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BOUNDEDNESS IN TOPOLOGICAL SPACES

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Abstract. We investigate abstract boundedness in a topological space and demonstrate the importance of this notion in selection principles theory. Some applications to function spaces and hyperspaces are given. This approach sheds more light on several results scattered in the literature.

1. Introduction

The notion of abstract boundedness in a topological space was introduced and studied by S.T. Hu in 1949 [9]. A modified version of his notion is a useful tool in investigation of different hyperspace topologies, including the Hausdorff-Bourbaki, Attouch-Wets, bounded (proximal) Vietoris and Wijsman topology (see [20, p. 53], [6]). For instance, in [6] it was shown that the boundedness in a metric space generated by closed balls is a powerful device for investigation of the Wijsman hyperspace topology and offers deeper and simpler proofs without epsilonetics.

A family $\mathbb{B}$ of nonempty *closed* subsets of a space X is said to be an *abstract boundedness*, or simply *boundedness*, if it is *closed for finite unions, closed hereditary and contains all singletons*. The families $\mathsf{CL}(X)$, $\mathbb{F}(X)$ and $\mathbb{K}(X)$ of all nonempty closed, all nonempty finite and all nonempty compact subsets of a Hausdorff topological space X, the family of all (totally) bounded subsets of a metric or uniform space are examples of boundedness. But there are other important examples of boundedness: the family of closed countably compact (Lindelöf, Čech-complete, zero-dimensional, meager, nowhere dense, topologically bounded) subsets of a space.

If $\mathbb{B}$ is a boundedness in a space X and $\mathcal{U}$ is an open cover of X, then $\mathcal{U}$ is said to be a $\mathbb{B}$-*cover* if each $B \in \mathbb{B}$ is contained in an element of $\mathcal{U}$ and $X \notin \mathcal{U}$. (Therefore, we have $X \notin \mathbb{B}$ and X is infinite.) In particular, $\mathbb{F}(X)$-covers are called ω-*covers*. $\mathcal{U}$ is called a γ-*cover* [8] ($\gamma_{\mathbb{B}}$-*cover*) if it is infinite and each $x \in X$ (each $B \in \mathbb{B}$) is

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not contained in at most finitely many elements of $\mathcal{U}$. For a given boundedness $\mathbb{B}$ in a space denote by $\mathcal{O}_{\mathbb{B}}$ (Γ, $\Gamma_{\mathbb{B}}$) the collection of all $\mathbb{B}$-covers (γ-covers, $\gamma_{\mathbb{B}}$-covers), while Ω is the collection of ω-covers of the space. Observe that each infinite subset of a γ-cover ($\gamma_{\mathbb{B}}$-cover) is still a γ-cover ($\gamma_{\mathbb{B}}$-cover). So, we may suppose that such covers are countable. Each set from $\mathbb{B}$ is contained in infinitely many elements of a $\mathbb{B}$-cover of the space.

In this paper $\mathbb{B}$ will be a fixed boundedness in a space X. We consider only Hausdorff spaces in which each $\mathbb{B}$-cover contains a countable $\mathbb{B}$-subcover; such spaces are called $\mathbb{B}$-*Lindelöf*. $\mathbb{F}(X)$-Lindelöf spaces are called ω-Lindelöf.

Our approach allows us to unify and extend many results that appear in the literature (see, for example, [2, 3, 4, 5, 12, 16, 21]).

2. Notation and terminology

We use the usual topological notation and terminology, mainly as in [7]. We also need notation concerning selection principles, games and partition relations (see [10, 14, 15, 17, 18, 19, 22, 25, 26]).

Let $\mathcal{A}$ and $\mathcal{B}$ be collections whose elements are families of subsets of an infinite set X. Then:

1. ([11, 24]) $\mathsf{S}_1(\mathcal{A},\mathcal{B})$ denotes the selection principle:

- For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(b_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $b_n \in A_n$ and $\{b_n : n \in \mathbb{N}\} \in \mathcal{B}$.

2. ([11, 24]) $\mathsf{S}_{fin}(\mathcal{A},\mathcal{B})$ denotes the selection hypothesis:

- For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbb{N})$ of finite (not necessarily non-empty) sets such that for each $n \in \mathbb{N}$, $B_n \subset A_n$ and $\bigcup_{n\in\mathbb{N}} B_n$ is an element of $\mathcal{B}$.

3. The symbol $\mathsf{G}_1(\mathcal{A},\mathcal{B})$ [24] denotes an infinitely long game for two players, ONE and TWO, which play a round for each positive integer. In the n-th round ONE chooses a set $A_n \in \mathcal{A}$, and TWO responds by choosing an element $b_n \in A_n$. TWO wins a play $(A_1, b_1; \cdots; A_n, b_n; \cdots)$ if $\{b_n : n \in \mathbb{N}\} \in \mathcal{B}$; otherwise, ONE wins.

Recall that a strategy of a player is a function σ from the set of all finite sequences of moves of player's opponent into the set of legal moves of the strategy owner.

4. [24] For positive integers n and m the *ordinary partition symbol* $\mathcal{A} \to (\mathcal{B})^n_m$ denotes the statement:

- For each $A \in \mathcal{A}$ and for each function $f : [A]^n \to \{1, \cdots, m\}$ there are a set $B \subset A$, $B \in \mathcal{B}$, and an $i \in \{1, \cdots, m\}$ such that for each $Y \in [B]^n$, $f(Y) = i$.

Here $[A]^n$ denotes the set of n-element subsets of A. We call f a "coloring" and say that "B is homogeneous of color i for f".

5. [16] The symbol $\alpha_i(\mathcal{A},\mathcal{B})$, $i = 2, 3, 4$, denotes the following selection hypothesis:

For each sequence $(A_n : n \in \mathbb{N})$ of infinite elements of $\mathcal{A}$ there is an element $B \in \mathcal{B}$ such that:

- $\alpha_2(\mathcal{A}, \mathcal{B})$: for each $n \in \mathbb{N}$ the set $A_n \cap B$ is infinite;
- $\alpha_3(\mathcal{A}, \mathcal{B})$: for infinitely many $n \in \mathbb{N}$ the set $A_n \cap B$ is infinite;
- $\alpha_4(\mathcal{A}, \mathcal{B})$: for infinitely many $n \in \mathbb{N}$ the set $A_n \cap B$ is nonempty.

3. General results on boundedness

In this section we consider boundedness in an arbitrary Hausdorff topological space. Let us begin with a result whose proof is quite similar to the proofs of Theorem 1.1 in [11] or Theorem 5 in [13].

Proposition 3.1. *For a space X the following are equivalent:*

(1) *X satisfies $\mathsf{S}_{fin}(\mathcal{O}_{\mathbb{B}}, \Gamma)$;*

(2) *X satisfies $\mathsf{S}_1(\mathcal{O}_{\mathbb{B}}, \Gamma)$.*

Theorem 3.2. *For a $\mathbb{B}$-Lindelöf space X the following are equivalent:*

(1) *X satisfies $\alpha_2(\mathcal{O}_{\mathbb{B}}, \Gamma)$;*

(2) *X satisfies $\alpha_3(\mathcal{O}_{\mathbb{B}}, \Gamma)$;*

(3) *X satisfies $\alpha_4(\mathcal{O}_{\mathbb{B}}, \Gamma)$;*

(4) *X satisfies $\mathsf{S}_1(\mathcal{O}_{\mathbb{B}}, \Gamma)$;*

(5) ONE *has no winning strategy in the game $\mathsf{G}_1(\mathcal{O}_{\mathbb{B}}, \Gamma)$;*

(6) *For all $n, m \in \mathbb{N}$, X satisfies $\mathcal{O}_{\mathbb{B}} \to (\Gamma)^n_m$.*

Proof. (3) $\Rightarrow$ (4): Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of $\mathbb{B}$-covers of X. Assume that for each $n \in \mathbb{N}$ we have $\mathcal{U}_n = \{U_{n,m} : m \in \mathbb{N}\}$. For every $n \in \mathbb{N}$ define

$$\mathcal{V}_n = \{U_{1,m_1} \cap \cdots \cap U_{n,m_n} : n < m_1 < m_2 < \cdots < m_n, i \leq n\} \setminus \{\emptyset\}.$$

Then each $\mathcal{V}_n$ is a $\mathbb{B}$-cover of X. By (3) and the fact that each infinite subset of a γ-cover is also a γ-cover, there is an increasing sequence $n_1 < n_2 < \cdots$ in $\mathbb{N}$ and a γ-cover $\mathcal{V} = \{V_{n_i} : i \in \mathbb{N}\}$ such that for each $i \in \mathbb{N}$, $V_{n_i} \in \mathcal{V}_{n_i}$. Let for each $i \in \mathbb{N}$,

$$V_{n_i} = U_{1,m_1} \cap \cdots \cap U_{n_i,m_{n_i}},\ j \leq n_i.$$

Put $n_0 = 0$. For each $i \geq 0$ and each n with $n_i < n \leq n_{i+1}$ let W_n be the n-th coordinate in the representation of $V_{n_{i+1}}$:

$$W_n = U_{n,m_{n_{i+1}}}.$$

For each $n \in \mathbb{N}$, $W_n \in \mathcal{U}_n$ and the set $\mathcal{W} := \{W_n : n \in \mathbb{N}\}$ is a γ-cover of X. Therefore, X satisfies $\mathsf{S}_1(\mathcal{O}_{\mathbb{B}}, \Gamma)$.

(4) $\Rightarrow$ (5): Let σ be a strategy for ONE in $\mathsf{G}_1(\mathcal{O}_{\mathbb{B}}, \Gamma)$ and let the first move of ONE be a $\mathbb{B}$-cover $\sigma(\emptyset) = \{U_{(1)}, U_{(2)}, \cdots, U_{(n)}, \cdots\}$. Suppose that for each finite

sequence s of natural numbers of length at most m, U_s has been already defined. Then define $\{U_{(n_1,\cdots,n_m,k)} : k \in \mathbb{N}\}$ to be the set

$$\sigma(U_{(n_1)}, U_{(n_1,n_2)}, \cdots, U_{(n_1,\cdots,n_m)}) \setminus \{U_{(n_1)}, U_{(n_1,n_2)}, \cdots, U_{(n_1,\cdots,n_m)}\}.$$

Because each set from $\mathbb{B}$ belongs to infinitely many elements of a $\mathbb{B}$-cover, we have that for each s a finite sequence of natural numbers, the set $\{U_{s\frown(n)} : n \in \mathbb{N}\}$ is a $\mathbb{B}$-cover. Apply (4) and for each s choose $n_s \in \mathbb{N}$ such that $\{U_{s\frown(n_s)} :$ s a finite sequence of natural numbers$\}$ is a γ-cover of X. Then inductively define a sequence $n_1 = n_\emptyset$, $n_{k+1} = n_{(n_1,\cdots,n_k)}$, for $k \geq 1$. Then

$$U_{(n_1)}, U_{(n_1,n_2)}, \cdots, U_{(n_1,\cdots,n_k)}, \cdots$$

is a γ-cover of X. Since it is actually a sequence of moves of TWO in a play of the game $\mathsf{G}_1(\mathcal{O}_\mathbb{B}, \Gamma)$, σ is not a winning strategy for ONE.

(5) $\Rightarrow$ (6): It follows from [18, Th. 1] because $\mathcal{O}_\mathbb{B}$ is a persistent family in terminology of [18].

(6) $\Rightarrow$ (4): Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of countable $\mathbb{B}$-covers of X and suppose that for each $n, \mathcal{U}_n = \{U_{n;m} : m \in \mathbb{N}\}$. Consider the set $\mathcal{V}$ of all nonempty sets of the form $U_{1;n} \cap U_{n;k}$, $n, k \in \mathbb{N}$. It is understood that $\mathcal{V}$ is a $\mathbb{B}$-cover of X. Define $f : [\mathcal{V}]^2 \to \{1, 2\}$ by

$$f(\{U_{1;n_1} \cap U_{n_1;k}, U_{1;n_2} \cap U_{n_2;m}\}) = \begin{cases} 1, & \text{if } n_1 = n_2, \\ 2, & \text{otherwise.} \end{cases}$$

As $\mathcal{O}_\mathbb{B} \to (\Gamma)^2_2$ is satisfied there are $j \in \{1, 2\}$ and a collection $\mathcal{H} \subset \mathcal{V}$, $\mathcal{H} \in \Gamma$, homogeneous for f of color j. Consider two possibilities:

(i) $j = 1$: Then there is some n such that $H \subset U_{1,n}$ for each $H \in \mathcal{H}$. But, this means that $\mathcal{H}$ is not a γ-cover of X and we have a contradiction; so, this case is impossible.

(ii) $j = 2$: For each $H \in \mathcal{H}$ choose, whenever it is possible, $U_{n;k_n}$ to be the second coordinate in the chosen representation of H; otherwise let $U_{n;k_n}$ be an arbitrary element in $\mathcal{U}_n$. Let $\mathcal{G}$ be the set of all U_{n,k_n}'s chosen in this way. Then $\mathcal{G}$ is a γ-cover of X witnessing for $(\mathcal{U}_n : n \in \mathbb{N})$ that X satisfies $\mathsf{S}_1(\mathcal{O}_\mathbb{B}, \Gamma)$.

(5) $\Rightarrow$ (1): Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of $\mathbb{B}$-covers of X and let for each $n \in \mathbb{N}$, $\mathcal{U}_n = \{U_{n,m} : m \in \mathbb{N}\}$. Define the following strategy σ for ONE. In the first round ONE plays $\sigma(\emptyset) = \mathcal{U}_1$. Assuming that the set $U_{1,m_{i_1}} \in \mathcal{U}_1$ is TWO's response, ONE plays $\sigma(U_{1,m_{i_1}}) = \mathcal{V}(1, m_{i_1}) = \{U_{1,m} : m > m_{i_1}\}$, still a $\mathbb{B}$-cover of X. If TWO chooses a set $U_{1,m_{i_2}} \in \mathcal{V}(1, m_{i_1})$, then ONE plays $\sigma(U_{1,m_{i_1}}, U_{1,m_{i_2}}) = \mathcal{V}(1, m_{i_2}) = \{U_{1,m} : m > m_{i_2}\}$; this set is still a $\mathbb{B}$-cover of X. Then TWO chooses a set $U_{1,m_{i_3}} \in \sigma(U_{1,m_{i_1}}, U_{1,m_{i_2}})$. And so on. (By this procedure we actually form a sequence of $\mathbb{B}$-covers from each $\mathcal{U}_n$ and apply (5) to these new $\mathbb{B}$-covers.)

Since σ is not a winning strategy for ONE, there is a σ-play

$$\sigma(\emptyset), U_{1,m_{i_1}}; \sigma(U_{1,m_{i_1}}), U_{1,m_{i_2}}; \sigma(U_{1,m_{i_1}}, U_{1,m_{i_2}}), U_{1,m_{i_3}}; \cdots$$

lost by ONE. That means that the sequence $\mathcal{V}$ consisting of TWO's moves is a γ-cover of X. Of course, it contains infinitely many elements from each $\mathcal{U}_n$, $n \in \mathbb{N}$, and thus $\mathcal{V}$ witnesses for the original sequence $(\mathcal{U}_n : n \in \mathbb{N})$ that X satisfies $\alpha_2(\mathcal{O}_\mathbb{B}, \Gamma)$. ∎

In a quite similar way one proves the following theorem.

Theorem 3.3. *For a $\mathbb{B}$-Lindelöf space X the following are equivalent:*

(1) *X satisfies $\alpha_2(\mathcal{O}_{\mathbb{B}}, \Gamma_{\mathbb{B}})$;*

(2) *X satisfies $\alpha_3(\mathcal{O}_{\mathbb{B}}, \Gamma_{\mathbb{B}})$;*

(3) *X satisfies $\alpha_4(\mathcal{O}_{\mathbb{B}}, \Gamma_{\mathbb{B}})$;*

(4) *X satisfies $\mathsf{S}_1(\mathcal{O}_{\mathbb{B}}, \Gamma_{\mathbb{B}})$;*

(5) ONE *has no winning strategy in the game $\mathsf{G}_1(\mathcal{O}_{\mathbb{B}}, \Gamma_{\mathbb{B}})$;*

(6) *For all $n, m \in \mathbb{N}$, X satisfies $\mathcal{O}_{\mathbb{B}} \to (\Gamma_{\mathbb{B}})^n_m$.*

We also have the following result.

Theorem 3.4. *For a space X the following statements are equivalent:*

(1) *X satisfies $\alpha_2(\Gamma_{\mathbb{B}}, \Gamma)$;*

(2) *X satisfies $\alpha_3(\Gamma_{\mathbb{B}}, \Gamma)$;*

(3) *X satisfies $\alpha_4(\Gamma_{\mathbb{B}}, \Gamma)$;*

(4) *X satisfies $\mathsf{S}_1(\Gamma_{\mathbb{B}}, \Gamma)$;*

(5) *ONE has no winning strategy in the game $\mathsf{G}_1(\Gamma_{\mathbb{B}}, \Gamma)$ on X.*

Proof. We must prove (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (1).

(3) $\Rightarrow$ (4): Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of $\gamma_{\mathbb{B}}$-covers of X, and $\mathcal{U}_n = \{U_{n,m} : m \in \mathbb{N}\}$. For all $n, m \in \mathbb{N}$ define

$$V_{n,m} = U_{1,m} \cap U_{2,m} \cap \cdots \cap U_{n,m}.$$

Then for each n the set $\mathcal{V}_n = \{V_{n,m} : m \in \mathbb{N}\}$ is a $\gamma_{\mathbb{B}}$-cover of X. By (3) applied to the sequence $(\mathcal{V}_n : n \in \mathbb{N})$ there is an increasing sequence $n_1 < n_2 < \cdots$ in $\mathbb{N}$ and a γ-cover $\mathcal{V} = \{V_{n_i,m_i} : i \in \mathbb{N}\}$ such that for each $i \in \mathbb{N}$, $V_{n_i,m_i} \in \mathcal{V}_{n_i}$. Let $n_0 = 0$. For each $i \geq 0$, each j with $n_i < j \leq n_{i+1}$ and each $V_{n_{i+1},m_{i+1}} = U_{1,m_{i+1}} \cap \cdots \cap U_{n_{i+1},m_{i+1}}$ put

$$W_j = U_{j,m_{i+1}}.$$

For each $j \in \mathbb{N}$, $W_j \in \mathcal{U}_j$ and the set $\{W_j : j \in \mathbb{N}\}$ belongs to Γ. Therefore, X satisfies $\mathsf{S}_1(\Gamma_{\mathbb{B}}, \Gamma)$.

(4) $\Rightarrow$ (5): Let σ be a strategy for ONE in $\mathsf{G}_1(\Gamma_{\mathbb{B}}, \Gamma)$. Let the first move of ONE be $\sigma(\emptyset) = \{U_{(1)}, U_{(2)}, \cdots, U_{(n)}, \cdots\}$, a $\gamma_{\mathbb{B}}$-cover of X. Suppose that for each finite sequence s of natural numbers of length $\leq m$, U_s has been defined. Define now $\{U_{(n_1,\cdots,n_m,k)} : k \in \mathbb{N}\}$ as the set

$$\sigma(U_{(n_1)}, U_{(n_1,n_2)}, \cdots, U_{(n_1,\cdots,n_m)}) \setminus \{U_{(n_1)}, U_{(n_1,n_2)}, \cdots, U_{(n_1,\cdots,n_m)}\}.$$

Clearly, we have that for each finite sequence s of natural numbers, the set $\{U_{s\frown(n)} : n \in \mathbb{N}\}$ is a $\gamma_{\mathbb{B}}$-cover of X. By (4) for each s we may choose $n_s \in \mathbb{N}$ such that $\{U_{s\frown(n_s)} : s$ a finite sequence of natural numbers$\}$ is a γ-cover of X. Define inductively the sequence $n_1 = n_\emptyset$, $n_{k+1} = n_{(n_1,\cdots,n_k)}$, for $k \geq 1$. Then the sequence

$$U_{(n_1)}, U_{(n_1,n_2)}, \cdots, U_{(n_1,\cdots,n_k)}, \cdots$$

of moves of TWO is in fact a γ-cover of X which shows that σ is not a winning strategy for ONE in the game $\mathsf{G}_1(\Gamma_{\mathbb{B}}, \Gamma)$.

(5) $\Rightarrow$ (1): Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of $\gamma_{\mathbb{B}}$-covers of X; suppose $\mathcal{U}_n = \{U_{n,m} : m \in \mathbb{N}\}$, $n \in \mathbb{N}$. A strategy σ for ONE will be defined in this way. ONE's first move is $\sigma(\emptyset) = \mathcal{U}_1$. Let $U_{1,m_{i_1}} \in \mathcal{U}_1$ be TWO's response. Then ONE looks at the $\gamma_{\mathbb{B}}$-cover $\mathcal{V}(1, m_{i_1}) = \{U_{1,m} : m > m_{i_1}\}$ and plays $\sigma(U_{1,m_{i_1}}) = \mathcal{V}(1, m_{i_1})$. If TWO takes a set $U_{1,m_{i_2}} \in \mathcal{V}(1, m_{i_1})$, then ONE plays $\sigma(U_{1,m_{i_1}}, U_{1,m_{i_2}}) = \mathcal{V}(1, m_{i_2}) = \{U_{1,m} : m > m_{i_2}\}$, still a $\gamma_{\mathbb{B}}$-cover of X. TWO chooses a set $U_{1,m_{i_3}} \in \sigma(U_{1,m_{i_1}}, U_{1,m_{i_2}})$, and so on as in the proof of Theorem 3.2.

σ is not a winning strategy for ONE, so that there exists a σ-play

$$\sigma(\emptyset), U_{1,m_{i_1}}; \sigma(U_{1,m_{i_1}}), U_{1,m_{i_2}}; \sigma(U_{1,m_{i_1}}, U_{1,m_{i_2}}), U_{1,m_{i_3}}; \cdots$$

lost by ONE. In other words, TWO's moves $U_{1,m_{i_1}}, U_{1,m_{i_2}}, U_{1,m_{i_3}}, \cdots$ form a sequence which is a γ-cover of X and, obviously, it contains infinitely many elements from each $\mathcal{U}_n$, $n \in \mathbb{N}$. So, that sequence shows that (1) holds. ∎

Similarly one proves the following theorem.

THEOREM 3.5. *For a space X the following statements are equivalent:*

(1) *X satisfies $\alpha_2(\Gamma_{\mathbb{B}}, \Gamma_{\mathbb{B}})$;*

(2) *X satisfies $\alpha_3(\Gamma_{\mathbb{B}}, \Gamma_{\mathbb{B}})$;*

(3) *X satisfies $\alpha_4(\Gamma_{\mathbb{B}}, \Gamma_{\mathbb{B}})$;*

(4) *X satisfies $\mathsf{S}_1(\Gamma_{\mathbb{B}}, \Gamma_{\mathbb{B}})$;*

(5) *ONE has no winning strategy in the game $\mathsf{G}_1(\Gamma_{\mathbb{B}}, \Gamma_{\mathbb{B}})$ played on X.*

We omit the proof of the following statement.

THEOREM 3.6. *For a space X and $n, k \in \mathbb{N}$ the following are equivalent:*

(1) *$\mathsf{S}_1(\mathcal{O}_{\mathbb{B}}, \mathcal{O}_{\mathbb{B}})$;*

(2) *X satisfies $\mathcal{O}_{\mathbb{B}} \to (\mathcal{O}_{\mathbb{B}})^n_k$.*

We shall consider now another class of spaces.

The symbol $\mathsf{U}_{fin}(\Gamma_{\mathbb{B}}, \mathcal{O}_{\mathbb{B}})$ denotes the selection principle:

- For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of $\gamma_{\mathbb{B}}$-covers of X there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that each $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$, and either $\{\cup\mathcal{V}_n : n \in \mathbb{N}\}$ is a $\mathbb{B}$-cover for X, or for some $n \in \mathbb{N}$, $X = \cup\mathcal{V}_n$.

An open cover $\mathcal{U}$ of a space X is said to be *large* if each point of X belongs to infinitely many elements of $\mathcal{U}$. A countable large cover $\mathcal{U}$ of X is $\mathbb{B}$-*weakly groupable* if there is a partition of $\mathcal{U}$ into infinitely many finite, pairwise disjoint subsets $\mathcal{U}_n$

such that each $B \in \mathbb{B}$ is contained in $\cup\mathcal{U}_n$ for some $n \in \mathbb{N}$. Let $\Lambda^{\mathbb{B}-wgp}$ denote the family of $\mathbb{B}$-weakly groupable large covers of a space.

THEOREM 3.7. *For a space X the following assertions are equivalent:*

(1) *X has property $\mathsf{U}_{fin}(\Gamma_\mathbb{B}, \mathcal{O}_\mathbb{B})$;*

(2) *X has property $\mathsf{S}_{fin}(\Gamma_\mathbb{B}, \Lambda^{\mathbb{B}-wgp})$;*

(3) *For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of $\gamma_\mathbb{B}$-covers of X there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$, $\mathcal{V}_n$'s are pairwise disjoint and for each $B \in \mathbb{B}$ there exists some $n \in \mathbb{N}$ with $B \subset \cup\mathcal{V}_n$.*

Proof. (1) $\Rightarrow$ (2): Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of $\gamma_\mathbb{B}$-covers of X. Because each infinite subset of a $\gamma_\mathbb{B}$-cover is also a $\gamma_\mathbb{B}$-cover one can suppose that $\mathcal{U}_n$'s are pairwise disjoint. Moreover, without loss of generality, we may assume that for each $n \in \mathbb{N}$, no finite subset of $\mathcal{U}_n$ is a cover of X.

Apply (1) to find a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ of finite sets such that for each $n \in \mathbb{N}$, $\mathcal{V}_n \subset \mathcal{U}_n$ and $\{\cup\mathcal{V}_n : n \in \mathbb{N}\}$ is a $\mathbb{B}$-cover of X. This implies that $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is a large cover of X, while its partition $\{\mathcal{V}_n : n \in \mathbb{N}\}$ shows that $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is in fact $\mathbb{B}$-weakly groupable.

(2) $\Rightarrow$ (3): Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of (countable) $\gamma_\mathbb{B}$–covers of X. As in (1) $\Rightarrow$ (2) one can suppose that $\mathcal{U}_n$'s are pairwise disjoint. For $n \in \mathbb{N}$ let $\mathcal{U}_n = \{U_{n,m} : m \in \mathbb{N}\}$. Define new $\gamma_\mathbb{B}$-covers $\mathcal{V}_n$, $n \in \mathbb{N}$, as follows:

$$\mathcal{V}_n = \{U_{1,m} \cap U_{2,m} \cap \cdots \cap U_{n,m} : m \in \mathbb{N}\} \setminus \{\emptyset\}.$$

We again may suppose that $\mathcal{V}_{n_1} \cap \mathcal{V}_{n_2} = \emptyset$ for $n_1 \neq n_2$.

Apply (2) to the sequence $(\mathcal{V}_n : n \in \mathbb{N})$ and choose a sequence $(\mathcal{W}_n : n \in \mathbb{N})$ of finite sets such that for each $n \in \mathbb{N}$, $\mathcal{W}_n \subset \mathcal{V}_n$ (so $\mathcal{W}_n$'s are pairwise disjoint) and $\bigcup_{n\in\mathbb{N}} \mathcal{W}_n$ is a $\mathbb{B}$-weakly groupable large cover of X. This means that $\bigcup_{n\in\mathbb{N}} \mathcal{W}_n = \bigcup_{n\in\mathbb{N}} \mathcal{S}_n$, where $\mathcal{S}_n$'s are finite, pairwise disjoint and each $B \in \mathbb{B}$ is contained in $\cup\mathcal{S}_n$ for some $n \in \mathbb{N}$.

Because sets $\mathcal{W}_n$ and $\mathcal{S}_n$ are finite, there are $p \in \mathbb{N}$ such that $\mathcal{W}_1 \cap \mathcal{S}_q = \emptyset$ for $q > p$. Let p_1 be the smallest such natural number. Define $\mathcal{T}_1$ to be the set of all $U_{1,m}$ such that $U_{1,m}$ is a term in the above representation of a member of $\mathcal{S}_q$ for some $q \leq p_1$. Let $p_2 > p_1$ be the minimal natural number such that $\mathcal{W}_2 \cap \mathcal{S}_q = \emptyset$ whenever $q > p_2$. Let $\mathcal{T}_2$ be the set of all $U_{2,m}$ such that $U_{2,m}$ is a term in the representation of an element of $\mathcal{S}_q$ for some $q \leq p_2$. And so on.

The obtained sequence $(\mathcal{T}_n : n \in \mathbb{N})$ is such that for each $n \in \mathbb{N}$ $\mathcal{T}_n$ is a finite subset of $\mathcal{U}_n$, and thus $\mathcal{T}_{n_1} \cap \mathcal{T}_{n_2} = \emptyset$ for $n_1 \neq n_2$. We are going to prove that the sequence $(\mathcal{T}_n : n \in \mathbb{N})$ witnesses that (3) is satisfied.

Let $B \in \mathbb{B}$. There is $n \in \mathbb{N}$ with $B \subset \cup\mathcal{S}_n$. Choose the smallest k such that $n \leq p_k$. Then $(\mathcal{W}_1 \cup \cdots \cup \mathcal{W}_{k-1}) \cap \mathcal{S}_n = \emptyset$. It follows that each element S in $\mathcal{S}_n$ has in its representation a set of the form $U_{k,j}$ (note that such elements are in $\mathcal{T}_k$) and therefore we have $\cup\mathcal{S}_n \subset \cup\mathcal{T}_k$. It follows $B \subset \cup\mathcal{T}_k$ which means that (3) holds.

(3) $\Rightarrow$ (1): It is clear by the definition of $\mathsf{U}_{fin}(\Gamma_\mathbb{B}, \mathcal{O}_\mathbb{B})$. ∎

4. Boundedness and function spaces

All spaces in this section are assumed to be Tychonoff. For a boundedness $\mathbb{B}$ in a space X, by $\mathsf{C}_b(X)$ we denote the space of continuous real-valued functions on X in the topology whose basic open sets are of the form

$$W(B_1, \dots, B_n; V_1, \dots, V_n) := \{f \in C(X) : f(B_i) \subset V_i, i = 1, \dots, n\},$$

where $B_1, \dots, B_n \in \mathbb{B}$ and $V_1 \dots, V_n$ are open in $\mathbb{R}$. For a function $f \in \mathsf{C}_b(X)$, a set $B \in \mathbb{B}$ and a positive real number ε we let

$$W(f; B; \varepsilon) := \{g \in \mathsf{C}_b(X) : |g(x) - f(x)| < \varepsilon, \ \forall x \in B\}.$$

The standard local base of a point $f \in \mathsf{C}_b(X)$ consists of the sets $W(f; B; \varepsilon)$, where B is a set from $\mathbb{B}$ and ε is a positive real number.

The symbol $\underline{\mathsf{0}}$ denotes the constantly zero function in $\mathsf{C}_b(X)$. The space $\mathsf{C}_b(X)$ is homogeneous so that we may consider the point $\underline{\mathsf{0}}$ when studying local properties of $\mathsf{C}_b(X)$.

For a space X and a point $x \in X$ the symbol Ω_x denotes the set $\{A \subset X \setminus \{x\} : x \in \overline{A}\}$.

A space X has *countable fan tightness* if for each $x \in X$ we have that $\mathsf{S}_{fin}(\Omega_x, \Omega_x)$ holds. X has *countable strong fan tightness* if for each $x \in X$ the selection principle $\mathsf{S}_1(\Omega_x, \Omega_x)$ holds.

THEOREM 4.1. *If a Tychonoff space X has property $\mathsf{S}_1(\mathcal{O}_\mathbb{B}, \mathcal{O}_\mathbb{B})$, then $\mathsf{C}_b(X)$ has countable strong fan tightness.*

Proof. Let $(A_n : n \in \mathbb{N})$ be a sequence of subsets of $\mathsf{C}_b(X)$ the closures of which contain $\underline{\mathsf{0}}$. Fix n. For every $B \in \mathbb{B}$ the neighborhood $W = W(\underline{\mathsf{0}}; B; 1/n)$ of $\underline{\mathsf{0}}$ intersects A_n so that there exists a function $f_{B,n} \in A_n$ such that $|f_{B,n}(x)| < 1/n$ for each $x \in B$. Since $f_{B,n}$ is a continuous function there are neighborhoods O_x, $x \in B$, such that for $U_{B,n} = \bigcup_{x \in B} O_x \supset B$ we have $f_{B,n}(U_{B,n}) \subset (-1/n, 1/n)$. Let $\mathcal{U}_n = \{U_{B,n} : B \in \mathbb{B}\}$. For each $n \in \mathbb{N}$, $\mathcal{U}_n$ is a $\mathbb{B}$-cover of X. Applying that X is an $\mathsf{S}_1(\mathcal{O}_\mathbb{B}, \mathcal{O}_\mathbb{B})$-set, choose $U_{B_n,n} \in \mathcal{U}_n$, $n \in \mathbb{N}$, such that $\{U_{B_n,n} : n \in \mathbb{N}\}$ is a $\mathbb{B}$-cover of X. Look at the corresponding functions $f_{B_n,n}$ in A_n.

Let us show $\underline{\mathsf{0}} \in \overline{\{f_{B_n,n} : n \in \mathbb{N}\}}$. Let $W = W(\underline{\mathsf{0}}; C; \varepsilon)$, $C \in \mathbb{B}$, be a neighborhood of $\underline{\mathsf{0}}$ in $\mathsf{C}_b(X)$ and let m be a natural number such that $1/m < \varepsilon$. Since $C \in \mathbb{B}$ there is $j \in \mathbb{N}$, $j \geq m$ such that one can find a $U_{B_j,j}$ with $C \subset U_{B_j,j}$. We have

$$f_{B_j,j}(C) \subset f_{B_j,j}(U_{B_j,j}) \subset (-1/j, 1/j) \subset (-1/m, 1/m) \subset (-\varepsilon, \varepsilon),$$

i.e. $f_{B_j,j} \in W$. ■

In a similar way one proves the following theorem.

THEOREM 4.2. *If a Tychonoff space X has property $\mathsf{S}_{fin}(\mathcal{O}_\mathbb{B}, \mathcal{O}_\mathbb{B})$, then $\mathsf{C}_b(X)$ has countable fan tightness.*

Let us point out that for some boundedness the converses of Theorems 4.1 and 4.2 are also true (see, for example, [23, 1, 12]). But we have not yet investigated other significant cases for which the converse holds.

Recall now the following definition. A space X has *countable T-tightness* if for each uncountable regular cardinal ρ and each increasing sequence $(A_\alpha : \alpha < \rho)$ of closed subsets of X, the set $\cup\{A_\alpha : \alpha < \rho\}$ is closed.

THEOREM 4.3. *If a Tychonoff space X satisfies the condition*

($*$) *for each uncountable regular cardinal ρ and each increasing sequence $(\mathcal{U}_\alpha : \alpha < \rho)$ of families of open subsets of X such that $\bigcup_{\alpha<\rho}\mathcal{U}_\alpha$ is a $\mathbb{B}$-cover of X there is a $\beta < \rho$ so that $\mathcal{U}_\beta$ is a $\mathbb{B}$-cover of X,*

then $\mathsf{C}_b(X)$ has countable T-tightness.

Proof. Let $(A_\alpha : \alpha < \rho)$ be an increasing sequence of closed subsets of $\mathsf{C}_b(X)$, with ρ a regular uncountable cardinal. We shall prove that the set $A := \bigcup_{\alpha<\rho} A_\alpha$ is closed. Let $f \in \overline{A}$. For each $n \in \mathbb{N}$ and each set $B \in \mathbb{B}$ the neighborhood $W(f; B; 1/n)$ of f intersects A. Put

$$\mathcal{U}_{n,\alpha} = \{(f-g)^{\leftarrow}(-1/n, 1/n) : g \in A_\alpha\}$$

and

$$\mathcal{U}_n = \bigcup_{\alpha<\rho} \mathcal{U}_{n,\alpha}.$$

Let us check that for each $n \in \mathbb{N}$, $\mathcal{U}_n$ is a $\mathbb{B}$-cover of X. Let B be a set from $\mathbb{B}$. The neighborhood $W := W(f; B; 1/n)$ of f intersects A, i.e. there is $g \in A$ such that $|f(x) - g(x)| < 1/n$ for all $x \in B$; this means $B \subset (f-g)^{\leftarrow}(-1/n, 1/n) \in \mathcal{U}_n$.

By ($*$) there is $\mathcal{U}_{n,\beta_n} \subset \mathcal{U}_n$ which is a $\mathbb{B}$-cover of X. Put $\beta_0 = \sup\{\beta_n : n \in \mathbb{N}\}$. Since ρ is a regular uncountable cardinal, $\beta_0 < \rho$. It is easy to verify that for each n the set $\mathcal{U}_{n,\beta_0}$ is a $\mathbb{B}$-cover of X. Let us show that $f \in A_{\beta_0}$. Take a neighborhood $W(f; C; \varepsilon)$, $C \in \mathbb{B}$, of f and let m be a positive integer such that $1/m < \varepsilon$. Since $\mathcal{U}_{m,\beta_0}$ is a $\mathbb{B}$-cover of X one can find $g \in A_{\beta_0}$ such that $C \subset (f-g)^{\leftarrow}(-1/m, 1/m)$. Then $g \in W(f; C; 1/m) \cap A_{\beta_0} \subset W(f; C; \varepsilon) \cap A_{\beta_0}$, i.e. $f \in \overline{A_{\beta_0}} = A_{\beta_0}$ and thus $f \in A$. So, A is closed. ∎

A space X is called a *selectively strictly A-space* [21] if for each sequence $(A_n : n \in \mathbb{N})$ of subsets of X and each point $x \in X$ such that $x \in \overline{A_n} \setminus A_n$ for each $n \in \mathbb{N}$, there is a sequence $(C_n : n \in \mathbb{N})$, where for each n $C_n \subset A_n$, and $x \in \overline{\bigcup_{n\in\mathbb{N}} C_n} \setminus \bigcup_{n\in\mathbb{N}} \overline{C_n}$.

Let us say that a space X has property $S_{\mathbb{B}}(X)$ if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of $\mathbb{B}$-covers of X there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that $\mathcal{V}_n \subset \mathcal{U}_n$ for each n, no $\mathcal{V}_n$ is a $\mathbb{B}$-cover of X, and $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is a $\mathbb{B}$-cover for X.

THEOREM 4.4. *If a Tychonoff space X has the property $S_{\mathbb{B}}(X)$, then $\mathsf{C}_b(X)$ is a selectively strictly A-space.*

Proof. Let $(A_n : n \in \mathbb{N})$ be a sequence of subsets of $\mathsf{C}_b(X)$ with $\underline{0} \in \overline{A_n} \setminus A_n$, $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ let

$$\mathcal{U}_n = \{f^{\leftarrow}(-1/n, 1/n) : f \in A_n\}.$$

It is easy to see that for each n, $\mathcal{U}_n$ is a $\mathbb{B}$-cover of X.

Case 1: $X \in \mathcal{U}_n$ for infinitely many n.

Then there exist an increasing sequence $n_1 < n_2 < \cdots < n_k < \dots$ in $\mathbb{N}$ and $f_{n_k} \in A_{n_k}$, $k \in \mathbb{N}$, such that $f_{n_k}^{\leftarrow}(-1/n_k, 1/n_k) = X$. Put $C_{n_k} = \{f_{n_k}\}$, $k \in \mathbb{N}$, and $C_n = \emptyset$ for $n \neq n_k$, $k \in \mathbb{N}$. Then $\underline{0} \in \overline{\bigcup_{n\in\mathbb{N}} C_n} \setminus \bigcup_{n\in\mathbb{N}} \overline{C_n}$.

Case 2: $X \in \mathcal{U}_n$ for finitely many n.

Without loss of generality one may suppose that $X \notin \mathcal{U}_n$ for each n. Using the fact that X has property $S_{\mathbb{B}}(X)$, choose a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ as in the definition of the property $S_{\mathbb{B}}(X)$. For each $V \in \mathcal{V}_n$ pick a function $f_V \in A_n$ with $V = f_V^{\leftarrow}(-1/n, 1/n)$ and put $C_n = \{f_V : V \in \mathcal{V}_n\}$. Let us show that $(C_n : n \in \mathbb{N})$ is the required sequence of subsets of A_n's.

First, $\underline{0} \notin \overline{C_n}$, $n \in \mathbb{N}$. Otherwise, the fact $\underline{0} \in \overline{C_m}$ for some m would imply that the corresponding $\mathcal{V}_m$ is a $\mathbb{B}$-cover of X which is a contradiction.

Let now $W = W(\underline{0}, B, 1/n)$, with $B \in \mathbb{B}$, be a neighborhood of $\underline{0}$. Since $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is a $\mathbb{B}$-cover of X there is $m > n$ such that for some $V \in \mathcal{V}_m$ we have $B \subset V = f_V^{\leftarrow}(-1/m, 1/m)$, and $f_V \in C_m$. Clearly, $f_V \in W$ and so $\underline{0} \in \overline{\bigcup_{n\in\mathbb{N}} C_n}$ which completes the proof. ■

5. Boundedness and hyperspaces

Let $\mathbb{B}$ be a fixed boundedness in a space X. If A is a subset of X and $\mathcal{A}$ a family of subsets of X, then we write

$$A^+ = \{B \in \mathbb{B} : B \subset A\},\ \mathcal{A}^+ = \{A^+ : A \in \mathcal{A}\}.$$

The *upper Vietoris topology* τ_{V^+} on $\mathbb{B}$ is the topology whose basic sets are of the form U^+, U open in X.

We need the following lemma.

LEMMA 5.1. *For a space X and an open cover $\mathcal{W}$ of $(\mathbb{B}, \tau_{\bigvee^+})$ the following holds: $\mathcal{W}$ is an ω-cover of $(\mathbb{B}, \tau_{\bigvee^+})$ if and only if $\mathcal{U}(\mathcal{W}) := \{U \subset X : U$ is open in X and $U^+ \subset W$ for some $W \in \mathcal{W}\}$ is a $\mathbb{B}$-cover of X.*

Proof. Let $\mathcal{W}$ be an ω-cover of $(\mathbb{B}, \tau_{\bigvee^+})$ and let $B \in \mathbb{B}$. Then there exists $W \in \mathcal{W}$ such that $B \in W$ and thus there is an open set $U \subset X$ with $B \in U^+ \subset W$. Clearly, $U \in \mathcal{U}(\mathcal{W})$. On the other hand, $B \subset U$, i.e. $\mathcal{U}(\mathcal{W})$ is a $\mathbb{B}$-cover of X.

Conversely, let $\mathcal{U}(\mathcal{W})$ be a $\mathbb{B}$-cover of X and let $\{B_1, \cdots, B_m\}$ be a finite subset of $(\mathbb{B}, \tau_{\bigvee^+})$. Then $B = \bigcup_{i=1}^m B_i$ is in $\mathbb{B}$ and thus B is contained in some $U \in \mathcal{U}(\mathcal{W})$; pick $W \in \mathcal{W}$ such that $U^+ \subset W$. From $B_i \subset U$ for each $i \leq m$, it follows $\{B_1, \cdots, B_m\} \subset U^+ \subset W$ which just means that $\mathcal{W}$ is an ω-cover of $(\mathbb{B}, \tau_{\bigvee^+})$. ■

THEOREM 5.2. *A space X is $\mathbb{B}$-Lindelöf if and only if $(\mathbb{B}, \tau_{\bigvee^+})$ is ω-Lindelöf.*

Proof. Let X be a $\mathbb{B}$-Lindelöf space and let $\mathcal{W}$ be an ω-cover of $(\mathbb{B}, \tau_{\bigvee^+})$. By Lemma 5.1, $\mathcal{U}(\mathcal{W})$ is a $\mathbb{B}$-cover of X. Choose a countable family $\{U_i : i \in \mathbb{N}\} \subset \mathcal{U}(\mathcal{W})$ which is a $\mathbb{B}$-cover of X. For each $i \in \mathbb{N}$ choose $W_i \in \mathcal{W}$ such that $U_i^+ \subset W_i$. Again by Lemma 5.1 $\{W_i : i \in \mathbb{N}\} \subset \mathcal{W}$ is an ω-cover of $(\mathbb{B}, \tau_{\bigvee^+})$.

Let us show the converse. Let $\mathcal{U}$ be a $\mathbb{B}$-cover of X. It is easy to check that $\mathcal{U}^+$ is an ω-cover of $(\mathbb{B}, \tau_{\bigvee^+})$. Choose a countable collection $\{U_i^+ : i \in \mathbb{N}\} \subset \mathcal{U}^+$ which is an ω-cover of $(\mathbb{B}, \tau_{\bigvee^+})$. Then $\{U_i : i \in \mathbb{N}\} \subset \mathcal{U}$ is a $\mathbb{B}$-cover of X, i.e. X is a $\mathbb{B}$-Lindelöf space. ■

THEOREM 5.3. *For a $\mathbb{B}$-Lindelöf space X the following are equivalent:*

(1) $(\mathbb{B}, \tau_{\bigvee^+})$ *satisfies* $\mathsf{S}_1(\Omega, \Gamma)$;

(2) X *satisfies* $\mathsf{S}_1(\mathcal{O}_{\mathbb{B}}, \Gamma_{\mathbb{B}})$.

Proof. (1) $\Rightarrow$ (2): Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of $\mathbb{B}$-covers of X. Then $(\mathcal{U}_n^+ : n \in \mathbb{N})$ is a sequence of ω-covers of $(\mathbb{B}, \tau_{\bigvee^+})$. To check this, fix n and let $\{B_1, \cdots, B_m\}$ be a finite subset of $\mathbb{B}$. Then $B = B_1 \cup \cdots \cup B_m$ is in $\mathbb{B}$ and thus there is $U \in \mathcal{U}$ with $B \subset U$. This means that for each $i \leq m$, $B_i \subset U$, i.e. $B_i \in U^+$. Therefore $\{B_1, \cdots, B_m\} \subset U^+$ and $\mathcal{U}_n$ is an ω-cover of $\mathbb{B}$. By (1) for each n, one can choose an element U_n^+ in $\mathcal{U}_n^+$ such that the set $\mathcal{U}^+ = \{U_n^+ : n \in \mathbb{N}\}$ is a γ-cover of $(\mathbb{B}, \tau_{\bigvee^+})$. Let us prove that $\{U_n : n \in \mathbb{N}\}$ is a $\gamma_{\mathbb{B}}$-cover of X. Let $B \in \mathbb{B}$. Then there is $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$ we have $B \in U_n^+$, i.e. $B \subset U_n$. It shows that $\{U_n : n \in \mathbb{N}\}$ is indeed a $\gamma_{\mathbb{B}}$-cover of X, i.e. that (2) holds.

(2) $\Rightarrow$ (1): Let $(\mathcal{W}_n : n \in \mathbb{N})$ be a sequence of ω-covers of $(\mathbb{B}, \tau_{\bigvee^+})$. For each n let

$$\mathcal{U}_n = \{U \subset X : U \text{ is open in } X \text{ and } U^+ \subset W \text{ for some } W \in \mathcal{W}_n\}.$$

It is easy to prove that each $\mathcal{U}_n$ is a $\mathbb{B}$-cover of X. Apply (2) to the sequence $(\mathcal{U}_n : n \in \mathbb{N})$ to find a sequence $(U_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $U_n \in \mathcal{U}_n$ and the set $\mathcal{U} = \{U_n : n \in \mathbb{N})$ is a $\gamma_{\mathbb{B}}$-cover of X. For each $U_n \in \mathcal{U}$ take an element $W_n \in \mathcal{W}_n$ with $U_n^+ \subset W_n$. We claim that $\{W_n : n \in \mathbb{N}\}$ is a γ-cover of $(\mathbb{B}, \tau_{\bigvee^+})$ which witnesses for $(\mathcal{W}_n : n \in \mathbb{N})$ that (1) is satisfied. Let $B \in \mathbb{B}$. Then there is n_0 such that for each $n \geq n_0$, $B \subset U_n$, i.e. $B \in U_n^+ \subset W_n$. ■

REFERENCES

[**1**] A.V. Arhangel'skiĭ, *Topological Function Spaces*, Kluwer Academic Publishers, 1992.

[**2**] A. Caserta, G. Di Maio, Lj.D.R. Kočinac, E. Meccariello, *Applications of k-covers* II, Topology Appl. **153** (2006), 3277–3293.

[**3**] G. Di Maio, Lj.D.R. Kočinac, *Some covering properties of hyperspaces*, Topology Appl., to appear.

[**4**] G. Di Maio, Lj.D.R. Kočinac, E. Meccariello, *Applications of k-covers*, Acta Mathematica Sinica, English Series **22** (2006), 1151–1160.

[**5**] G. Di Maio, Lj.D.R. Kočinac, E. Meccariello, *Selection principles and hyperspace topologies*, Topology Appl. **153** (2005), 912–923.

[**6**] G. Di Maio, E. Meccariello, S. Naimpally, *Bombay topologies*, Applied General Topology **4** (2003), 421–444.

[**7**] R. Engelking, *General Topology*, PWN, Warszawa, 1977.

[**8**] J. Gerlits, Zs. Nagy, *Some properties of* $C(X)$, I, Topology Appl. **14** (1982), 151–161.

[**9**] S.T. Hu, *Boundedness in a topological space*, Journal Math. Pures Appl. **28** (1949), 287–320.

[**10**] W. Hurewicz, *Über die Verallgemeinerung des Borelschen Theorems*, Math. Zeitschrift **24** (1925), 401–425.

[**11**] W. Just, A.W. Miller, M. Scheepers, P.J. Szeptycki, *The combinatorics of open covers* II, Topology Appl. **73** (1996), 241–266.

[**12**] Lj.D.R. Kočinac, *Closure properties of function spaces*, Applied General Topology **4** (2003), 255–261.

[**13**] Lj.D.R. Kočinac, *γ-sets, γ_k-sets and hyperspaces*, Math. Balkanica **19** (2005), 109–118.

[**14**] Lj.D.R. Kočinac, *Some covering properties in topological and uniform spaces*, Proceedings of the Steklov Institute of Mathematics **252** (2006), 122–137.

[**15**] Lj.D.R. Kočinac, *Selection principles in hyperspaces*, In: *Selection Principles and Covering Properties in Topology* (Lj.D.R. Kočinac, ed.), Quaderni di Matematica, Vol. 18, Caserta, 2006, pp. 157–194.

[**16**] Lj.D.R. Kočinac, *Selection principles related to α_i-properties*, Taiwanese J. Math., to appear (avaliable at: `http://arxiv.math.GN/0608107`).

[**17**] Lj.D. Kočinac, M. Scheepers, *Function spaces and a property of Reznichenko*, Topology Appl. **123** (2002), 135–143.

[**18**] Lj.D.R. Kočinac, M. Scheepers, *Combinatorics of open covers* (VII): *Groupability*, Fundamenta Math. **179** (2003), 131–155.

[**19**] K. Menger, *Einige Überdeckungssätze der Punktmengenlehre*, Sitzungsberichte Abt. 2a, Mathematik, Astronomie, Physik, Meteorologie und Mechanik (Wiener Akademie, Wien) **133** (1924), 421–444.

[**20**] S.A. Naimpally, B.D. Warrack, *Proximity Spaces*, Cambridge Tracts in Mathematics, No. **59**, Cambridge, 1970.

[**21**] V. Pavlović, *Selectively strictly A function spaces*, Note di Matematica, to appear.

[**22**] F. Rothberger, *Eine Verschärfung der Eigenschaft C*, Fundamenta Math. **30** (1938), 50–55.

[**23**] M. Sakai, *Property C'' and function spaces*, Proc. Am. Math. Soc. **104** (1988), 917–919.

[**24**] M. Scheepers, *Combinatorics of open covers* I: *Ramsey Theory*, Topology Appl. **69** (1996), 31–62.

[**25**] M. Scheepers, *Selection principles and covering properties in Topology*, Note di Matematica **22**:2 (2003/2004), 3–41.

[**26**] B. Tsaban, *Some new directions in infinite-combinatorial topology*, In: *Set Theory* (J. Bagaria and S. Todorčević, eds.), Trends in Mathematics, Birkhäuser, 2006, 225–255.

[**27**] B. Tsaban, *On the Kočinac α_i properties*, Topology Appl. **155** (2007), 141–145.

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