts model: when the inverse temperature β is small enough, the unique Gibbs measure for the q-state Potts model at inverse temperature β satisfy the following result for the empirical distributions:

there exists a constant $\chi_\beta > 0$ such that

$$\frac{n(\Lambda_t) - |\Lambda_t|\nu}{\sqrt{|\Lambda_t|}} \Longrightarrow \mathcal{N}(0, \frac{\chi_\beta}{q^2}(qI - J)),$$

where J is the $q\times q$ matrix each of whose entries is equal to 1, $\nu = (\frac{1}{q}, \dots, \frac{1}{q})$, and the vector $n(\Lambda_t) = (n_1(\Lambda_t), n_2(\Lambda_t), \dots, n_q(\Lambda_t))$ consists of the numbers of occurring of each of the q states.

In the region of phase transition, the situation is more complex. For example, under the Gibbs measure which is obtained as the limit of finite size Gibbs measures with a constant boundary condition "1", we have, provided that β is large enough:

$$\frac{n(\Lambda_t) - |\Lambda_t|(1, 1-\theta, 1-\theta, \dots, 1-\theta)}{\sqrt{|\Lambda_t|}} \Longrightarrow \mathcal{N}(0, C)$$

for some matrix C and some constant θ. For the Gibbs measure which is obtained as the limit of finite size Gibbs measures with free boundary condition, a central limit theorem is also proved, but the limit is not Gaussian (except in the case of the Ising model).

As in the first part, the region of validity of the theorems is optimal when $d = 2$: the results hold for each $\beta \neq \beta_c$ as soon as $q = 2$ or $q \geq 26$.

Actually, when $d = 2$, the results of the first part (and, consequently, the results on the last one) need an exponential inequality relative to FK-percolation which has been proved to hold in the whole subcritical zone when $q = 1, 2$ or $q \geq 26$. In fact, it is widely believed that this holds in the whole subcritical zone for each $q \geq 1$. If this challenge were performed, this would automatically give the optimality of the results presented here for each value of q.

2. Notations and preliminaries

Graph theoretical notations

For $x \in \mathbb{Z}^d$, let us denote $\|x\| = \sum_{i=1}^d |x_i|$ and consider the graph $\mathbb{L}^d = (\mathbb{Z}^d, \mathbb{E}^d)$, with

$$\mathbb{E}^d = \{\{x,y\}; x, y \in \mathbb{Z}^d \text{ and } \|x-y\| = 1\}.$$

For $x \in \mathbb{Z}^d$ and $r \in [0, +\infty)$, we note $B(x,r) = \{y \in \mathbb{Z}^d; \|x-y\| \leq r\}$. If $e = \{x,y\} \in \mathbb{E}^d$, then x and y are called *neighbours*.

In the following, the expression "subgraph of $\mathbb{L}^d$" will always be employed for each graph of the form $G = (\mathbb{Z}^d, E)$ where E is a subset of $\mathbb{E}^d$. We denote by $\mathcal{S}(\mathbb{L}^d)$ the set of all subgraphs of $\mathbb{L}^d$.

Set $\Omega = \{0,1\}^{\mathbb{E}^d}$. An edge $e \in \mathbb{E}^d$ is said to be *open* in the configuration ω if $\omega(e) = 1$, and *closed* otherwise.

There is a natural bijection between $\mathcal{S}(\mathbb{L}^d)$ and Ω, that is $E \mapsto (\mathbb{1}_{e\in E})_{e\in\mathbb{E}^d}$. Consequently, we sometimes identify $\mathcal{S}(\mathbb{L}^d)$ and Ω and say "random graph measure" rather than "measure on Ω".

A *path* is a sequence $\gamma = (x_1, e_1, x_2, e_2, \dots, x_n, e_n, x_{n+1})$ such that x_i and x_{i+1} are neighbours and e_i is the edge between x_i and x_{i+1}. We will also sometimes describe γ only by the vertices it visits $\gamma = (x_1, x_2, \dots, x_n, x_{n+1})$ or by its edges $\gamma = (e_1, e_2, \dots, e_n)$. The number n of edges in γ is called the *length* of γ and is denoted by $|\gamma|$. We will also consider *cycles*, that are paths for which the visited vertices are all distinct, except that $x_1 = x_{n+1}$. A path is said to be *open* in the configuration ω if all its edges are open in ω. If Λ_1 and Λ_2 are two subsets of $\mathbb{Z}^d$, we denote by $d(\Lambda_1, \Lambda_2)$ the length of the shortest path from Λ_1 to Λ_2.

The *clusters* of a configuration ω are the connected components of the graph induced on $\mathbb{Z}^d$ by the open edges in ω. For x in $\mathbb{Z}^d$, we denote by $C(x)$ the cluster containing x. In other words, $C(x)$ is the set of points in $\mathbb{Z}^d$ that are linked to x by an open path.

We note $x \leftrightarrow \infty$ to say that $|C(x)| = +\infty$. In the whole paper, we will note $C_\infty = \{x \in \mathbb{Z}^d; x \leftrightarrow \infty\}$.

We say that two bonds e and e' of $\mathbb{E}^d$ are neighbours if $e \cap e'$ is not empty. This also gives a notion of connectedness in $\mathbb{E}^d$ in the usual way.

For each subset Λ of $\mathbb{Z}^d$, we denote by $\partial\Lambda$ the boundary of Λ:

$$\partial\Lambda = \{y \in \Lambda^c; \exists x \in \Lambda \text{ with } \|x - y\| = 1\}$$

and by $\mathbb{E}_\Lambda$ the set of inner bonds of Λ:

$$\mathbb{E}_\Lambda = \{e \in \mathbb{E}^d; e \subset \Lambda\}.$$

Note that if Λ and Λ' are disjoint sets, then $\mathbb{E}_\Lambda$ and $\mathbb{E}_{\Lambda'}$ are disjoint too.

For each $E \subset \mathbb{E}^d$, we denote by $\sigma(E)$ the σ-field generated by the projections $(\omega_e)_{e\in E}$. A subset A of Ω is said to be a local event if there exists a finite subset E of $\mathbb{E}^d$ such that A is $\sigma(E)$-measurable.

When $\Lambda \subset \mathbb{Z}^d$, we also use the notation $\sigma(\Lambda)$ instead of $\sigma(\mathbb{E}_\Lambda)$.

We sometimes consider another set of bonds on $\mathbb{Z}^d$, that is

$$\mathbb{F}^d = \{\{x, y\}; x, y \in \mathbb{Z}^d \text{ and } \|x - y\|_\infty = 1\},$$

where $\|x\|_\infty = \max(|x_i|; 1 \le i \le d)$. If $e = \{x, y\} \in \mathbb{F}^d$ then x and y are called $*$-*neighbours*. Similarly, we define the notion of $*$-paths, $*$-cycles, $*$ connected sets,... exactly in the same way as for the graph $\mathbb{L}^d$.

A subset Λ of $\mathbb{Z}^d$ is said to be a box if it can be written in the following form: $\Lambda = \big([a_1, \dots b_1] \times \dots\dots [a_d, \dots b_d]\big) \cap \mathbb{Z}^d$ for some real numbers $a_1, \dots, a_d, b_1, \dots b_d$.

For our central limit theorems, we will use boxes $(\Lambda_t)_{t\ge 1}$, with

$$\Lambda_t = \{x \in \mathbb{Z}^d; \|x\|_\infty \le t\}.$$

Let X and S be arbitrary sets. Each $\omega \in X^S$ can be considered as a map from S to X. We will denote ω_Λ its restriction to Λ. Then, when A and B are two disjoint subsets of S and $(\omega, \eta) \in X^A \times X^B$, $\omega\eta$ denotes the concatenation of ω and η, that is the element $z \in X^{A\cup B}$ such that

$$z_i = \begin{cases} \omega_i & \text{if } i \in A \\ \eta_i & \text{if } i \in B. \end{cases}$$

2.1. **FK Random cluster measures.**

Let $0 \le p \le 1$ and $q > 0$.

For each configuration $\eta \in \Omega$ and each connected subset E of $\mathbb{E}^d$ we define the random-cluster measure $\phi^\eta_{E,p,q}$ with boundary condition η on $(\Omega, \mathcal{B}(\Omega))$ by

$$\phi^\eta_{E,p,q}(\omega) = \begin{cases} \dfrac{1}{Z^\eta_{E,p,q}} \Big\{ \displaystyle\prod_{e\in E} p^{\omega(e)}(1-p)^{1-\omega(e)} \Big\} q^{k_E(\omega_E \eta_{E^c})} & \text{if } \omega_{E^c} = \eta_{E^c}, \\ 0 & \text{otherwise,} \end{cases}$$

where $k_E(\omega)$ is the number of components of the graph ω which intersect $\cup_{e\in E} e$. $Z^{\eta}_{E,p,q}$ is the renormalizing constant

$$Z^{\eta}_{E,p,q} = \sum_{\omega\in\{0,1\}^E} \Big\{ \prod_{e\in E} p^{\omega(e)}(1-p)^{1-\omega(e)} \Big\} q^{k_E(\omega_E\eta_{E^c})}.$$

For each $b \in \{0,1\}$, we will simply denote by $\phi^b_{E,p,q}$ the measure $\phi^{\eta}_{E,p,q}$ corresponding to the configuration η which is such that $\eta_e = b$ for each $e \in \mathbb{E}^d$.

When Λ is a finite subset of $\mathbb{Z}^d$, we also use the notation $\phi_{\Lambda,p,q}$ instead of $\phi_{\mathbb{E}_\Lambda,p,q}$

A probability measure ϕ on $(\Omega, \mathcal{B}(\Omega))$ is called a *random-cluster measure* with parameters p and q if for each measurable set A and each finite subset Λ of $\mathbb{Z}^d$, we have the D.L.R. condition:

$$\phi(A) = \int_\Omega \phi^{\eta}_{\Lambda,p,q}(A)\ d\phi(\eta)$$

The set of such measures is denoted by $\mathcal{R}_{p,q}$.

Let $b \in \{0,1\}$. If $(\Lambda_n)_{n\geq 1}$ is an increasing sequence of volumes tending to $\mathbb{Z}^d$, it is known that the sequence $\phi^b_{\Lambda_n,p,q}$ has a weak limit which does not depend on the sequence $(\Lambda_n)_{n\geq 1}$. We denote by $\phi^b_{p,q}$ this limit. The following facts are well known; refer to the recent summary of Grimmett [Gri03] for complete references.

- $\phi^b_{p,q}$ is a translation invariant ergodic measure.
- $\phi^b_{p,q} \in \mathcal{R}_{p,q}$.
- Let us note $\theta^b(p,q) = \phi^b_{p,q}(0 \leftrightarrow \infty)$. There exists $p_c(q) \in (0,1)$, such that for each $b \in \{0,1\}$ we have $\theta^b(p,q) = 0$ for $p < p_c(q)$ and $\theta^b(p,q) > 0$ for $p > p_c(q)$. Moreover, $\mathcal{R}_{p,q}$ is a singleton as soon as $p < p_c(q)$.

FKG inequalities and stochastic comparison

There is a natural partial order $\prec$ on $\Omega = \{0,1\}^{\mathbb{E}^d}$: for ω and ω' in Ω, we say that $\omega \prec \omega'$ holds if and only if $\omega_e \leq \omega'_e$ for each $e \in \mathbb{E}^d$. Consequently, we say that a function $f : \omega \to \mathbb{R}$ is increasing if $f(\omega) \leq (\omega')$ as soon as $\omega \prec \omega'$. If ϕ is a probability measure and f, g two bounded measurable functions, we note

$$\text{Cov}_\phi(f,g) = \int fg\ d\phi - \Big(\int f\ d\phi\Big)\Big(\int g\ d\phi\Big).$$

If A and B are measurable events, we also note $\text{Cov}_\phi(A,B) = \text{Cov}_\phi(1\!\!1_A, 1\!\!1_B) = \phi(A\cap B) - \phi(A)\phi(B)$.

We say that a measure ϕ on $(\Omega, \mathcal{B}(\Omega))$ satisfies the FKG inequalities if for each pair of increasing bounded functions f and g, we have $\text{Cov}_\phi(f,g) \geq 0$.

It is well known that $\phi^b_{p,q}$ satisfies the FKG inequalities if $q \geq 1$.

An event A is said to be increasing (*resp.* decreasing) if $1\!\!1_A$ (*resp.* $1 - 1\!\!1_A$) is an increasing function. Of course, if ϕ satisfies the FKG inequalities and A and B are increasing events, we have $\text{Cov}_\phi(A,B) \geq 0$.

Let us first recall the concept of stochastic domination: we say that a probability measure μ dominates a probability measure ν, if

$$\int f\ d\nu \leq \int f\ d\mu$$

holds whenever f is an increasing function. Thus, we write $\nu \prec \mu$.

The following stochastic comparison for random cluster measures is well known: for $q' \geq q, q' \geq 1$ and $\frac{p'}{q'(1-p')} \geq \frac{p}{q(1-p)}$, we have $\phi^0_{p,q} \prec \phi^0_{p',q'}$.

Exponential bounds
Grimmett and Piza [GP97] also introduced another critical probability: Let us define

$$Y(p,q) = \limsup_{n\to\infty} \Big\{ n^{d-1}\phi^0_{p,q}\big(0 \leftrightarrow \partial B(0,n)\big)\Big\}$$

and $p_g(q) = \sup\big\{p : Y(p,q) < \infty\big\}$. We have $0 < p_g(q) \leq p_c(q)$.

Grimmett and Piza proved the following exponential bound:

Proposition 1. *Let $q \geq 1$, $d \geq 2$. For $p < p_g(q)$, there exists a constant $\gamma = \gamma(p,q) > 0$ with*

$$\phi_{p,q}(0 \leftrightarrow \partial B(0,n)) \leq e^{-\gamma n} \text{ for large } n. \tag{1}$$

(Note that we can write $\phi_{p,q}$ instead of $\phi^0_{p,q}$ because of the uniqueness of the random cluster measure for $p < p_c$.)

It is conjectured that $p_g(q) = p_c(q)$ for each $q \geq 1$.

When $d = 2$, this widely believed conjecture has already be proved for $q = 1, q = 2$ and $q \geq 26$ – see the Saint-Flour notes by Grimmett ([Gri97]).

2.2. **The Potts model.** Let us recall the definition of Gibbs measure in the context of the Potts model. Let $q \geq 2$ and $\beta > 0$. We denote by S_q a set of cardinality q.

For a finite subset Λ of $\mathbb{Z}^d$, the Hamiltonian on the volume Λ is defined by

$$H_\Lambda = 2 \sum_{\substack{e=\{x,y\}\in\mathbb{E}^d\\ e\cap\Lambda\neq\varnothing}} 1\!\!1_{\{\omega(x)\neq\omega(y)\}}.$$

Then, we can define for each bounded measurable function f and for each $\omega \in S_q^{\mathbb{Z}^d}$,

$$\Pi_\Lambda f(\omega) = \frac{1}{Z_\Lambda(\omega)} \sum_{\eta\in S_q^\Lambda} \exp(-\beta H_\Lambda(\eta_\Lambda\omega_{\Lambda^c}))f(\eta_\Lambda\omega_{\Lambda^c}),$$

where

$$Z_\Lambda(\omega) = \sum_{\eta\in S_q^\Lambda} \exp(-\beta H_\Lambda(\eta_\Lambda\omega_{\Lambda^c})).$$

For each ω, we will denote by $\Pi_\Lambda(\omega)$ the measure on $S_q^{\mathbb{Z}^d}$ which is associated to the map $f \mapsto \Pi_\Lambda f(\omega)$. A measure μ on $S_q^{\mathbb{Z}^d}$ is said to be a Gibbs measure for the q-state Potts model at inverse temperature β when for each bounded measurable function f and each finite subset Λ of $\mathbb{Z}^d$, we have

$$E_\mu(f|(X_i)_{i\in\Lambda^c}) = \Pi_\Lambda f \quad \mu \text{ a.s.}$$

For each $z \in S_q$, let us denote by $\Pi_\Lambda(z)$ the measure $\Pi_\Lambda(\omega)$ where ω is the element of $S_q^{\mathbb{Z}^d}$ with $\omega_x = z$ for each $x \in \mathbb{Z}^d$. It is known that for each $\beta > 0$ and each $z \in S_q$, the sequence $(\Pi_\Lambda(z))_\Lambda$ converges when Λ tends to $\mathbb{Z}^d$. Let us denote by $\mathsf{WPt}_{q,\beta,z}$ this limit. By the general theory of Gibbs measures, this limit is necessarily a Gibbs measure – see for example Georgii's book [Geo88].

Note that although each $\mathsf{WPt}_{q,\beta,z}$ is a pure phase (*i.e.* an extremal Gibbs measure), the converse is not necessarily true.

Those Gibbs measures can be characterized by the fact that they can be obtained from a random-cluster measure by a procedure described below:

Proposition 2. *Let $q \in \{2, 3, \ldots\}$, $p \in [0, 1]$, S_q be a finite set with $|S_q| = q$ and $z \in S_q$. Pick a random edge configuration $X \in \{0, 1\}^{\mathbb{E}^d}$ according to the random-cluster measure $\phi^1_{p,q}$. Then, for each finite connected component $\mathcal{C}$ of X independently, pick a spin uniformly from S_q, and assign this spin to all vertices of $\mathcal{C}$. Finally assign the value z to all vertices of infinite connected components. The $S_q^{\mathbb{Z}^d}$-valued random spin configuration arising from this procedure is then distributed according to the Gibbs measure $\mathsf{WPt}_{q,\beta,z}$ for the q-state Potts model at inverse temperature $\beta := -\frac{1}{2}\log(1-p)$.*

3. Central Limit Theorem for the random cluster measure

The aim of this section is to prove the following theorem:

Theorem 1. *For each $q \geq 1$, there exists $p_r(q) < 1$ such that, for $p > p_r(q)$, $\mathcal{R}_{p,q}$ consists of a unique measure $\phi_{p,q}$ which satisfies the following: if C_∞ denotes the infinite cluster for FK percolation, then*

$$\frac{|\Lambda_n \cap C_\infty| - \theta(p,q)|\Lambda_n|}{|\Lambda_n|^{1/2}} \Longrightarrow \mathcal{N}(0, \sigma^2_{p,q}),$$

where $\theta(p,q) = \phi_{p,q}(0 \in C_\infty)$ and

$$\sigma^2_{p,q} = \sum_{k \in \mathbb{Z}^d} \big(\phi_{p,q}(0 \leftrightarrow \infty \text{ and } k \leftrightarrow \infty) - \theta(p,q)^2\big).$$

Particularly, we have

- $p_r(1) = p_c(\mathbb{Z}^d)$
- $p_r(q) = \frac{q(1-p_g)}{p_g + q(1-p_g)}$ *for $d = 2$.*
- $p_r(q) = \frac{\sqrt{q}}{1+\sqrt{q}}$ *for $d = 2$ and ($q = 1, q = 2$ or $q \geq 26$).*

We begin with a general theorem which gives sufficient conditions for having a central limit theorem for the fluctuations of the size of the intersection of large boxes with the infinite clusters. This will tell us what sort of estimates about random cluster measures can help us.

Theorem 2. *Let ϕ be a translation-invariant ergodic measure on $\mathcal{S}(\mathbb{L}^d)$. We suppose that ϕ satisfies the FKG inequalities and that we have $\theta_\phi = \phi(0 \leftrightarrow \infty) > 0$.*

For each $n \in \mathbb{Z}^d$ and $r > 0$, let us note the event $D_{n,r} = \{|C(n)| > r\}$. We suppose also that there exists a sequence $(r_n)_{n \in \mathbb{Z}^d}$ such that the following assumptions together hold:

- (m)
$$\sum_{n \in \mathbb{Z}^d} \phi(+\infty > |C(0)| \geq r_n) < +\infty$$
- (c)
$$\sum_{n \in \mathbb{Z}^d} Cov_\phi(D_{0,r_n}, D_{n,r_n}) < +\infty.$$

Then, we have

- (S^*)
$$\sigma_\phi^2 = \sum_{k\in\mathbb{Z}^d} \left(\phi(0\leftrightarrow\infty \textit{ and } k\leftrightarrow\infty) - \theta_\phi^2\right) < +\infty.$$
- (CLT)
$$\frac{|\Lambda_n \cap C_\infty| - \theta_\phi|\Lambda_n|}{|\Lambda_n|^{1/2}} \Longrightarrow \mathcal{N}(0,\sigma_\phi^2).$$

Proof.
$$|\Lambda_n \cap C_\infty(\omega)| - \theta_\phi|\Lambda_n| = \sum_{k\in\Lambda_n} f(T^k\omega),$$
where T^k is the translation operator defined by $T^k(\omega) = (\omega_{k+e})_{e\in\mathbb{E}^d}$ and $f = 1\!\!1_{\{|C(0)|=+\infty\}} - \theta_\phi$. Moreover, f is an increasing function and ϕ satisfies the F.K.G. inequalities. Then, $(f(T^k\omega))_{k\in\mathbb{Z}^d})$ is a stationary random field of square integrable variables satisfying the F.K.G. inequalities. Therefore, according to Newman's theorem [New80], the Central Limit Theorem is true if we prove that the quantity

$$\sum_{k\in\mathbb{Z}^d} \mathrm{Cov}(f, f\circ T^k) \tag{2}$$

is finite, which is just proving (S^*).

Let us define $B = \{|C(0)| = +\infty\}$, and for each $n \in \mathbb{Z}^d$, $A_n = \{|C(n)| = +\infty\}$, $\tilde{A}_n = \{|C(n)| \geq r_n\}$ and $\tilde{B}_n = \{|C(0)| \geq r_n\}$ Since $B \subset \tilde{B}_n$ and $A_n \subset \tilde{A}_n$, one has $1\!\!1_{\tilde{B}_n} = 1\!\!1_B + 1\!\!1_{\tilde{B}_n\backslash B}$ and $1\!\!1_{\tilde{A}_n} = 1\!\!1_{A_n} + 1\!\!1_{\tilde{A}_n\backslash A_n}$. It follows that

$$\mathrm{Cov}(1\!\!1_{\tilde{A}_n}, 1\!\!1_{\tilde{B}_n}) - \mathrm{Cov}(1\!\!1_{A_n}, 1\!\!1_B) = \mathrm{Cov}(1\!\!1_{\tilde{A}_n\backslash A_n}, 1\!\!1_{\tilde{B}_n}) + \mathrm{Cov}(1\!\!1_{\tilde{B}_n\backslash B}, 1\!\!1_{A_n}),$$

and hence that

$$\begin{aligned} |\mathrm{Cov}(1\!\!1_{\tilde{A}_n}, 1\!\!1_{\tilde{B}_n}) - \mathrm{Cov}(1\!\!1_{A_n}, 1\!\!1_B)| &\leq P(\tilde{A}_n\backslash A_n) + P(\tilde{B}_n\backslash B) \\ &\leq 2P(+\infty > |C(0)| \geq r_n) \end{aligned}$$

It follows that

$$\sigma_\phi^2 \leq 2\sum_{n\in\mathbb{Z}^d} P(+\infty > |C(0)| \geq r_n) + \sum_{n\in\mathbb{Z}^d} |\mathrm{Cov}(D_{0,r_n}, D_{n,r_n})| < +\infty.$$

□

As we mentioned in the introduction, the idea of using Newman's theorem to prove Central Limit Theorems for the density of infinite clusters in percolation models satisfying the F.K.G. inequalities is not new; indeed it has already been pointed out by Newman and Schulman [NS81a, NS81b] that $(S^*) + (FKG) \Longrightarrow (CLT)$. The interest of our theorem is that it gives a concrete way to prove that assumption (S^*) is satisfied; basically, it splits a problem about infinite clusters into two problems relative to finite clusters:

- The existence of sufficiently high moments
- A control of the correlation for the appearance of reasonably large clusters in two points which are separated by a large distance – note that we can rewrite $\mathrm{Cov}(D_{0,r_n}, D_{n,r_n})$ as $\mathrm{Cov}(D^c_{0,r_n}, D^c_{n,r_n})$.

This method gives an alternative proof of a recent result that Zhang [Zha01] obtained by martingale techniques.

Corollary 1. *In the case of Bernoulli percolation, the density of the infinite cluster in large boxes satisfies a central limit theorem for each $p > p_c$.*

Proof. Simply take $r_n = \|n\|/3$. The convergence (m) follows for example from Kesten and Zhang[KZ90]: there exists $\eta(p) > 0$ such that

$$\forall n \in \mathbb{Z}_+ \quad P(|C(0)| = n) \leq \exp(-\eta(p)n^{(d-1)/d}).$$

Of course, such a sharp estimate is not necessary for our purpose. Estimates derived from [CCN87] or [CCG$^+$89] would have been sufficient. The convergence of (c) is an evidence since D_{0,r_n} and D_{n,r_n} are independent for $\|n\| > 12$.

□

3.1. **The case of the square lattice.** Let $\mathbb{Z}^2_* = \mathbb{Z}^2 + (1/2, 1/2)$, $\mathbb{E}^2_* = \{\{a,b\}; a, b \in \mathbb{Z}^2_*$ and $\|a-b\| = 1\}$ and $\mathbb{L}^2_* = (\mathbb{Z}^2_*, \mathbb{E}^2_*)$. It is easy to see that $\mathbb{L}^2_*$ is isomorphic to $\mathbb{L}^2$.

For each bond $e = \{a, b\}$ of $\mathbb{L}^2$ (*resp.* $\mathbb{L}^2_*$), let us denote by $s(e)$ the only subset $\{i, j\}$ of $\mathbb{Z}^2_*$ (*resp.* $\mathbb{Z}^2$) such that the quadrangle $aibj$ is a square in $\mathbb{R}^2$. s is clearly an involution.

For finite $A \subset \mathbb{Z}^2_*$, we denote by $\mathcal{P}\text{eierls}(A)$ the Peierls contour associated to A, that is

$$\mathcal{P}\text{eierls}(A) = \{e \in \mathbb{E}^2; 1\!\!1_A \text{is not constant on } s(e)\}.$$

If on the plane $\mathbb{R}^2$ we draw the edges which are in $\mathcal{P}\text{eierls}(A)$, we obtain a family of curves – the so-called Peierls contours – which are exactly the boundary of the subset of $\mathbb{R}^2$: $A + ([-1/2, 1/2] \times [-1/2, 1/2])$.

It is known that, provided that $A \subset \mathbb{Z}^2_*$ is a bounded connected subset of $\mathbb{L}^2_*$, there exists a unique set of bonds $\Gamma(A) \subset \mathcal{P}\text{eierls}(A) \subset \mathbb{E}^2$ which form a cycle surrounding A, in the following sense that every infinite connected subset of bonds $D \subset \mathbb{E}^2_*$ satisfying $D \cap A \neq \varnothing$ also satisfies $D \cap s(\Gamma(A)) \neq \varnothing$.

Now consider the map

$$\begin{array}{ccc} \{0,1\}^{\mathbb{E}^2} & \to & \{0,1\}^{\mathbb{E}^2_*} \\ \omega & \mapsto & \omega^* = (1 - \omega_{s(e)})_{e \in \mathbb{E}^2_*} \end{array}$$

For $\eta \in \{0,1\}^{\mathbb{E}^2_*}$, we also denote by η^* the only $\omega \in \{0,1\}^{\mathbb{E}^2}$ such that $\omega^* = \eta$.

For each subset A of $\{0,1\}^{\mathbb{E}^2}$ (*resp.* $\{0,1\}^{\mathbb{E}^2_*}$), we denote by A^* the set $\{\omega^*; \omega \in A\}$.

The following planar duality between planar random cluster measures is now well known: let us define p^* to be the unique element of $[0,1]$ which satisfies $F(p)F(p^*) = 1$, with $F(x) = \frac{1}{\sqrt{q}} \frac{x}{1-x}$. and also define a map t by

$$\begin{array}{ccc} \{0,1\}^{\mathbb{E}^2} & \to & \{0,1\}^{\mathbb{E}^2_*} \\ \omega & \mapsto & (\omega_{e+(1/2,1/2)})_{e \in \mathbb{E}^2_*} \end{array}$$

Then, for each $p \in [0,1]$, $b \in \{0,1\}$ and each event A, we have

$$\phi^b_{p,q}(A) = t\phi^{1-b}_{p^*,q}(A^*).$$

Let us define

$$p_r(q) = p_g(q)^*. \tag{3}$$

Since $p_g(q) > 0$, we have $p_r(q) < 1$. Note that it is believed that $p_g(q) = p_c(q)$. As was noted by Grimmett and Piza [GP97], the fact that $p_g(q) = p_c(q)$ would imply that p_c is the solution of the equation $x = x^*$, *i.e.* $p_c = \frac{\sqrt{q}}{1+\sqrt{q}}$. So, it follows that we have $p_r(q) = p_c(q)$ provided that $p_g(q) = p_c(q)$. We recall that when $d = 2$, this conjecture has already be proved for $q = 1, 2$ and $q \geq 26$.

Lemma 1. *Let $d = 2$ and $p < p_g(q)$. There exists $K \in (0, +\infty)$ and $\gamma(p, q) > 0$ with*

$$\forall n \in \mathbb{Z}_+ \quad \phi_{p,q}(|C(0)| \geq n) \leq K \exp(-\gamma(p,q)\sqrt{n/2}).$$

Proof. Suppose that $n \geq 16$ and denote by r the integer part of $\sqrt{n/2} - 1$. Let $T = \{k \in \mathbb{Z}_+; C(0) \cap \partial B(0,k) \neq \varnothing\}$ and $R = \max T$.
Suppose $|C(0)| \geq n$: we have $C(0) \subset B(0,R)$, so

$$n \leq |C(0)| \leq |B(0,R)| = 1 + 2R(R+1).$$

It follows that $r \leq R$. Since $C(0)$ is connected, we have $0 \leftrightarrow \partial B(0,r)$. Then $\phi_{p,q}(|C(0)| \geq n) \leq \phi_{p,q}(0 \leftrightarrow \partial B(0,r))$. The result follows then from Proposition 1. □

When $d = 2$, it is known that $p_g \leq p_c \leq \frac{\sqrt{q}}{1+\sqrt{q}}$. It follows that $p_g^* \geq p_c^* \geq \frac{\sqrt{q}}{1+\sqrt{q}}$. Then $(p > p_g^*) \Longrightarrow (p > p_c^*)$. By a duality argument, the uniqueness of the random cluster measure for $p < p_c$ implies the uniqueness of the random cluster measure for $p > p_c^*$, and then for $p > p_r(q)$. Then, we simply write $\phi_{p,q}$ without any superscript.

Lemma 2. *Let $d = 2$ and $p > p_r(q)$. There exists $K \in (0, +\infty)$ with*

$$\forall n \geq 1 \quad \phi_{p,q}(|C(0)| = n) \leq Kne^{-\gamma(p^*,q)\sqrt{n}}.$$

Note that $\gamma(p^*, q) > 0$.

Proof. Here we use a duality argument. Let $p > p_r(q)$ and note $A = \{|C(0)| = n\}$. We have $\phi_{p,q}(A) = t\phi_{p^*,q}(A^*)$. In this case

$$t^{-1}(A^*) = \left\{ \begin{array}{c} \text{there exists at least one open cycle surrounding } (0,1/2) \\ \text{Those of these cycles which minimizes the distance to } (0,1/2) \\ \text{surrounds exactly } n \text{ closed bonds.} \end{array} \right\}.$$

The number of bonds used by this cycle is at least $2n + 2$. Moreover, the position of the first intersection of this cycle with the positive x-axis is at most n. So

$$t^{-1}(A^*) \subset \bigcup_{k=1}^{n} \{|C((k,0))| \geq 2n\}.$$

It follows then from lemma 1 that

$$\phi_p(A) = t\phi_{p^*,q}(A^*) \leq Kne^{-\gamma(p^*,q)\sqrt{n}}.$$

□

We must also recall a decoupling property of the random cluster measure which will be very useful.

Lemma 3. ***Decoupling lemma***
*Let F be a finite connected subset of $\mathbb{E}^d$ such that $\mathbb{E}^d \backslash F$ has two connected components in $\mathbb{L}^d$. We denote by $Int(F)$ (*resp. *$Ext(F)$) the bounded (*resp. *unbounded) connected component of $\mathbb{E}^d \backslash F$ and $\overline{Int}(F) = Int(F) \cup F$ (*resp. *$\overline{Ext}(F) = Ext(F) \cup F$).*

Now consider the event $W_F = \{\forall e \in F; \omega_e = 1\}$.

Then, for each $\sigma(\overline{Int}(F))$ measurable local event M_1 and each each $\sigma(\overline{Ext}(F))$ measurable local event M_2, we have the decoupling property:

$$\forall b \in \{0,1\} \quad \phi^b_{p,q}(M_1 \cap M_2 | W_F) = \phi^b_{p,q}(M_1|W_F)\phi^b_{p,q}(M_2|W_F) \tag{4}$$

Proof. Let $E' = \mathbb{E}_{\Lambda_t}$ be a large box such that M_1 and M_2 are $\sigma(E')$-measurable: for each $\omega \in W_F$ with $\omega_e = b$ for e outside of E' , we have

$$Z^b_{E',p,q}\phi^b_{E',p,q} = \left\{ \prod_{e \in E'} p^{\omega(e)}(1-p)^{1-\omega(e)} \right\} q^{k_{E'}(\omega_{E'}\eta_{E'^c})}$$

For such an ω, we can write

$$k_{E'}(\omega_{E'}\eta_{E'^c}) = k_{\mathrm{Int}(F)}(\omega_{\mathrm{Int}(F)}1_{\overline{\mathrm{Ext}}(F)}) + k_{E' \backslash \overline{\mathrm{Int}}(F)}(\omega_{E' \backslash \overline{\mathrm{Int}}(F)}1_{\overline{\mathrm{Int}}(F)}b_{\overline{\mathrm{Ext}}(E')}) - 1$$

It follows that, conditioning by W_F, $\omega_{\mathrm{Int}(F)}$ and $\omega_{E' \backslash \overline{\mathrm{Int}}(F)}$ are independent under $\phi^b_{E',p,q}$:

$$\phi^b_{E',p,q}(M_1 \cap M_2|W_F) = \phi^b_{E',p,q}(M_1|W_F)\phi^b_{E',p,q}(M_2|W_F).$$

Letting t tend to infinity, we obtain equation (4). □

Note that in the two-dimensional lattice, the set of bonds of a cycle satisfies the assumptions of the decoupling lemma.

The goal of the next lemma is to bound the covariance of two decreasing events that are defined by the state of the bonds in two boxes separated by a large distance. It is clear that it does not pretend to originality and that its use could have been replaced by those of an analogous result of the literature, *e.g.* Theorem 3.4 of Alexander [Ale98] joined to its Remark 3.5. Nevertheless, we preferred to present our lemma because its proof is rather short and allows an instructive comparison with the case of a higher dimension which will be studied later.

Lemma 4. *Let $q \geq 1$. For each $p > p_r(q)$, there exists $C > 0$ and $\alpha > 0$ such that for each couple of boxes $\Lambda_1, \Lambda_2 \subset \mathbb{Z}^2$ and each pair of monotone events A and B, with A (*resp. *B) $\sigma(\Lambda_1)$ (*resp. *$\sigma(\Lambda_2)$) measurable, we have*

$$|Cov_\phi(A,B)| = |\phi_{p,q}(A \cap B) - \phi_{p,q}(A)\phi_{p,q}(B)| \leq C|\partial\Lambda_1|e^{-\alpha d(\Lambda_1,\Lambda_2)}.$$

Proof. Since $\mathrm{Cov}_\phi(A,B) = -\mathrm{Cov}_\phi(A^c,B) = \mathrm{Cov}_\phi(A^c,B^c) = -\mathrm{Cov}_\phi(A,B^c)$, we can assume that A and B are decreasing events. We can also assume without loss of generality that $\Lambda_1 = \{-n,\dots,n\} \times \{-p,\dots,p\}$. Put $\Lambda_1^* = \{-n+1/2,\dots,n-1/2\} \times \{-p+1/2,\dots,p-1/2\}$. For $x \in \mathbb{Z}^2_*$ and $\omega \in \Omega$, let us define $C^*(x)(\omega)$ to be the connected component of x in the configuration ω^*. Let now be F the random subset of $\mathbb{E}^2$ defined by

$$F(\omega) = \Gamma(\bigcup_{y \in \Lambda_1^*} C^*(y)(\omega))$$

and consider the event $V = \{\overline{\mathrm{Int}}(F) \cap \Lambda_2 = \varnothing\}$. The following facts are elementary, but relevant:

- For every cycle $\gamma \subset \mathbb{E}^2$ surrounding the origin, the event $\{F = \gamma\}$ is $\sigma(\overline{\mathrm{Int}}(\gamma))$-measurable.
- For any subset γ of $\mathbb{E}^2$, $\{F = \gamma\} \subset W_\gamma = \{\forall e \in \gamma; \omega_e = 1\}$.

By the decoupling lemma, we know that if T is a $\sigma(\overline{\mathrm{Ext}}(\gamma))$-measurable event and R a $\sigma(\overline{\mathrm{Int}}(\gamma))$-measurable local event, we have the decoupling property:

$$\phi_{p,q}(R \cap T|W_\gamma) = \phi_{p,q}(R|W_\gamma)\phi_{p,q}(T|W_\gamma).$$

So, if $\overline{\mathrm{Int}}(\gamma) \cap \Lambda_2 = \varnothing$ we have

$$\begin{aligned}
\phi_{p,q}(A \cap B \cap \{F = \gamma\}) &= \phi_{p,q}(A \cap B \cap \{F = \gamma\} \cap W_\gamma) \\
&= \phi_{p,q}(W_\gamma)\phi_{p,q}(A \cap \{F = \gamma\}) \cap B|W_\gamma) \\
&= \phi_{p,q}(W_\gamma)\phi_{p,q}(A \cap \{F = \gamma\})|W_\gamma)\phi_{p,q}(B|W_\gamma) \\
&= \phi_{p,q}(A \cap \{F = \gamma\}) \cap W_\gamma)\phi_{p,q}(B|W_\gamma) \\
&= \phi_{p,q}(A \cap \{F = \gamma\})\phi_{p,q}(B|W_\gamma) \\
&\leq \phi_{p,q}(A \cap \{F = \gamma\})\phi_{p,q}(B).
\end{aligned}$$

The last inequality follows from the fact that B is a decreasing event, whereas W_γ is an increasing one.

If we sum over suitable values of γ, we get

$$\phi_{p,q}(A \cap B \cap V) \leq \phi_{p,q}(A \cap V)\phi_{p,q}(B) \leq \phi_{p,q}(A)\phi_{p,q}(B).$$

Since A and B are decreasing events, they are positively correlated, then

$$0 \leq \phi_{p,q}(A \cap B) - \phi_{p,q}(A)\phi_{p,q}(B) \leq \phi_{p,q}(A \cap B \cap V^c) \leq \phi_{p,q}(V^c).$$

Since

$$V^c \subset \bigcup_{y \in \partial\Lambda_1^*} \{y \leftrightarrow \partial B(y, d(\Lambda_1, \Lambda_2))\},$$

the result follows from the inequality of Grimmett and Piza. $\square$

3.2. **The general case.** The goal of the next lemma consists in bounding the covariance of two monotone events that are defined by the state of the bonds in two boxes separated by a large distance.

Its proof, unlike the one of lemma 4, can not use duality arguments. We nevertheless attempt to present this proof in a form which is as close as possible to those of lemma 4 to highlight the differences and the similarities between them.

Note that the proof is inspired by the proof of Grimmett [Gri95] for the uniqueness of the random-cluster when p is large.

Lemma 5. *Let $q \geq 1$. There exists $p'(q) < 1$ such that for each $p > p'(q)$, there exists $\alpha(p, q) > 0$ such that for each couple of finite connected volumes $\Lambda_1 \subset \mathbb{Z}^d$ and $\Lambda_2 \subset \mathbb{Z}^d$ and each pair of monotone events A and B, with A (*resp. *B) $\sigma(\Lambda_1)$ (*resp. *$\sigma(\Lambda_2)$) measurable, we have*

$$|Cov_\phi(A, B)| = |\phi_{p,q}(A \cap B) - \phi_{p,q}(A)\phi_{p,q}(B)| \leq C|\partial\Lambda_1|e^{-\alpha d(\Lambda_1, \Lambda_2)}.$$

Proof. We begin by a topological remark: let D be a finite connected subset of $\mathbb{Z}^d$. Since D is bounded, D^c has only finitely many connected components and exactly one of them is unbounded.

We denote by $\mathrm{Fill}(D)$ the reunion of D with the finite connected components of D^c. Clearly, $\mathrm{Fill}(D)$ is connected, too. Let us note $D' = \partial\mathrm{Fill}(D)$. It is easy to see that D' surrounds D, in the following sense that every infinite path starting in D must meet D'. It is important to note that D' is $*$-connected. Although it seems to be evident, the proof of this fact is rather arduous, see lemma 2.1 in Deuschel and Pisztora [DP96].

Let us now define $W(D) = \{e \in \mathbb{E}^d; e \cap D' \neq \varnothing\}$. Since D' is $*$-connected, $W(D)$ is connected in $\mathbb{L}^d$; $W(D)$ also surrounds D. Note that $W(D)$ is analogous to a surrounding Peierls contour in the two dimensional lattice.

As before, suppose now that A and B are decreasing events. Given a configuration ω, say that a point $x \in \mathbb{Z}^d$ is wired if every bond $e = \{a, b\}$ with $\|x - a\|_\infty \le 1$ and $\|x - b\|_\infty \le 1$ satisfies $\omega_e = 1$. Otherwise, say that x is free. Let us define $D(\omega)$ to be the set of points in $\mathbb{Z}^d \backslash \Lambda_1$ which can be connected to Λ_1 using only free vertices – the origin of the path in Λ_1 does not need to be free. By definition of D, $(\Lambda_1 \cup D)$ is a connected set.

Let us consider the random set $F = W(D)$ and define the event $V = \{\Lambda_2 \cap \mathrm{Fill}(F) = \varnothing\}$.

Note that V is an increasing event.

The following facts are elementary, but relevant:

- For any set γ satisfying the decoupling lemma and surrounding Λ_1, the event $\{F = \gamma\}$ is $\sigma(\overline{\mathrm{Int}}(\gamma))$-measurable.
- For any subset γ of $\mathbb{E}^d$, $\{F = \gamma\} \subset W_\gamma = \{\forall e \in \gamma; \omega_e = 1\}$.

As previously, if T is a $\sigma(\overline{\mathrm{Ext}}(\gamma))$-measurable event and R a $\sigma(\overline{\mathrm{Int}}(\gamma))$-measurable local event, then we have the following decoupling property:

$$\phi^0_{p,q}(R \cap T | W_\gamma) = \phi^0_{p,q}(R|W_\gamma)\phi^0_{p,q}(T|W_\gamma).$$

So, if γ does neither touch nor surround Λ_2, we have

$$\begin{aligned}
\phi^0_{p,q}(A \cap B \cap \{F = \gamma\}) &= \phi^0_{p,q}(A \cap B \cap \{F = \gamma\} \cap W_\gamma) \\
&= \phi^0_{p,q}(W_\gamma)\phi^0_{p,q}(A \cap \{F = \gamma\}) \cap B|W_\gamma) \\
&= \phi^0_{p,q}(W_\gamma)\phi^0_{p,q}(A \cap \{F = \gamma\})|W_\gamma)\phi^0_{p,q}(B|W_\gamma) \\
&= \phi^0_{p,q}(A \cap \{F = \gamma\}) \cap W_\gamma)\phi^0_{p,q}(B|W_\gamma) \\
&= \phi^0_{p,q}(A \cap \{F = \gamma\})\phi^0_{p,q}(B|W_\gamma) \\
&\le \phi^0_{p,q}(A \cap \{F = \gamma\})\phi^0_{p,q}(B).
\end{aligned}$$

If we sum over suitable values of γ, we get

$$\phi^0_{p,q}(A \cap B \cap V) \le \phi^0_{p,q}(A \cap V)\phi^0_{p,q}(B) \le \phi^0_{p,q}(A)\phi^0_{p,q}(B).$$

Since A and B are decreasing events, they are positively correlated, then

$$0 \le \phi^0_{p,q}(A \cap B) - \phi^0_{p,q}(A)\phi^0_{p,q}(B) \le \phi^0_{p,q}(A \cap B \cap V^c) \le \phi^0_{p,q}(V^c).$$

Since V is an increasing event, we can use the stochastic domination of a product measure by $\phi^0_{p,q}$: $\phi^0_{r,1} \prec \phi^0_{p,q}$, with $r = p/(p + (1-p)q)$, thus $\phi^0_{p,q}(V^c) \le \phi^0_{r,1}(V^c)$.

It remains to prove, for large p, the existence of C and $\alpha > 0$ such that $\phi^0_{r,1}(V^c) \le C|\partial\Lambda_1|e^{-\alpha n}$, where $n = d(\Lambda_1, \Lambda_2) - 2$.

Since the random field $(1\!\!1_{x \text{ is free}})_{x \in \mathbb{Z}^d}$ is M-dependent with $\lim_{r \to 1} \phi^0_{r,1}(0 \text{ is free}) = 0$, it follows from a theorem of Liggett, Schonmann and Stacey [LSS97] that the field $(1\!\!1_{x \text{ is free}})_{x \in \mathbb{Z}^d}$ is stochastically dominated by a product of Bernoulli measures of parameter $\frac{1}{2d}$ as soon as p is large enough.

Now, a counting argument gives

$$\phi^0_{r,1}(V^c)| \le |\partial\Lambda_1|(2d)(2d-1)^n(1/(2d))^n,$$

where $n = d(\Lambda_1, \Lambda_2) - 1$, which completes the proof. □

We can now prove Theorem 1.

Proof. The uniqueness of the random cluster measure for p close to 1 has been proved by Grimmett [Gri95]. To prove that the uniqueness holds for $p > p_r(q)$ in the cases where we have announced a convenient value for $p_r(q)$, simply note that when $q = 1$, the uniqueness is obvious and that we have already remarked that there was uniqueness for $d = 2$ and $p > p_g^*$.

Let us now prove the Central Limit Theorem. We will apply Theorem 2 to the sequence $r_n = \|n\|/4$.

Let us show that

$$\sum_{n\in\mathbb{Z}^d} P(+\infty > |C(0)| \geq \frac{\|n\|}{4})$$

converges.

- For $d \geq 3$ and p sufficiently close to 1, this follows from the estimate of Pisztora [Pis96]: for each $b \in \{0,1\}$ and each $p > p_{\text{slab}}$, there exists a constant $a = a(p,q)$ with

$$\forall n \geq 0 \quad \phi^b_{p,q}(|C| = n) \leq \exp\big(-an^{(d-1)/d}\big). \tag{5}$$

- For $d = 2$ and $p > p_g(q)^*$, it follows from our lemma 2.

Now, it remains to prove that

$$\sum_{n\in\mathbb{Z}^d} \text{Cov}(1\!\!1_{\tilde{A}_n}, 1\!\!1_{\tilde{B}_n}) < +\infty, \tag{6}$$

with $\tilde{A}_n = \{|C(0)| \geq r_n\}$ and $\tilde{B}_n = \{|C(n)| \geq r_n\}$.

Put $\Lambda_n = B(n, \|n\|/3)$ and $\Lambda'_n = B(0, \|n\|/3)$. It is clear that $\tilde{A}_n$ (*resp.* $\tilde{B}_n$) is $\sigma(\Lambda_n)-$ (*resp.* $\sigma(\Lambda'_n)-$) measurable. It is obvious that $\tilde{A}_n$ and $\tilde{B}_n$ are increasing events. Then, we can apply lemma 5. Since $d(\Lambda_n, \Lambda'_n) \geq n/3$, it follows that

$$0 \leq \text{Cov}(1\!\!1_{\tilde{A}_n}, 1\!\!1_{\tilde{B}_n}) \leq Kn^{d-1}e^{-\frac{\alpha(p,q)}{3}n},$$

which forms a convergent series as soon as $p > p'(q)$. When $d = 2$, the result follows similarly from lemma 4. □

Before considering applications of our theorem 1, we try to highlight the nature of the gap between our theorem and our natural wish (that is $p_r(q) = p_c(q)$ or at least $p_{\text{slab}}(q)$). Actually, we need a substitute for lemma 5 for $p > p_c$. The main tool for proving our lemma 5 was to exhibit sets satisfying the assumptions of the decoupling lemma. Is it always possible to produce with large probability such sets? An affirmative answer could also lead to a proof of the famous conjecture about the uniqueness of the random cluster measure. This clearly shows the difficulty of the task.

4. Random coloring of clusters

If G is a subgraph of $\mathbb{L}^d$, $s \in \mathbb{R}$ and if ν is a probability measure on $\mathbb{R}$, we define the color-probability $P^{G,\nu,s}$ as follows: $P^{G,\nu,s}$ is the unique measure on $(\mathbb{R}^{\mathbb{Z}^d}, \mathcal{B}(\mathbb{R}^{\mathbb{Z}^d}))$ under which the canonical projections X_i – defined, as usual by $X_i(\omega) = \omega_i$ – satisfy

- For each $i \in \mathbb{Z}^d$, the law of X_i is
  - ν if $|C(i)| < +\infty$.
  - δ_s otherwise.
- For each independent set $S \subset \mathbb{Z}^d$, the variables $(X_i)_{i\in S}$ are independent.

- For each connected set $S \subset \mathbb{Z}^d$, the variables $(X_i)_{i \in S}$ are identical.

Let ϕ be a measure on $\mathcal{S}(\mathbb{L}^d)$ which satisfies the following assumptions:

- (E): ϕ is a translation-invariant ergodic measure on $\mathcal{S}(\mathbb{L}^d)$.
- (M): $\exists \alpha > d \quad \sum_{k=1}^{+\infty} k^\alpha P(|C(0)| = k) < +\infty$.

In this case, we define

$$\chi^f(\phi) = \sum_{k=1}^{+\infty} k P(|C(0)| = k). \tag{7}$$

When $\theta_\phi = \phi(0 \leftrightarrow \infty) > 0$, the following assumption will also be considered:

$$(CLT) : \exists \sigma_\phi^2 > 0, \frac{|\Lambda_n \cap C_\infty| - \theta_\phi |\Lambda_n|}{|\Lambda_n|^{1/2}} \Longrightarrow \mathcal{N}(0, \sigma_\phi^2).$$

The randomized color-measure associated to ϕ is defined by

$$P^{\phi,\nu,s} = \int_{\mathcal{S}(\mathbb{L}^d)} P^{G,\nu,s} \ d\phi(G).$$

We emphasize that the results which will be proved here are not restricted to the case where ϕ is a random cluster measure. Nevertheless, to motivate this amount of generality, let us give at once some examples of models covered by randomized color-measures when $\phi = \phi_{p,q}^b$.

- The case $q = 1$ is a generalization of the divide and color model of Häggström [Häg01], which has already been studied in a earlier paper of the author [Gar01].
- The most celebrated of the randomized color-measure is obtained when $q \geq 2$ is an integer and $\nu = \frac{1}{q}(\delta_1 + \delta_2 + \cdots + \delta_q)$. In this case $P^{\phi,\nu,s}$ is the Gibbs measure $\mathsf{WPt}_{q,\beta,s}$ for the q-state Potts model on $\mathbb{Z}^d$ at inverse temperature $\beta := -\frac{1}{2}\log(1-p)$, according to Proposition 2. It includes of course the case of the Ising model.
- If $n_1, n_2, \ldots n_k$ are positive integers with $n_1 + n_2 + \cdots + n_k = q$ and we take $\nu = \frac{1}{q}(n_1\delta_1 + n_2\delta_2 + \ldots n_k\delta_k)$, we obtain a fuzzy Potts model. It obviously follows from the previous example and the definition of a fuzzy Potts model, see [MVV95, Häg99, Häg03].

We begin with a general property of randomized color-measures.

Theorem 3. *$P^{\phi,\nu,s}$ is translation invariant and the action of $\mathbb{Z}^d$ on $P^{\phi,\nu,s}$ is ergodic.*

Proof. Let $\Omega = \{0,1\}^{\mathbb{E}^d}$, $\Omega_t = [0,1]^{\mathbb{Z}^d}$, $\Omega_c = \mathbb{R}^{\mathbb{Z}^d}$, $\Omega_3 = \Omega \times \Omega_t \times \Omega_c$. As usually, $\mathbb{Z}^d$ acts on Ω_3 by translation, with for each $k, n, p \in \mathbb{Z}^d, (\omega, \omega_t, \omega_c) \in \Omega_3, \{x, y\} \in \mathbb{E}^d$:

$$k.(\omega, \omega_t, \omega_c)(\{x, y\}, n, p) = (\omega, \omega_t, \omega_c)(\{k + x, k + y\}, k + n, k + p).$$

Let us denote by $U[0,1]$ the uniform distribution on $[0,1]$ and consider the action of $\mathbb{Z}^d$ on $(\Omega_3, \mathcal{B}(\Omega_3), \phi \otimes U[0,1]^{\otimes \mathbb{Z}^d} \otimes \nu^{\otimes \mathbb{Z}^d})$. Since $\phi \otimes U[0,1]^{\otimes \mathbb{Z}^d} \otimes \nu^{\otimes \mathbb{Z}^d}$ is a direct product of an ergodic measure by two mixing measures, it follows that the action of $\mathbb{Z}^d$ on Ω_3 is ergodic – see Brown [Bro76] for instance.

Let us define $f : \Omega_3 \to \mathbb{R}$ by $f(\omega, \omega_t, \omega_c) = s$ if $|C(0)(\omega)| = +\infty$ or if there exist $x, y \in C(0)(\omega)$, with $x \neq y$ and $\omega_t(y) = \omega_t(x)$. (Note that the second case actually not happens under $\phi \otimes U[0,1]^{\otimes \mathbb{Z}^d} \otimes \nu^{\otimes \mathbb{Z}^d}$.) Otherwise, we define

$f(\omega,\omega_t,\omega_c)$ to be equal to $\omega_c(x)$, where x is the unique element of $C(0)(\omega)$ such that $(\omega_t)(x) = \max\{\omega_t(y); y \in C(0)\}$.

Now, if we define $(X_k)_{k\in\mathbb{Z}^d}$, by $X_k(\omega,\omega_t,\omega_c) = f(k.(\omega,\omega_t,\omega_c))$, it is not difficult to see that the law of $(X_k)_{k\in\mathbb{Z}^d}$ under $\phi\otimes U[0,1]^{\otimes\mathbb{Z}^d}\otimes\nu^{\otimes\mathbb{Z}^d}$ is $P^{\phi,\nu,s}$. Since a factor of an ergodic system is an ergodic system – see also Brown [Bro76] – , it follows that $P^{\phi,\nu,s}$ is ergodic under the action of $\mathbb{Z}^d$. □

4.1. **Normal fluctuations of sums for color-measures.** We will first present a "quenched" central limit theorem:

Theorem 4. *Suppose that ν is a probability measure on $\mathbb{R}$ with a second moment ant that ϕ satisfies (E) and (M). We put $m = \int_{\mathbb{R}} x\ d\nu(x)$ and $\sigma^2 = \int_{\mathbb{R}}(x-m)^2\ d\nu(x)$.*

For ϕ-almost all subgraphs G of $\mathbb{L}^d$, the following holds:

$$\frac{1}{|\Lambda_n|^{1/2}}\Big(\sum_{x\in\Lambda_n\backslash C_\infty}(X(x)-m)\Big) \Longrightarrow \mathcal{N}(0,\chi^f(\phi)\sigma^2)$$

where $C_\infty(G) = \{x\in\mathbb{Z}^d; x\leftrightarrow\infty\}$.

The following lemma will be very useful.

Lemma 6. *For each subgraph G of $\mathbb{L}^d$, let us denote by $(A_i)_{i\in J}$ the partition of G into connected components.*

Suppose that ϕ satisfies (E) and (M). Then, we have for ϕ-almost all G:

$$\lim_{n\to\infty}\frac{1}{|\Lambda_n|}\sum_{i\in J;|A_i|<+\infty}|A_i\cap\Lambda_n|^2 = \chi^f(\phi),$$

where

$$\chi^f(\phi) = \sum_{k=1}^{+\infty}k\phi(|C(0)|=k).$$

Proof. Let us define $C'(x)$ by

$$C'(x) = \begin{cases} C(x) & \text{if } |C(x)|<+\infty \\ \varnothing & \text{otherwise}\end{cases}$$

and $C'_n(x) = C'(x)\cap\Lambda_n$.

It is easy to see that

$$\sum_{i\in J;|A_i|<+\infty}|A_i\cap\Lambda_n|^2 = \sum_{x\in\Lambda_n}|C'_n(x)|. \tag{8}$$

We have $|C'_n(x)|\le|C(x)|$, and the equality holds if and only if $C'(x)\subset\Lambda_n$.

The quantity residing in connected components intersecting the boundary of Λ_n can be controlled using the conditions on the moments of the size of finite clusters.

We have

$$\begin{aligned}\sum_{k=1}^{+\infty}n^{d-1}\phi(+\infty>|C(0)|\ge n^{d/\alpha}) &\le \int_0^{+\infty}x^{d-1}\phi(+\infty>|C(0)|\ge n^{d/\alpha}) \\ &= \frac{1}{d}\int_{|C(0)|<+\infty}|C(0)|^\alpha\ d\phi.\end{aligned}$$

It follows from a standard Borel-Cantelli argument that for ϕ-almost all G, there exists a (random) N such that

$$\forall n \geq N \quad (\|x\|_\infty = n) \Longrightarrow (|C'(x)| \leq n^{d/\alpha}). \tag{9}$$

Thus, for large n and for each $x \in \Lambda_{n-n^{d/\alpha}}$, $C'(x)$ is completely inside Λ_n, which means that $C'(x) = C'_n(x)$. Then,

$$\begin{aligned} \sum_{x\in\Lambda_{n-n^{d/\alpha}}} |C'(x)| &\leq \sum_{x\in\Lambda_n} |C'_n(x)| \leq \sum_{x\in\Lambda_n} |C'(x)| \\ \frac{1}{|\Lambda_n|} \sum_{x\in\Lambda_{n-n^{d/\alpha}}} |C'(x)| &\leq \frac{1}{|\Lambda_n|} \sum_{x\in\Lambda_n} |C'_n(x)| \leq \frac{1}{|\Lambda_n|} \sum_{x\in\Lambda_n} |C'(x)|. \end{aligned}$$

By the ergodic Theorem, we have ϕ-almost surely:

$$\lim_{n\to+\infty} \frac{1}{|\Lambda_n|} \sum_{x\in\Lambda_n} |C'(x)| = \int |C'(0)| \, d\phi = \chi^f(\phi).$$

Since $\lim_{n\to+\infty} \frac{|\Lambda_{n-n^{d/\alpha}}|}{|\Lambda_n|} = 1$, the result follows.

□

Remark: Besides technical details, the key point of this proof is the identity (8). It is interesting to note that Grimmett [Gri99] used an analogous trick to prove that $\lim_{n\to+\infty} k(n)/|\Lambda_n| = \kappa(p)$ almost surely, when $k(n)$ is the number of open clusters in Λ_n.

We can now prove Theorem 4.

Proof. Let $(a_i)_{i\geq 1}$ be a family of elements of $\mathbb{Z}^d$ which represent the connected components of the graph G.

$$\sum_{x\in\Lambda_n\backslash C_\infty} (X(x) - m) = \sum_{i=1}^{+\infty} |C'_n(a_i)|(X(a_i) - m)$$

Then

$$\frac{1}{|\Lambda_n|^{1/2}} \sum_{x\in\Lambda_n\backslash C_\infty} (X(x) - m) = \big(\frac{s_n^2}{|\Lambda_n|}\big)^{1/2} \frac{1}{s_n} \sum_{i=1}^{+\infty} |C'_n(a_i)|(X(a_i) - m),$$

with

$$s_n^2 = \sum_{i=1}^{+\infty} |C'_n(a_i)|^2.$$

By lemma 6, we have for ϕ-almost all G $\lim_{n\to+\infty} \frac{s_n^2}{|\Lambda_n|} = \chi^f(\phi)$.

Now, it remains to prove that

$$\frac{1}{s_n} \sum_{i=1}^{+\infty} |C'_n(a_i)|(X(a_i) - m) \Longrightarrow \mathcal{N}(0, \sigma^2). \tag{10}$$

Therefore, we will prove that for ϕ-almost all G, the sequence $Y_{n,k} = |C'_n(a_i)|(X(a_i) - m)$ satisfies the Lindeberg condition. For each $\epsilon > 0$, we have

$$\begin{aligned}\sum_{k=1}^{+\infty} \frac{1}{s_n^2} \int_{|Y_{n,k}|\ge \epsilon s_n} Y_{n,k}^2 \ dP^{G,\nu} &= \sum_{k=1}^{+\infty} \frac{|C'_n(a_k)|^2}{s_n^2} \int_{|C'_n(a_k)||x|\ge \epsilon s_n} (x-m)^2 \ d\nu(x) \\ &\le \int_{|x|\ge \frac{\epsilon}{\eta_n}} (x-m)^2 \ d\nu(x),\end{aligned}$$

with $\eta_n = \frac{\sup_{k\ge 1} |C'_n(a_k)|}{s_n}$. Thus, the Lindeberg condition is fulfilled if $\lim \eta_n = 0$. But we have already seen that $s_n \sim (\chi^f(\phi)|\Lambda_n|)^{1/2}$, whereas equation (9) gives $\sup_{k\ge 1} |C'_n(a_k)| = O(n^{d/\alpha}) = o(n^{d/2})$. This concludes the proof.

□

We can now pass to the "annealed" central limit theorem.

Theorem 5. *Let ϕ be a measure on $\mathcal{S}(\mathbb{L}^d)$ that satisfies (E) and (M). Let ν be a probability measure on $\mathbb{R}$ with a second moment. We put $m = \int_{\mathbb{R}} x \ d\nu(x)$ and $\sigma^2 = \int_{\mathbb{R}} (x-m)^2 \ d\nu(x)$. Let also $s \in \mathbb{R}$.*

- *If $\theta_\phi = 0$, then we have under $P^{\phi,\nu,s}$*

$$\frac{1}{|\Lambda_n|^{1/2}} \sum_{x\in\Lambda_n} (X(x) - m) \Longrightarrow \mathcal{N}(0, \chi^f(\phi)\sigma^2)$$

- *If $(\theta_\phi > 0)$ and (CLT) hold , then we have under $P^{\phi,\nu,s}$*

$$\frac{1}{|\Lambda_n|^{1/2}} \quad \sum_{x\in\Lambda_n} X(x) - ((1-\theta_\phi)m + \theta_\phi s)|\Lambda_n| \quad \Longrightarrow \mathcal{N}(0, \chi^f(\phi)\sigma^2 + (s-m)^2\sigma_\phi^2).$$

Proof. When $\theta_\phi = 0$, the result easily follows from Theorem 4. So, let us suppose that $(\theta_\phi > 0)$ and (CLT) hold. In this proof, it will be useful to consider G as a random variable. Let $\Omega' = \mathcal{S}(\mathbb{L}^d) \times \mathbb{R}^{\mathbb{Z}^d}$ and define the probability $\mathbb{P}$ on $\mathcal{B}(\Omega')$ as a skew-product: for measurable $A \times B \in \mathcal{B}(\mathcal{S}(\mathbb{L}^d)) \times \mathcal{B}(\mathbb{R}^{\mathbb{Z}^d})$, we have $\mathbb{P}(A \times B) = \int_A P^{G,\nu,s}(B) \ d\phi(G)$. Then, the law of the marginals G and X are $\mathbb{P}_G = \phi$ and $\mathbb{P}_X = P^{\phi,\nu,s}$. As usually, the letter $\mathbb{E}$ will be used to denote an expectation – or a conditional expectation – under $\mathbb{P}$.

Rearranging the terms of the sum, we easily obtain

$$\sum_{x\in\Lambda_n} X(x) - ((1-\theta_\phi)m + \theta_\phi s)|\Lambda_n| = \sum_{x\in\Lambda_n\backslash C_\infty} (X(x)-m) + (s-m)(|C_\infty \cap \Lambda_n| - |\Lambda_n|\theta_\phi)$$

We will now put

$$Q_n = \frac{1}{|\Lambda_n|^{1/2}} \quad \sum_{x\in\Lambda_n} X(x) - ((1-\theta_\phi)m + \theta_\phi s)|\Lambda_n| \quad ,$$

and define

$$\forall t \in \mathbb{R} \quad \phi_{n,s}(t) = \mathbb{E} \ \exp(iQ_n t).$$

Thereby, we have

$$\phi_{n,s}(t) = \mathbb{E} \ \exp(-\frac{it}{|\Lambda_n|^{1/2}} \sum_{x\in\Lambda_n\backslash C_\infty} (X(x) - m) + (s-m)(|C_\infty \cap \Lambda_n| - |\Lambda_n|\theta_\phi))$$

Conditioning by $\sigma(G)$ and using the fact that C_∞ is $\sigma(G)$-measurable, we get $\phi_{n,s}(t) = \mathbb{E}\ f_n(t,.)g_n((s-m)t,.)$, with

$$\begin{aligned} f_n(t,\omega) &= \mathbb{E}\ \exp(-\frac{it}{|\Lambda_n|^{1/2}} \sum_{x\in\Lambda_n\backslash C_\infty} (X(x)-m)|\sigma(G) \\ &= \int \exp(-\frac{it}{|\Lambda_n|^{1/2}} \sum_{x\in\Lambda_n\backslash C_\infty(\omega)} (X(x)-m)\ dP^{G(\omega),\nu} \end{aligned}$$

and

$$g_n(t,\omega) = \exp(-\frac{it}{|\Lambda_n|^{1/2}}(|C_\infty(\omega)\cap\Lambda_n| - |\Lambda_n|\theta_\phi)).$$

By Theorem 4 we have for each $t\in\mathbb{R}$ and $P^{\phi,\nu,s}$-almost all ω: $\lim_{n\to+\infty} f_n(t,\omega) = \exp(-\frac{t^2}{2}\chi^f(\phi)\sigma^2)$ Then, by dominated convergence

$$\lim_{n\to+\infty} \mathbb{E}\ (f_n(t,.) - \exp(-\frac{t^2}{2}\chi^f(\phi)\sigma^2))g_n((s-m)t,.) = 0.$$

Next

$$\begin{aligned} \lim_{n\to+\infty} \mathbb{E}\ f_n(t,.)g_n((s-m)t,.) &= \lim_{n\to+\infty} \exp(-\frac{t^2}{2}\chi^f(\phi)\sigma^2)\mathbb{E}\ g_n((s-m)t,.) \\ &= \exp(-\frac{t^2}{2}\chi^f(\phi)\sigma^2)\exp(-\frac{t^2}{2}(s-m)^2\sigma_\phi^2) \end{aligned}$$

where the last equality follows from Proposition 1. We have just proved that

$$\lim_{n\to\infty} \phi_{n,s}(t) = \exp(-\frac{t^2}{2}(\chi^f(\phi)\sigma^2 + (s-m)^2\sigma_\phi^2)).$$

The result now follows from the Theorem of Levy. □

4.2. **Fluctuation of the empirical vector associated to coloring models.** We now specialize to the case where ν has a finite support (in other words, there is only a finite number of colors) and obtain theorems on the frequencies of occurring of each of the states in large boxes.

Definition Let q be an integer with $q\ge 2$ and consider a finite set $S_q = \{a_1,a_2,\dots,a_q\}$. For every $z\in S_q$ and each vector $\nu\in\mathbb{R}_+^q$ with $\nu_1+\dots+\nu_q = 1$, we denote by $\text{Color}_{\phi,\nu}^z$ the measure $P^{\phi,\nu',z}$, with $\nu' = \sum_{i=1}^q \nu_i\delta_{a_i}$.

For $\omega\in S_q^{\mathbb{Z}^d}$, and $\Lambda\subset\mathbb{Z}^d$, we note $n(\Lambda)(\omega) = (n_1(\Lambda)(\omega),\dots,n_q(\Lambda)(\omega))$, with $n_k(\Lambda)(\omega) = |\{x\in\Lambda;\omega_x = k\}|$. We also denote by $(e_1,\dots,e_q)$ the canonical basis of $\mathbb{R}^q$.

Theorem 6. *Let ϕ be a measure on $\mathcal{S}(\mathbb{L}^d)$ that satisfies (E),(M) and moreover $\theta_\phi = 0$ or (CLT). Let q be an integer with $q\ge 2$, $z\in S_q$, and $\nu\in\mathbb{R}_+^q$ with $\nu_1+\dots+\nu_q = 1$. Then, under $\text{Color}_{\phi,\nu}^z$, we have*

$$\frac{n(\Lambda_t) - |\Lambda_t|((1-\theta_\phi)\nu + \theta_\phi e_z)}{\sqrt{|\Lambda_t|}} \Longrightarrow \mathcal{N}(0,C),$$

where C is the matrix associated to the quadratic form

$$Q(b) = \chi^f(\phi)(\langle D_\nu b,b\rangle - \langle\nu,b\rangle^2) + \sigma_\phi^2\langle e_z-\nu,b\rangle^2,$$

with $D_\nu = \mathsf{diag}(\nu_1,\dots,\nu_q)$. In other words, C is the matrix of the map

$$b \mapsto \chi^f(\phi)(D_\nu b - \langle \nu, b\rangle \nu) + \sigma_\phi^2 \langle e_z - \nu, b\rangle (e_z - \nu).$$

Proof. Let us note $Q_t = \frac{n(\Lambda_t)-|\Lambda_t|((1-\theta_\phi)\nu+\theta_\phi e_z)}{\sqrt{|\Lambda_t|}}$ and let L be a random vector following $\mathcal{N}(0,C)$, where C is the covariance matrix denoted above.

We will prove

$$\forall b \in \mathbb{R}^d \quad \langle Q_t, b\rangle \Longrightarrow \langle L, b\rangle.$$

Using the theorem of Levy, it is easy to see that it is equivalent to say that $Q_t \Longrightarrow L$.

Let now $b \in \mathbb{R}^d$. For $x \in \mathbb{Z}^d$, let us note $Y_x = b_{X_x}$. We have

$$\begin{aligned}
\langle n(\Lambda_t), q\rangle &= \sum_{k=1}^q n_k(\Lambda_t) b_k = \sum_{k=1}^q \sum_{x\in\Lambda_t} \delta_{X_x}(k) b_k \\
&= \sum_{x\in\Lambda_t} \sum_{k=1}^q \delta_{X_x}(k) b_k = \sum_{x\in\Lambda_t} b_{X_x} \\
&= \sum_{x\in\Lambda_t} Y_x
\end{aligned}$$

Now put $m = \langle \nu, b\rangle$ and $s = b_r = \langle e_z, b\rangle$. We have

$$\langle Q_n, b\rangle = \frac{\Big(\sum_{x\in\Lambda_t} Y_x\Big) - |\Lambda_t|((1-\theta_\phi)m + \theta_\phi s)}{\sqrt{|\Lambda_t|}}.$$

Now if we define μ to be the image of ν by $k \mapsto b_k$, it is not difficult to see that the mean of μ is m and that the law of $(Y_k)_{k\in\mathbb{Z}^d}$ under $\text{Color}^z_{\phi,\nu}$ is $P^{\phi,\mu,s}$. Then, it follows from Theorem 5 that $\langle Q_n, b\rangle \Longrightarrow \mathcal{N}(0, Q(b))$, with $Q(b) = \chi^f(\phi)\sigma^2 + (s-m)^2\sigma_\phi^2)$, where σ^2 is the variance of ν. Finally, we get the explicit form

$$Q(b) = \chi^f(\phi)(\langle D_\nu b, b\rangle - \langle \nu, b\rangle^2) + \sigma_\phi^2 \langle e_z - \nu, b\rangle^2,$$

with $D_\nu = \mathsf{diag}(\nu_1,\dots,\nu_q)$. This concludes the proof. □

We are now interested in having, when $\theta_\phi > 0$, a version of theorem 6 in which the observed quantity does not depend on r. There are several reasons to motivate such a theorem: if we want to use this central limit theorem to test if a concrete physical system conforms to this model (have in mind an Ising or a Potts model for instance), we have a priori no reason to guess the r phase of the underlying theoretical system. There is also a theoretical motivation for such a theorem: if we get a theorem which does not depend on r, it will be easy to transfer it to any measure which resides in the convex hull of the measures $(\text{Color}^z_{\phi,\nu})_{z\in S_q}$.

Theorem 7. *Let ϕ be a measure on $\mathcal{S}(\mathbb{L}^d)$ that satisfies (E), (M) , $\theta_\phi > 0$ and (CLT). Let q be an integer with $q \ge 2$, $z \in S_q$, and $\nu \in \mathbb{R}^q_+$ with $\nu_1 + \dots + \nu_q = 1$. For $\Lambda \subset \mathbb{Z}^d$, we denote by R_Λ an element of S_q which realizes the maximum of $(n_\Lambda(k) - |\Lambda|(1-\theta_\phi)\nu(k))_{k\in S_q}$. Then, under $\text{Color}^z_{\phi,\nu}$, we have*

$$\frac{n(\Lambda_t) - |\Lambda_t|((1-\theta_\phi)\nu + \theta_\phi e_{R_{\Lambda_t}})}{\sqrt{|\Lambda_t|}} \Longrightarrow \mathcal{N}(0,C),$$

where C is the matrix associated to the quadratic form

$$Q(b) = \chi^f(\phi)(\langle D_\nu b, b\rangle - \langle \nu, b\rangle^2) + \sigma_\phi^2 \langle e_z - \nu, b\rangle^2,$$

with $D_\nu = \mathsf{diag}(\nu_1, \dots, \nu_q)$. In other words, C is the matrix of the map

$$b \mapsto \chi^f(\phi)(D_\nu b - \langle \nu, b\rangle \nu) + \sigma_\phi^2 \langle e_z - \nu, b\rangle (e_z - \nu).$$

Proof. Since $\text{Color}^z_{\phi,\nu}$ is ergodic, $\frac{n(\Lambda_t)}{|\Lambda_t|} = \frac{1}{|\Lambda_t|}\sum_{x\in\Lambda_t} e_{\omega_x}$ almost surely converges to the mean value of e_{ω_0}, that is $(1-\theta_\phi)\nu + \theta_\phi e_z$.
Asymptotically, we have $n(\Lambda_t) - |\Lambda_t|(1-\theta_\phi)\nu \sim |\Lambda_t|\theta_\phi e_z$. It follows that

$$R_{\Lambda_t} = z \quad \text{for large } t \quad \text{Color}^z_{\phi,\nu} - \text{almost surely.} \tag{11}$$

Now let g be a bounded continuous function on $\mathbb{R}^d$:

$$\begin{aligned}
\mathbb{E}\ g(\frac{n(\Lambda_t) - |\Lambda_t|((1-\theta_\phi)\nu + \theta_\phi e_{R_{\Lambda_t}})}{\sqrt{|\Lambda_t|}}) = \\
& \mathbb{E}\ g(\frac{n(\Lambda_t) - |\Lambda_t|((1-\theta_\phi)\nu + \theta_\phi e_z)}{\sqrt{|\Lambda_t|}}) \\
+ \quad \mathbb{E} \quad & g(\frac{n(\Lambda_t) - |\Lambda_t|((1-\theta_\phi)\nu + \theta_\phi e_{R_{\Lambda_t}})}{\sqrt{|\Lambda_t|}}) \\
& - g(\frac{n(\Lambda_t) - |\Lambda_t|((1-\theta_\phi)\nu + \theta_\phi e_z)}{\sqrt{|\Lambda_t|}}) \quad .
\end{aligned}$$

The first term of the sum converges to the integral of g under $\mathcal{N}(0, C)$ by Theorem 6 and the second one converges to 0 by dominated convergence. It follows that $\mathbb{E}\ g(\frac{n(\Lambda_t) - |\Lambda_t|((1-\theta_\phi)\nu + \theta_\phi e_{R_{\Lambda_t}})}{\sqrt{|\Lambda_t|}})$ converges to the integral of g under $\mathcal{N}(0, C)$ for any bounded continuous function g, which is exactly the weak convergence to $\mathcal{N}(0, C)$. $\square$

We can now pass to the case where the color which is used to paint the infinite cluster is also randomized.

Theorem 8. *Let ϕ be a measure on $\mathcal{S}(\mathbb{L}^d)$ that satisfies (E),(M),$\theta_\phi > 0$ and (CLT). Let q be an integer with $q \geq 2$, $z \in S_q$, and $\nu \in \mathbb{R}^q_+$ with $\nu_1 + \dots + \nu_q = 1$. For $\Lambda \subset \mathbb{Z}^d$, we denote by R_Λ an element of S_q which realizes the maximum of $(n_\Lambda(k) - |\Lambda|(1-\theta_\phi)\nu(k))_{k\in S_q}$.*

Let γ be a measure on S_q and $\Phi_\gamma = \int \text{Color}^z_{\phi,\nu}\ d\gamma(z)$. Then, under Φ_γ, we have

$$\frac{n(\Lambda_t) - |\Lambda_t|((1-\theta_\phi)\nu + \theta_\phi e_{R_{\Lambda_t}})}{\sqrt{|\Lambda_t|}} \Longrightarrow \mu,$$

where μ is the law of $X + S(e_Z - \nu)$, where X, S and Z are independent, with $X \sim \mathcal{N}(0, C'), S \sim \mathcal{N}(0, \sigma_\phi^2)$ and $Z \sim \gamma$. C' is the matrix associated to the quadratic form

$$Q(b) = \chi^f(\phi)(\langle D_\nu b, b\rangle - \langle \nu, b\rangle^2)$$

with $D_\nu = \mathsf{diag}(\nu_1, \dots, \nu_q)$. In other words, C' is the matrix of the map

$$b \mapsto \chi^f(\phi)(D_\nu b - \langle \nu, b\rangle \nu).$$

Proof. The theorem just follows from Levy's theorem and a straightforward computation of characteristic functions. $\square$

5. Applications to Potts and Ising models

Let q be an integer with $q \ge 2$ and note $\nu = \frac{1}{q}(e_1 + e_2 + \cdots + e_q) = \frac{1}{q}(1, \ldots, 1)$. According to Proposition 2, $\text{Color}^z_{\phi,\nu}$ is the Gibbs measure $\mathsf{WPt}_{q,\beta,s}$ for the q-state Potts model on $\mathbb{Z}^d$ at inverse temperature $\beta := -\frac{1}{2}\log(1-p)$,

Thus, we will note $\beta_c = -\frac{1}{2}\ln(1-p_c)$, $\beta_g = -\frac{1}{2}\ln(1-p_g)$ and $\beta_r = -\frac{1}{2}\ln(1-p_r(q))$.

By Aizenman, Chayes, Chayes and Newman [ACCN88], the Gibbs measure at inverse temperature β is unique if and only if $\phi^1_{p,q}(0 \leftrightarrow \infty) = 0$, so there is uniqueness of the Gibbs measure for $\beta < \beta_c$ and phase transition for $\beta > \beta_c$.

5.1. Central limit theorems for Potts models.

In the uniqueness zone, we obtain a simple result:

Theorem 9. *Let $\beta < \beta_g$. There is a unique Gibbs measure for the q-state Potts model at inverse temperature β. If we note $p = 1 - \exp(-2\beta)$ and $\nu = \frac{1}{q}(e_1 + e_2 + \cdots + e_q)$, we have the following results for the empirical distributions:*

$$\frac{n(\Lambda_t) - |\Lambda_t|\nu}{\sqrt{|\Lambda_t|}} \Longrightarrow \mathcal{N}(0, \frac{\chi(p,q)}{q^2}(qI - J)),$$

where J is the $q \times q$ matrix whose each entry is equal to 1, and

$$\chi(p,q) = \sum_{k=1}^{+\infty} k\phi^1_{p,q}(|C(0)| = k).$$

We will apply Theorem 6. Since $\phi^1_{p,q}(0 \leftrightarrow \infty) = 0$, $\mathsf{WPt}_{q,\beta,z}$ does not depends on z. The assumption of ergodicity (E) is satisfied by $\phi^1_{p,q}$. In this case, $\theta_\phi = \theta^1(p,q) = 0$, so we just have to check assumption (M), which here is a consequence of the exponential inequality of Grimmett and Piza, which is satisfied as soon as $p < p_g$, or equivalently $\beta < \beta_g = -\frac{1}{2}\ln(1-p_g)$. Then, Theorem 8 applies.

If $p > p_c$, then the Gibbs measures $(\mathsf{WPt}_{q,\beta,z})_{z \in S_q}$ are all distinct – this can be seen as a consequence of Equation (11). Since they are ergodic by theorem 3, they are affinely independent.

Then, in the case $\beta > \beta_r$, we will obtain central limit theorems relative to the empirical distribution for a q-dimensional convex set of Gibbs measures:

Theorem 10. *Let $\beta > \beta_r$ and let Φ_γ be a Gibbs measure for the q states Potts model at inverse temperature β which can be written in the form*

$$\Phi_\gamma = \int \mathsf{WPt}_{q,\beta,z}\ d\gamma(z).$$

For $\Lambda \subset \mathbb{Z}^d$, we denote by R_Λ an element of S_q which realizes the maximum of $(n_\Lambda(k))_{k \in S_q}$. Let us note $p = 1 - \exp(-2\beta)$ and $\nu = \frac{1}{q}(e_1 + e_2 + \cdots + e_q)$. Then, under Φ_γ, we have

$$\frac{n(\Lambda_t) - |\Lambda_t|((1-\theta_\phi)\nu + \theta_\phi e_{R_{\Lambda_t}})}{\sqrt{|\Lambda_t|}} \Longrightarrow \mu,$$

where μ is the law of $X + S(e_Z - \nu)$, when X, S and Z are independent, with $X \sim \mathcal{N}(0, \frac{\chi^f(p,q)}{q^2}(qI - J))$, $S \sim \mathcal{N}(0, \sigma^2_{p,q})$ and $Z \sim \gamma$.

J is the $q \times q$ matrix whose each entry is equal to 1,

$$\begin{aligned} \chi^f(p,q) &= \sum_{k=1}^{+\infty} k\phi^1_{p,q}(|C(0)| = k), \\ \text{and } \sigma^2_{p,q} &= \sum_{k\in\mathbb{Z}^d} \big(\phi_{p,q}(0 \leftrightarrow \infty \text{ and } k \leftrightarrow \infty) - \theta(p,q)^2\big). \end{aligned}$$

When $\phi = \mathsf{WPt}_{q,\beta,z}$, the limit μ is Gaussian and

$$\frac{n(\Lambda_t) - |\Lambda_t|((1-\theta_\phi)\nu + \theta_\phi e_z)}{\sqrt{|\Lambda_t|}} \Longrightarrow \mu.$$

We will apply Theorems 7 and 8. The assumption of ergodicity (E) is again satisfied by $\phi^1_{p,q}$. By the inequality of Pisztora and lemma 2, (M) holds when $p > p_r(q)$ or, equivalently, when $\beta > \beta_r$. Since $\phi^1_{p,q}(0 \leftrightarrow \infty) > 0$, we must also check (CLT). By Theorem 1, (CLT) holds when $p > p_r(q)$, or, equivalently, when $\beta > \beta_r$. Then, Theorems 7 and 8 apply.

When γ is a Dirac measure, Z is constant, so μ is Gaussian.

Corollary 2. *For $\beta > \beta_r$, the Gibbs measure which is obtained as the limit of finite volume Gibbs measures with free boundary condition satisfies the conclusion of the preceding theorem where we take $\gamma = \frac{1}{q}(\sum_{k\in S_q}\delta_k)$.*

Since we have uniqueness of the infinite cluster in the random cluster model, we can consider that the law of $\Phi_\nu = \int \mathsf{WPt}_{q,\beta,z}\, d\nu(z)$ is obtained by coloring the connected components of the random cluster independently. Then, Φ_ν is just $\mathsf{FPt}^{\mathbb{Z}^d}_{q,\beta}$ in the terminology of Proposition 2.3 of [HJL02], *i.e.* the Gibbs measure which is obtained as the limit of finite size Gibbs measures with free boundary condition.

5.2. **Normal fluctuations of the magnetization in Ising models.** In spite of the fact that μ is in general not Gaussian, we can observe an intriguing fact when $q = 2$, *i.e.* for the Ising model. In this case $S(e_Z - \nu) = \epsilon S \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$, with $\epsilon = (-1)^{1\!\!1_{\{Z=1\}}} S$. But ϵS has the same law than S. It follows that μ does not depend on γ and is always Gaussian.

Note also that it is known that we have an exponential decay of the covariance in the Ising model at high temperature – the exact Ornstein-Zernike directional speed of decay has even be established by Campanino, Ioffe and Velenik [CIV03]. It follows that we have $p_c = p_g$ or equivalently $\beta_c = \beta_g$. Since the value of the critical point when $d = 2$ is the fixed point of $x \mapsto x^*$, we have even $\beta_r = \beta_c$ when $d = 2$.

In this model, it is most relevant to formulate the result in term of the magnetization $m_\Lambda = n(\Lambda).(1 - 1)$ rather than in terms of $n(\Lambda)$.

In this case, Theorem 10 admits the following reformulation.

Theorem 11. *Let $\beta > \beta_r$ and let Φ_γ be a Gibbs measure for the Ising model on $\{-1,+1\}^{\mathbb{Z}^d}$ at inverse temperature β which can be written in the form $\Phi_\gamma = \gamma\mathsf{WPt}_{2,\beta,1} + (1-\gamma)\mathsf{WPt}_{2,\beta,-1}$. Let us note $p = 1 - \exp(-2\beta)$ and $m_\Lambda = \frac{1}{|\Lambda|}\sum_{x\in\Lambda}\omega_x$. Then, under Φ_γ, we have*

$$\sqrt{|\Lambda_t|}\big(m_{\Lambda_t} - \mathsf{sign}(m_\Lambda)\theta(p,2)\big) \Longrightarrow \mathcal{N}(0, \chi^f(p,2) + \sigma^2_{p,2}),$$

$$\textit{where } \chi^f(p,q) = \sum_{k=1}^{+\infty} k\phi^1_{p,q}(|C(0)|=k)$$

$$\textit{and } \sigma^2_{p,q} = \sum_{k\in\mathbb{Z}^d} \big(\phi_{p,q}(0\leftrightarrow\infty \textit{ and } k\leftrightarrow\infty) - \theta(p,q)^2\big).$$

Note that $\beta_r = \beta_c$ when $d=2$.

Note that for $d=2$, Theorem 11 covers the whole set of Gibbs measure at temperature $\beta > \beta_c$. Indeed, $\mathsf{WPt}_{q,\beta,1}$ and $\mathsf{WPt}_{q,\beta,-1}$ are known to be the only two extremal Gibbs measures when $d=2$. (This celebrated result is due to Higuchi [Hig81] and Aizenman [Aiz80]. See also Georgii and Higuchi [GH00] for a modern proof.)

It follows that every Gibbs measure is a convex combination of $\mathsf{WPt}_{q,\beta,1}$ and $\mathsf{WPt}_{q,\beta,-1}$. We also note that $\theta(p,2)$ appears as the spontaneous magnetization in the "+" phase of the Ising model. Since the explicit expression of the spontaneous magnetization is known when $d=2$ – see Abraham and Martin-Löf [AML73], Aizenman [Aiz80], and also the bibliographical notes in Georgii [Geo88] for the whole long story of this result – , we get for $d=2$ and $p\ge p_c$ the formula $\theta(p,2) = \big(1-(\sinh 2\beta)^{-4}\big)^{1/8} = \big(1-16\frac{(1-p)^4}{p^4(2-p)^4}\big)^{1/8}$.

Of course we also have a reformulation of Theorem 9 in the high temperature regime $\beta < \beta_g = \beta_c$.

Theorem 12. *Let $\beta < \beta_c$ and let Φ be the unique Gibbs measure for the Ising model on $\{-1,+1\}^{\mathbb{Z}^d}$ at inverse temperature β. We note $p = 1-\exp(-2\beta)$ and $m_\Lambda = \frac{1}{|\Lambda|}\sum_{x\in\Lambda}\omega_x$.*

Then, under Φ, we have

$$\sqrt{|\Lambda_t|}\, m_{\Lambda_t} \Longrightarrow \mathcal{N}(0,\chi^f(p,2)),$$

where

$$\chi^f(p,q) = \sum_{k=1}^{+\infty} k\phi^1_{p,q}(|C(0)|=k).$$

Note that for the Gibbs measures $\mathsf{WPt}_{2,\beta,1}$ or $\mathsf{WPt}_{2,\beta,-1}$, the central limit theorems could be proved without the machinery of the above section: since the Ising model satisfies the F.K.G. inequalities, it follows from the theorem of Newman that is sufficient to prove that

$$\sum_{k\in\mathbb{Z}^d} \mathrm{Cov}(\sigma_0,\sigma_k) < +\infty. \tag{12}$$

Of course, this last result can be obtained independently from the present work – it is for example a consequence of Campanino, Ioffe and Velenik [CIV03].

Nevertheless, let us see how it can be obtained from the random cluster estimates of this paper:

It is not difficult to see that under $\mathsf{WPt}_{2,\beta,1}$ or $\mathsf{WPt}_{2,\beta,-1}$, we have $\mathrm{Cov}(\sigma_0,\sigma_k) = \phi_{p,2}(0\leftrightarrow k) - \phi_{p,2}(0\leftrightarrow\infty)^2$.

Now, $\phi_{p,2}(0 \leftrightarrow k) = \phi_{p,2}(0 \leftrightarrow k \text{ by a finite cluster}) + \phi_{p,2}(0 \leftrightarrow \infty, k \leftrightarrow \infty)$. Then

$$\begin{aligned}\sum_{k\in\mathbb{Z}^d} \mathrm{Cov}(\sigma_0,\sigma_k) &= \sum_{k\in\mathbb{Z}^d} \phi_{p,2}(0 \leftrightarrow k \text{ by a finite cluster}) \\ &\quad + \sum_{k\in\mathbb{Z}^d} \phi_{p,2}(0 \leftrightarrow \infty, k \leftrightarrow \infty) - \phi_{p,2}(0 \leftrightarrow \infty)^2 \\ &= \chi^f(p,2) + \sigma^2_{p,2}.\end{aligned}$$

Nevertheless, when $\beta > \beta_r$, the Gibbs measure "with free boundary conditions" $\mathsf{FPt}^{\mathbb{Z}^d}_{2,\beta}$ – which satisfies the assumptions of theorem 11 – does not have finite susceptibility: in this case

$$\begin{aligned}\mathrm{Cov}(\sigma_0,\sigma_k) = \mathbb{E}\ \sigma_0\sigma_k = \phi_{p,2}(0 \leftrightarrow k) &\geq \phi_{p,2}(0 \leftrightarrow \infty, k \leftrightarrow \infty) \\ &\geq \phi_{p,2}(0 \leftrightarrow \infty)^2 > 0,\end{aligned}$$

so the series $\sum_{k\in\mathbb{Z}^d} \mathrm{Cov}(\sigma_0,\sigma_k)$ diverges.

These results can be compared with a result of Martin-Löf [ML73]: he also proved a central limit theorem for the magnetization in Ising Models at very low temperature. Particularly, he relays the variance of the limiting normal measure to the second derivative at 0 of the thermodynamical function F. Nevertheless, his result is slightly different from ours, since he considers Gibbs measures in large boxes with boundary condition "+", whereas we consider here infinite Gibbs measures under the "+" phase.

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