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MEAN VALUE THEOREMS IN q-CALCULUS

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Abstract. In this paper, some properties of continuous functions in q-analysis are investigated. The behavior of q-derivative in a neighborhood of a local extreme point is described. Two theorems are proved which are q-analogons of the fundamental theorems of the differential calculus. Also, two q-integral mean value theorems are proved and applied to estimating remainder term in q-Taylor formula. Finally, the previous results are used in considering some new iterative methods for equation solving.

1. Introduction

At the last quarter of the XX century, q-calculus appeared as a connection between mathematics and physics ([5], [6]). It has a lot of applications in different mathematical areas, such as number theory, combinatorics, orthogonal polynomials, basic hyper-geometric functions and other sciences—quantum theory, mechanics and theory of relativity.

Let $q \in \mathbb{R}^+ \setminus \{1\}$. A q-natural number $[n]_q$ is defined by

$$[n]_q := 1 + q + \cdots + q^{n-1}, \quad n \in \mathbb{N}.$$

Generally, a q-complex number $[a]_q$ is $[a]_q := \dfrac{1-q^a}{1-q}, \quad a \in \mathbb{C}$. The factorial of a number $[n]_q$ is $[0]_q! := 1, \quad [n]_q! := [n]_q[n-1]_q \cdots [1]_q,\ n \in \mathbb{N}$.

Let q-derivative of a function $f(z)$ be

$$(D_q f)(z) := \frac{f(z) - f(qz)}{z - qz}, \quad z \neq 0, \quad (D_q f)(0) := \lim_{z \to 0} (D_q f)(z),$$

and high q-derivatives are

$$D_q^0 f := f, \qquad D_q^n f := D_q(D_q^{n-1} f), \quad n = 1, 2, 3, \ldots .$$

Notice, that a continuous function on an interval, which does not include 0, is continuous q-differentiable.

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2. Extreme values and q-derivative

We will consider relations between, on one side, extreme value of a continuous function and, on the other side, derivatives and q-derivatives.

THEOREM 2.1. *Let $f(x)$ be a continuous function on a segment $[a,b]$ and let $c \in (a,b)$ bea point of its local maximum.*

(i) If $0 < a < b$, then there exists $\hat{q} \in (0,1)$ such that

$$(D_q f)(c)\begin{cases} \geq 0, & \forall q \in (\hat{q},1) \\ \leq 0, & \forall q \in (1,\hat{q}^{-1}). \end{cases}$$

(ii) If $a < b < 0$, then there exists $\hat{q} \in (0,1)$ such

$$(D_q f)(c)\begin{cases} \leq 0, & \forall q \in (\hat{q},1) \\ \geq 0, & \forall q \in (1,\hat{q}^{-1}). \end{cases}$$

Furthermore, $\big(\forall q \in (\hat{q},1) \cup (1,\hat{q}^{-1})\big)\big(\exists \xi \in (a,b)\big)\ (D_q f)(\xi) = 0.$

Proof. Since the proofs of (i) and (ii) are very similar, we will expose only the first one. Since c is a point of local maximum of the function $f(x)$, there exists $\varepsilon > 0$, such that $f(x) \leq f(c)$, for all $x \in (c-\varepsilon, c+\varepsilon) \subset (a,b)$. Let $q_0 \in (0,1)$ such that $c - \varepsilon < q_0 c < c$. Now, for all $q \in (q_0,1)$, it is valid $qc < c$ and $f(qc) \leq f(c)$, wherefrom $(D_q f)(c) \geq 0$. In a similar way, there exists $q_1 \in (0,1)$ such that $c < c/q_1 < c + \varepsilon$ and for all $q \in (1, q_1^{-1})$ it holds $(D_q f)(c) \leq 0$. At last, denote by $\hat{q} = \max\{q_0, q_1\}$.

Let $q \in (\hat{q},1)$ be an arbitrary real number. Then $\eta = c/q \in (c, c+\varepsilon)$, wherefrom $f(c) \geq f(\eta)$, i.e., $f(q\eta) \geq f(\eta)$. From $q\eta < \eta$ we conclude $(D_q f)(\eta) \leq 0$. As $f(x)$ is a continuous function, $(D_q f)(x)$ is continuous in (a,b), too. Since $(D_q f)(c) \geq 0$, $(D_q f)(\eta) \leq 0$, where $c, \eta \in (a,b)$, there exists $\xi \in (c,\eta) \subset (a,b)$, such that $(D_q f)(\xi) = 0$. Analogously, for an arbitrary $q \in (1, \hat{q}^{-1})$, the number $\eta = c/q \in (c-\varepsilon, c)$, wherefrom $(D_q f)(\eta) \geq 0$. Since $(D_q f)(c) \leq 0$, we have proved the existence of a zero of $(D_q f)(x)$ for $q \in (1, \hat{q}^{-1})$. ■

EXAMPLE 2.1. Let us consider $f(x) = (x-1)(3-x) + 2$. Its maximum is at $c = 2$, but q-derivative is $(D_q f)(x) = -[2]x + 4$ and it vanishes at the point $\xi = 4/(1+q)$. So, here is $\hat{q} = 1/3$. For $q = 3/4$, we have $(D_{3/4} f)(2) = 1/2$, $\eta = 2\frac{2}{3}$ and $\xi = 2\frac{2}{7}$.

In a similar way, we can prove the next theorem.

THEOREM 2.2. *Let $f(x)$ be a continuous function on a segment $[a,b]$ and let $c \in (a,b)$ be a point of its local minimum.*

(i) If $0 < a < b$, then there exists $\hat{q} \in (0,1)$ such that

$$(D_q f)(c)\begin{cases} \leq 0, & \forall q \in (\hat{q},1) \\ \geq 0, & \forall q \in (1,\hat{q}^{-1}). \end{cases}$$

(ii) If $a < b < 0$, then there exists $\hat{q} \in (0,1)$ such that

$$(D_q f)(c) \begin{cases} \geq 0, & \forall q \in (\hat{q}, 1) \\ \leq 0, & \forall q \in (1, \hat{q}^{-1}). \end{cases}$$

Moreover, $(\forall q \in (\hat{q}, \hat{q}^{-1}))(\exists \xi \in (a,b))\ (D_q f)(\xi) = 0$.

REMARK. If $f(x)$ is differentiable for all $x \in (a,b)$, then $\lim_{q\uparrow 1} D_q f(x) = f'(x)$. So, if $c \in (a,b)$ is a point of local extreme of $f(x)$, we have $f'(c) = D_1 f(c) = 0$.

3. Some q-mean value theorems

By using the previous results, we can establish and prove analogons of well-known mean value theorems in q-calculus.

THEOREM 3.1. (q-Rolle) *Let $f(x)$ be a continuous function on $[a,b]$ satisfying $f(a) = f(b)$. Then there exists $\hat{q} \in (0,1)$ such that*

$$\big(\forall q \in (\hat{q}, 1) \cup (1, \hat{q}^{-1})\big)\big(\exists \xi \in (a,b)\big): \quad (D_q f)(\xi) = 0.$$

Proof. If $f(x)$ is not a constant function on $[a,b]$, then it attains its extreme value in some point in (a,b). But, according to Theorems 2.1–2, $(D_q)f(x)$ vanishes at a point $\xi \in (a,b)$. ■

THEOREM 3.2 (q-Lagrange) *Let $f(x)$ be a continuous function on $[a,b]$. Then there exists $\hat{q} \in (0,1)$ such that*

$$\big(\forall q \in (\hat{q}, 1) \cup (1, \hat{q}^{-1})\big)\big(\exists \xi \in (a,b)\big): \quad f(b) - f(a) = (D_q f)(\xi)(b-a).$$

Proof. The statement follows by applying the previous theorem to the function $f(x) - x(f(b) - f(a))/(b-a)$. ■

4. Mean value theorems for q-integrals

In q-analysis, we define q-integral by

$$I_q(f) = \int_0^a f(t)\, d_q(t) := a(1-q) \sum_{n=0}^{\infty} f(aq^n) q^n.$$

Notice that

$$I(f) = \int_0^a f(t)\, dt = \lim_{q\uparrow 1} I_q(f).$$

THEOREM 4.1 *Let $f(x)$ be a continuous function on a segment $[0,a]$ $(a > 0)$. Then*

$$(\forall q \in (0,1))(\exists \xi \in [0,a]): \quad I_q(f) = \int_0^a f(t)\, d_q(t) = a\ f(\xi).$$

Proof. Since $f(x)$ is a continuous function on the segment $[0,a]$, it attains its minimum m and maximum M and takes all values between. According to assumption $0<q<1$, we have $0<aq^n<a$ and $m \le f(aq^n) \le M$. Now,

$$a(1-q)\sum_{n=0}^{\infty} mq^n \le a(1-q)\sum_{n=0}^{\infty} f(aq^n)q^n \le a(1-q)\sum_{n=0}^{\infty} Mq^n,$$

wherefrom $m \le \frac{1}{a}\, I_q(f) \le M$. So, there exists $\xi \in [0,a]$ such that $a^{-1}\, I_q(f) = f(\xi)$. ∎

Moreover, if we define

$$\int_a^b f(t)\, d_q(t) := \int_0^b f(t)\, d_q(t) - \int_0^a f(t)\, d_q(t),$$

then the next theorem is valid.

THEOREM 4.2. *Let $f(x)$ be a continuous function on a segment $[a,b]$. Then there exists $\hat{q} \in (0,1)$ such that*

$$(\forall q \in (\hat{q},1))(\exists \xi \in (a,b)): \quad I_q(f) = \int_a^b f(t)\, d_q(t) = f(\xi)(b-a).$$

Proof. It is easy to prove that $\lim_{q\uparrow 1} I_q(f) = I(f)$, i.e.

$$(\forall \varepsilon > 0)(\exists q_0 \in (0,1))(\forall q \in (q_0,1)): \quad I(f) - \varepsilon < I_q(f) < I(f) + \varepsilon.$$

According to the well known mean value theorem for integrals, we have

$$(\exists c \in (a,b)): \quad I(f) = f(c)(b-a).$$

Let $\varepsilon \le (b-a)\ \min\{M - f(c), f(c) - m\}$, where m and M are the minimum and maximum of $f(x)$ on $[a,b]$. Now,

$$(\exists \hat{q} \in (0,1))(\forall q \in (\hat{q},1)): \ f(c) - \frac{\varepsilon}{b-a} < \frac{1}{b-a}\, I_q(f) < f(c) + \frac{\varepsilon}{b-a},$$

hence $m < I_q(f)/(b-a) < M$. Since $f(x)$ is a continuous function on the segment $[a,b]$, it takes all values between m and M, i.e.

$$(\exists \xi \in (a,b)): \frac{1}{b-a} I_q(f) = f(\xi),$$

what we wanted to prove. ∎

THEOREM 4.3 *Let $f(x)$ and $g(x)$ be some continuous functions on a segment $[a,b]$. Then there exists $\hat{q} \in (0,1)$ such that*

$$(\forall q \in (\hat{q},1))(\exists \xi \in (a,b)): \quad I_q(fg) = g(\xi) I_q(f).$$

Proof. According to the second mean value theorem for integrals, we have

$$(\exists c \in (a,b)): \quad I(fg) = g(c)I(f).$$

Hence $\lim_{q\uparrow 1} I_q(fg) = g(c)I(f) = g(c)\lim_{q\uparrow 1} I_q(f)$, i.e., $\lim_{q\uparrow 1} \dfrac{I_q(fg)}{I_q(f)} = g(c)$. Now,

$$(\exists q_0 \in (0,1))(\forall q \in (q_0,1)): \quad g(c) - \varepsilon < \frac{I_q(fg)}{I_q(f)} < g(c) + \varepsilon.$$

Since $g(x)$ is a continuous function on the segment $[a,b]$, it attains its minimum m_g and maximum M_g. Let $\varepsilon \le \min\{M_g - g(c), g(c) - m_g\}$. Hence

$$(\exists \hat{q} \in (0,1))(\forall q \in (\hat{q},1)): \quad m_g < \frac{I_q(fg)}{I_q(f)} < M_g.$$

Since $f(x)$ takes all values between m_g and M_g, we conclude that

$$(\exists \xi \in (a,b)): \frac{I_q(fg)}{I_q(f)} = g(\xi). \quad \blacksquare$$

5. Estimation of remainder term in q-Taylor formula

Let $f(x)$ be a continuous function on some interval (a,b) and $c \in [a,b]$. Jackson's q-Taylor formula (see [3], [4] and [2]) is given by

$$f(z) = \sum_{k=0}^{\infty} \frac{(D_q^k f)(c)}{[k]_q!}(z-c)^{(k)}, \quad z \in (a,b),$$

where

$$(z-c)^{(0)} = 1, \quad (z-c)^{(k)} = \prod_{i=0}^{k-1}(z - cq^i) \quad (k \in \mathbb{N}).$$

T. Ernst [2] have found the next q-Taylor formula

$$f(z) = \sum_{k=0}^{n-1} \frac{(D_q^k f)(c)}{[k]_q!}(z-c)^{(k)} + R_n(f,z,c,q), \tag{5.1}$$

where $R_n(f,z,c,q)$ is the remainder term determined by

$$R_n(f,z,c,q) = \int_{t=c}^{t=z} \frac{(z-t)^{(n)}}{z-t}\,\frac{(D_q^n f)(t)}{[n-1]_q!}\,d_q(t). \tag{5.2}$$

THEOREM 5.1. *Let $f(x)$ be a continuous function on $[a,b]$ and $R_n(f,z,c,q)$, $z, c \in (a,b)$ be the remainder term in q-Taylor formula. Then there exists $\hat{q} \in (0,1)$ such that for all $q \in (\hat{q},1)$, $\xi \in (a,b)$ can be found between c and z, which satisfies*

$$R_n(f,z,c,q) = \frac{(D_q^n f)(\xi)}{[n-1]_q!}\int_{t=c}^{t=z} \frac{(z-t)^{(n)}}{z-t}\,d_q(t). \tag{5.3}$$

Proof. Since $f(x)$ is a continuous function on $[a,b]$, it can be expanded by q-Taylor formula (5.1) with the remainder term (5.2). Notice that the functions

$$\frac{(z-t)^{(n)}}{z-t} = \prod_{i=1}^{n-1} (z - tq^i)$$

and $(D_q^n f)(t)/[n-1]_q!$ are continuous on the segment between c and z which is contained in (a,b). According to Theorem 4.3., there exists $\hat{q} \in (0,1)$, such that for all $q \in (\hat{q}, 1)$ can be found ξ between c and z such that (5.3) is valid. ■

THEOREM 5.2. *Let $f(x)$ be a continuous function on $[a,b]$ and $z, c \in (a,b)$. Then there exists $\hat{q} \in (0,1)$ such that for all $q \in (\hat{q},1)$, $\xi \in (a,b)$ can be found between c and z, which satisfies*

$$f(z) = \sum_{k=0}^{n-1} \frac{(D_q^k f)(c)}{[k]_q!}(z-c)^{(k)} + \frac{(D_q^n f)(\xi)}{[n]_q!}(z-c)^{(n)}.$$

Proof. Applying $\dfrac{(z-t)^{(n)}}{z-t} = -D_{q,t}\left(\dfrac{(z-t)^{(n)}}{[n]_q}\right)$, to the integral in (5.3) we have

$$\int_{t=c}^{t=z} \frac{(z-t)^{(n)}}{z-t}\, d_q(t) = -\int_{t=c}^{t=z} D_{q,t}\left(\frac{(z-t)^{(n)}}{[n]_q}\right) d_q(t)$$

$$= -\frac{(z-t)^{(n)}}{[n]_q}\Bigg|_{t=c}^{t=z} = \frac{(z-c)^{(n)}}{[n]_q}.$$

So, $R_n(f,z,c,q) = \dfrac{(D_q^n f)(\xi)}{[n]_q!}(z-c)^{(n)}$. ■

THEOREM 5.3 *Let $f(x)$ be a continuous function on $[a,b]$ and $R_n(f,z,c,q)$, $z, c \in (a,b)$ be the remainder term in q-Taylor formula. Then there exists $\hat{q} \in (0,1)$ such that for all $q \in (\hat{q},1)$, $\xi \in (a,b)$ can be found between c and z, which satisfies*

$$R_n(f,z,c,q) = \frac{(D_q^n f)(\xi)}{[n]_q!}(z-c)^{(n)}.$$

Proof. Since $f(x)$ is a continuous function on $[a,b]$, it can be expanded by q-Taylor formula (5.1) with the remainder term (5.2). Notice that the functions

$$\frac{(z-t)^{(n)}}{z-t} = \prod_{i=1}^{n-1} (z - tq^i)$$

and $(D_q^n f)(t)/[n-1]_q!$ are continuous on the segment between c and z which is contained in (a,b). According to Theorem 4.3., there exists $\hat{q} \in (0,1)$, such that for all $q \in (\hat{q},1)$, ξ between c and z can be found such that

$$R_n(f,z,c,q) = \frac{(D_q^n f)(\xi)}{[n-1]_q!} \int_{t=c}^{t=z} \frac{(z-t)^{(n)}}{z-t}\, d_q(t).$$

Applying

$$\frac{(z-t)^{(n)}}{z-t} = -D_{q,t}\left(\frac{(z-t)^{(n)}}{[n]_q}\right),$$

to the previous integral we have

$$\int_{t=c}^{t=z} \frac{(z-t)^{(n)}}{z-t}\, d_q(t) = -\int_{t=c}^{t=z} D_{q,t}\left(\frac{(z-t)^{(n)}}{[n]_q}\right) d_q(t)$$
$$= -\frac{(z-t)^{(n)}}{[n]_q}\Bigg|_{t=c}^{t=z} = \frac{(z-c)^{(n)}}{[n]_q}.$$

So, $R_n(f,z,c,q) = \dfrac{(D_q^n f)(\xi)}{[n]_q!}(z-c)^{(n)}$. ■

6. Application

Here we will apply the previous theorems in analyzing an iterative method for solving equations.

Suppose that an equation $f(x)=0$ has a unique isolated solution $x=\tau$. If x_n is an approximation for the exact solution τ, using Jackson's q-Taylor formula, we have

$$0 = f(\tau) \approx f(x_n) + (D_q f)(x_n)(\tau - x_n),$$

hence $\tau \approx x_n - \dfrac{f(x_n)}{(D_q f)(x_n)}$. So, we can construct *q-Newton method*

$$x_{n+1} = x_n - \frac{f(x_n)}{(D_q f)(x_n)}. \tag{6.1}$$

More simply, it looks like $x_{n+1} = x_n\left\{1 - \dfrac{1-q}{1-\frac{f(qx_n)}{f(x_n)}}\right\}$. This method written in the form

$$x_{n+1} = x_n - \frac{x_n - qx_n}{f(x_n) - f(qx_n)} f(x_n)$$

reminds to the method of chords (secants).

THEOREM 6.1. *Suppose that a function $f(x)$ is continuous on a segment $[a,b]$ and that the equation $f(x)=0$ has a unique isolated solution $\tau \in (a,b)$. Let the conditions*

$$|(D_q f)(x)| \geq M_1 > 0, \qquad |(D_q^2 f)(x)| \leq M_2$$

are satisfied for all $x \in (a,b)$. Then there exists $\hat{q} \in (0,1)$, such that for all $q \in (\hat{q},1)$, the iterations obtained by q-Newton method satisfy

$$|\tau - x_{k+1}| \leq \frac{M_2}{(1+q)M_1}|(\tau - x_k)^{(2)}|.$$

Proof. From the formulation of q-Newton method (6.1), we have

$$x_{k+1} - \tau = x_k - \tau - \frac{f(x_k)}{(D_q f)(x_k)},$$

hence $f(x_k) + (D_q f)(x_k)(\tau - x_k) = (D_q f)(x_k)(\tau - x_{k+1})$. By using q-Taylor formula at the point x_k of order $n = 2$ for $f(\tau)$ we have

$$f(\tau) = f(x_k) + (D_q f)(x_k)(\tau - x_k) + R_2(f, \tau, x_k, q).$$

Since $f(\tau) = 0$, we obtain $(D_q f)(x_k)(\tau - x_{k+1}) = -R_2(f, \tau, x_k, q)$, i.e.

$$|\tau - x_{k+1}| = \frac{|R_2(f, \tau, x_k, q)|}{|(D_q f)(x_k)|}.$$

According to Theorem 5.1., there exists $\hat{q} \in (0,1)$ such that for all $q \in (\hat{q}, 1)$, $\xi \in (a,b)$ can be found such that

$$R_2(f, \tau, x_k, q) = \frac{(D_q^2 f)(\xi)}{[2]_q} (\tau - x_k)^{(2)}.$$

Now,

$$|\tau - x_{k+1}| = \frac{|(D_q^2 f)(\xi)|}{|(D_q f)(x_k)|} \, \frac{|(\tau - x_k)^{(2)}|}{1+q}.$$

Using the conditions which function $f(x)$ and its q-derivatives satisfy we obtain the statement of the theorem. ■

REMARK. In our papers [7] and [8] we have discussed q-iterative methods in details.

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