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# COMPACT COMPOSITION OPERATORS ON HARDY-ORLICZ SPACES

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**Abstract.** In this paper, compact composition operators acting on Hardy-Orlicz spaces

$$H^\Phi = \Big\{ f \in H(\mathbb{D}) : \sup_{0<r<1} \int_{\partial\mathbb{D}} \Phi(\log^+ |f(re^{i\theta})|)\, d\sigma < \infty \Big\}$$

are studied. In fact, we prove that if $\Phi$ is a twice differentiable, non-constant, non-decreasing non-negative, convex function on $\mathbb{R}$, then the composition operator $C_\varphi$ induced by a holomorphic self-map $\varphi$ of the unit disk is compact on Hardy-Orlicz spaces $H^\Phi$ if and only if it is compact on the Hardy space $H^2$.

## 1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $\varphi$ be a holomorphic self-map of $\mathbb{D}$. Then the equation $C_\varphi f = f \circ \varphi$, for $f$ analytic in $\mathbb{D}$ defines a composition operator $C_\varphi$ with inducing map $\varphi$. As a consequence of the Littlewood subordination principle, every $\varphi$ induces a bounded composition operator on the classical Hardy spaces $H^p$ $(0 < p < \infty)$ and the weighted Bergman spaces $A^p_\alpha$ for all $p$ $(0 < p < \infty)$ and for all $\alpha$ $(-1 < \alpha < \infty)$ of the disk [4]. Amongst the nice composition operators on these spaces are compact composition operators.

The study of compact composition operators on $H^2$ was initiated by H. J. Schwartz [9] in his thesis in the late 1960's. This work was continued by Shapiro and Taylor [10], who showed that $C_\varphi$ is not compact whenever $\varphi$ has an angular derivative at some point of the unit circle. Non-existence of the angular derivative is not a sufficient condition for compactness of $C_\varphi$ in general. MacCluer and Shapiro [7] showed that the non-existence of the angular derivative is also a sufficient condition for compactness of $C_\varphi$ on the weighted Bergman spaces $A^p_\alpha$ but it fails to be a sufficient condition for compactness of $C_\varphi$ on Hardy spaces $H^p$. However the

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angular derivative condition does characterize the compactness of $C_\varphi$ on $H^p$ if the inducing map is univalent.

Finally J. H. Shapiro [12] in 1987 was able to discover the connection between the essential norm of a composition operator on the Hardy space $H^2$ and the Nevanlinna counting function for $\varphi$, which is defined as $N_\varphi(w) = \sum_{z\in\varphi^{-1}(w)} \log \frac{1}{|z|}$, and obtained the general expression

$$||C_\varphi||_e^2 = \limsup_{|w|\to 1^-} \frac{N_\varphi(w)}{\log \frac{1}{|w|}},$$

where by the essential norm of $C_\varphi$, we mean its distance, in the operator norm, from the space of compact operators on $H^2$. In particular, he proved that $C_\varphi$ is compact on $H^2$ if and only if $N_\varphi(w) = o\big(\log \frac{1}{|w|}\big)$ as $|w| \to 1$, thus providing a complete function theoretic characterization of compact composition operators in terms of the inducing map's Nevanlinna counting function $N_\varphi$.

Another solution to the compactness problem can be given by means of the positive measures $m_\lambda$ that are defined on the unit circle $\partial\mathbb{D}$ by the Poisson representation

$$\Re\frac{\lambda+\varphi(z)}{\lambda-\varphi(z)} = \int_{\partial\mathbb{D}} P(z,\zeta)\, dm_\lambda(\zeta)$$

for each $\lambda \in \partial\mathbb{D}$. These measures are often called the Aleksandrov measures of $\varphi$. In [2], Cima and Matheson showed that the essential norm of $C_\varphi$ on $H^2$ can also be expressed as

$$\|C_\varphi\|_e^2 = \sup_{\lambda\in\partial\mathbb{D}} \|\sigma_\lambda\|,$$

where $\sigma_\lambda$ is the singular part of $m_\lambda$. In particular, it follows that $C_\varphi$ is compact on the Hardy space $H^2$ if and only if all the measures $m_\lambda$ are absolutely continuous. If $\varphi$ is a holomorphic-self map of $\mathbb{D}$, then Liu, Cao and Wang [5] show that $C_\varphi$ is bounded on each of the Hardy-Orlicz spaces. Furthermore, they discuss the compactness of $C_\varphi$ on a particular subspace of $H^\Phi$, a question that is intimately related to the main result of this paper. In fact, we are inspired by the following results.

(1). If $C_\varphi$ is compact on one of the Hardy space $H^p$ for some $p$ $(0 < p < \infty)$, then it is compact on all of the Hardy spaces $H^p$ $(0 < p < \infty)$ [10].

(2). A holomorphic composition operator is compact on $L^1$ if and only if it is compact on $H^2$ [11].

(3). For an arbitrary $\varphi$ the compactness of $C_\varphi$ on Hardy spaces $H^p$ is quite different from the compactness of $C_\varphi$ on weighted Bergman spaces $A^p_\alpha$. For example, there exists inner function $\varphi$ such that $C_\varphi$ is compact on $A^p_\alpha$ for all $p$ $(0 < p < \infty)$ and for all $\alpha$ $(-1 < \alpha < \infty)$ but it is well known that no inner function can induce a compact composition operator on any Hardy space $H^p$ [7].

(4). That $C_\varphi$ is compact on the Nevanlinna class $N$ if and only if it is compact on $H^2$ [1].

All these results lead us to ask whether the compactness of $C_\varphi$ on Hardy-Orlicz spaces implies compactness of $C_\varphi$ on $H^2$ and conversely. The purpose of this paper is to give an affirmative answer to this question.

## 2. Preliminaries

Let $H(\mathbb{D})$ denote the space of all holomorphic functions on $\mathbb{D}$. Let $\sigma$ denote the normalized Lebesgue measure on the unit circle $\partial\mathbb{D}$, that is, $\sigma(\partial\mathbb{D}) = 1$. Let $ST^2(\mathbb{R})$ denote the class of strongly convex functions $\Phi : [-\infty, \infty) \to [0, \infty)$ (that is, $\Phi$ is non-negative, convex and nondecreasing with $\frac{\Phi(t)}{t} \to \infty$ as $t \to \infty$), which satisify

(i) $\Phi(t) = 0$ for all $t < 0$ with $\Phi(0) = \Phi'(0) = 0$,

(ii) $\Phi''$ exists for all $t > 0$ and,

(iii) $\Phi(2t) \leq C\Phi(t)$ for some positive constant $C$ and for all $t > 0$.

For $\Phi \in ST^2(\mathbb{R})$, we define the Hardy-Orlicz space $H^\Phi$ by

$$H^\Phi = \left\{ f \in H(\mathbb{D}) : \sup_{0<r<1} \int_{\partial\mathbb{D}} \Phi(\log^+ |f(re^{i\theta})|)\, d\sigma < \infty \right\}.$$

Although the integral expression above does not define a norm in $H^\Phi$, it holds that the distance

$$d(f, g) = \int_{\partial\mathbb{D}} \Phi(\log^+ |f(re^{i\theta}) - g(re^{i\theta})|)\, d\sigma$$

defines a translation-invariant metric on $H^\Phi$, and turns $H^\Phi$ into a complete metric space. Abusing notation, we will denote

$$\|f\|_\Phi = \sup_{0<r<1} \int_{\partial\mathbb{D}} \Phi(\log^+ |f(re^{i\theta})|)\, d\sigma,$$

for $f \in H^\Phi$. Obviously, the inequalitities

$$\log^+ x \leq \log(1 + x) \leq 1 + \log^+ x, \qquad x \geq 0$$

and

$$2\log^+ x \leq \log(1 + x^2) \leq 1 + 2\log^+ x, \qquad x \geq 0$$

and the fact that $\Phi$ is nondecreasing convex function imply that

$$\begin{aligned}\Phi(\log^+ x) &\leq \Phi(\log(1 + x)) \leq \Phi(1 + \log^+ x)\\ &\leq \frac{1}{2}\Phi(2) + \frac{1}{2}\Phi(2\log^+ x) \leq \frac{1}{2}\Phi(2) + \frac{1}{2}C\Phi(\log^+ x)\end{aligned}$$

and

$$\begin{aligned}\Phi(\log^+ x) &\leq \Phi(2\log^+ x) \leq \Phi(\log(1 + x^2)) \leq \Phi(1 + 2\log^+ x)\\ &\leq \frac{1}{2}\Phi(2) + \frac{1}{2}\Phi(4\log^+ x) \leq \frac{1}{2}\Phi(2) + \frac{1}{2}C\Phi(\log^+ x).\end{aligned}$$

Hence $f \in H^\Phi$ if and only if

$$\sup_{0<r<1} \int_{\partial\mathbb{D}} \Phi(\log(1 + |f(re^{i\theta})|))\, d\sigma < \infty$$

or if and only if

$$\sup_{0<r<1} \int_{\partial\mathbb{D}} \Phi(\log(1 + |f(re^{i\theta})|^2))\, d\sigma < \infty.$$

## 3. Compactness

As noted in the introduction, we want to prove the following result.

THEOREM 3.1. *Let $\Phi \in ST^2(\mathbb{R})$ and $\varphi$ be a holomorphic self-map of $\mathbb{D}$. Then $C_\varphi$ is compact on $H^\Phi$ if and only if $C_\varphi$ is compact on $H^2$.*

In order to prove the theorem, we need a series of lemmas.

First of all we recall the remarkable formula of C.S. Stanton for integral means of subharmonic functions in the disk $\mathbb{D}$ [13]. If $u$ is a positive subharmonic function on $\mathbb{D}$ and $\varphi$ is a holomorphic self-map of $\mathbb{D}$, then for $0 < r < 1$,

$$\frac{1}{2\pi} \int_0^{2\pi} u(\varphi(re^{i\theta}))\, d\theta = u(\varphi(0)) + \frac{1}{2\pi} \int_{r\mathbb{D}} N_\varphi(r, z)\, d\mu(z),$$

where $\mu$ is the Riesz measure of $u$, and $N_\varphi(r, \cdot)$ denotes the partial Nevanlinna counting function of $\varphi$ defined by

$$N_\varphi(r, z) = \sum_{w \in \varphi^{-1}(z), |w| \le r} \log \frac{r}{|w|}$$

for $r \in (0, 1)$. Let $f$ be an analytic map. Applying Stanton's formula to the subharmonic function $z \to \Phi(\log(1 + |f(z)|^2))$, we obtain

$$\frac{1}{2\pi} \int_0^{2\pi} \Phi(\log(1+|f(\varphi(re^{i\theta}))|^2))\, d\theta = \Phi(\log(1+|f(\varphi(0))|^2)) + \frac{1}{2\pi} \int_{r\mathbb{D}} N_\varphi(r, z)\, d\mu(z),$$

where $\mu$ is the Riesz measure of $\Phi(\log(1+|f(z)|^2))$. An easy calculation on the same lines as in [14] yields that, if $\Phi \in ST^2(\mathbb{R})$, $f \in H(\mathbb{D})$ and $g(z) = \Phi(\log(1+|f(z)|^2))$, $z \in \mathbb{D}$, then

$$\nabla^2 g(z) = 4\big[\Phi''(\log(1 + |f(z)|^2))|f(z)|^2 + \Phi'(\log(1 + |f(z)|^2))\big] \frac{|f'(z)|^2}{(1 + |f(z)|^2)^2}$$

and the Riesz measure $\mu_g$ of $g$ is given by

$$d\mu_g(z) = 4\big[\Phi''(\log(1 + |f(z)|^2))|f(z)|^2 + \Phi'(\log(1 + |f(z)|^2))\big] \frac{|f'(z)|^2}{(1 + |f(z)|^2)^2} dA(z),$$

where $dA(z)$ is the two dimensional area measure on $\mathbb{D}$.

Set

$$f^{\Phi}(z)=\frac{2}{\pi}\big[\Phi''(\log(1+|f(z)|^2))|f(z)|^2+\Phi'(\log(1+|f(z)|^2))\big]\frac{|f'(z)|^2}{(1+|f(z)|^2)^2}.$$

Thus, we have

$$\frac{1}{2\pi}\int_0^{2\pi}\Phi(\log(1+|f(\varphi(re^{i\theta}))|^2))\,d\theta = \Phi(\log(1+|f(\varphi(0))|^2))+\int_{r\mathbb{D}} f^{\Phi}(z)N_{\varphi}(r,z)\,dA(z).$$

Since $N_\varphi(r,z)$ increases monotonically to $N_\varphi(z)$, an application of Monotone convergence theorem yields

$$\lim_{r\to 1}\frac{1}{2\pi}\int_0^{2\pi}\Phi(\log(1+|f(re^{i\theta})|^2))\,d\theta = \Phi(\log(1+|f(\varphi(0))|^2))+\int_{\mathbb{D}} f^{\Phi}(z)N_{\varphi}(z)\,dA(z).$$

The important special case of the previous formula is obtained if we choose $\varphi$ to be the identity map.

$$\|f\|_{\Phi}\approx\Phi(\log(1+|f(0)|^2))+\int_{\mathbb{D}} f^{\Phi}(z)\log\frac{1}{|z|}\,dA(z).$$

The following lemma asserts that sequences that are norm bounded in $H^{\Phi}$ are uniformly bounded on compact subsets of $\mathbb{D}$. In other words, for $0<r<1$, there has to be a uniform bound for all point-evaluation functionals corresponding to points in $r\mathbb{D}$.

Lemma 3.2. *Let $\Phi\in ST^2(\mathbb{R})$. Then for $z=\rho e^{i\theta}\in\mathbb{D}$*

$$|f(z)|\le\exp\Big(\Phi^{-1}\Big(\frac{2\|f\|_{\Phi}}{1-\rho}\Big)\Big)$$

*for all $f\in H^{\Phi}$.*

*Proof.* Since $f$ is analytic

$$f(z)=\int_0^{2\pi}P(r,\theta-t)f(e^{it})d\sigma(t),$$

where $P(.,.)$ is the Poisson kernel. Replacing the unit disk $\mathbb{D}$ by $r\mathbb{D}$, where $0<r<1$ is arbitrarily fixed, we get for $0\le\rho<1$,

$$f(z)=\frac{1}{2\pi}\int_0^{2\pi}P(\rho e^{i\theta},re^{i\theta})f(e^{it})\,dt.$$

Since $\Phi(\log^+|f|)$ is convex and increasing, we have by Jensen's inequality

$$\Phi(\log^+|f(\rho e^{i\theta})|)\le\frac{1}{2\pi}\int_0^{2\pi}P(\rho e^{i\theta},re^{i\theta})\Phi(\log^+|f(re^{i\theta})|)\,dt.$$

Using the inequality $P(\rho e^{i\theta}, re^{i\theta}) \leq \dfrac{2}{r-\rho}$, we get

$$\Phi(\log^+ |f(\rho e^{i\theta})|) \leq \frac{2}{1-\rho}\|f\|_\Phi.$$

That is,

$$|f(z)| \leq \exp\left(\Phi^{-1}\left(\frac{2\|f\|_\Phi}{1-\rho}\right)\right). \quad ■$$

The following lemma characterizes the compactness of $C_\varphi$ on $H^\Phi$ in terms of sequential convergence.

Lemma 3.3. *Let $\varphi$ be a holomorphic self-map of $\mathbb{D}$. Then $C_\varphi$ is compact on $H^\Phi$ if and only if for every sequence $\{f_n\}$, which is norm bounded and converges to zero uniformly on compact subsets of $\mathbb{D}$, we have $\|f_n \circ \varphi\|_\Phi \to 0$.*

The proof is similar to that of Proposition 3.11 in [3]. So we omit the details.

In what follows we say that a positive Borel measure $\mu$ on $\overline{\mathbb{D}}$ is a vanishing Carleson measure if

$$\lim_{\delta \to 0} \frac{\mu(S(\delta, \zeta))}{\delta} = 0$$

uniformly in $\zeta \in \partial\mathbb{D}$ where $0 < \delta < 1$ and $S(\delta, \zeta) = \{z \in \mathbb{D} : |z - \zeta| < \delta\}$. The next criterion for compactness of $C_\varphi$ on $H^2$ which is due to Shapiro [12] and MacCluer [6] is useful in the proof of the main result.

Lemma 3.4. *For a holomorphic self-map $\varphi$ of $\mathbb{D}$, the following are equivalent:*
*(i) $C_\varphi$ is compact on $H^2$.*
*(ii) $N_\varphi(z) = o\Big(\log \frac{1}{|z|}\Big)$ as $|z| \to 1^-$.*
*(iii) The pull-back measure $\mu_\varphi = \sigma \circ \varphi^{-1}$ is a vanishing Carleson measure on $\overline{\mathbb{D}}$.*

We are now in a position to prove the main result of this paper.

*Proof of Theorem 3.1.* First assume that $C_\varphi$ is compact on $H^2$. The approach to the proof comes from [13, Chapter 10]. Fix a sequence $\{f_n\}$ that is bounded by a finite constant $M$ in $H^\Phi$ and converges to zero uniformly on compact subsets of $\mathbb{D}$. By Lemma 3.3, it is enough to show that $\|f_n \circ \varphi\|_\Phi \to 0$. Let $\epsilon > 0$ be given. Then it follows by Lemma 3.4, that we can choose $r$, $0 < r < 1$ such that

$$N_\varphi(z) < \epsilon \ \log \frac{1}{|z|}, \text{ whenever } r \leq |z| < 1.$$

Since $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, so is $f_n'$. Thus we can choose $\eta(\epsilon)$ so that

$$|f_n| < \sqrt{\epsilon} \text{ and } |f_n'| < \sqrt{\epsilon}$$

on $r\mathbb{D}$ whenever $n > \eta(\epsilon)$. Hence for such $n$ we have

$$\|C_\varphi f_n\|_\Phi \leq \Phi(\log(1 + |f_n(\varphi(0))|^2)) + \int_{\mathbb{D}} f_n^\Phi(z) N_\varphi(z) dA(z).$$

Since $|f_n(\varphi(0))| \to 0$ as $n \to \infty$ so

$$\Phi(\log(1 + |f_n(\varphi(0))|^2)) \to 0 \text{ as } n \to \infty$$

Thus it remains to show that

$$\lim_{n\to\infty} \int_{\mathbb{D}} f_n^{\Phi}(z) N_\varphi(z) dA(z) = 0.$$

Now

$$\int_{\mathbb{D}} f_n^{\Phi}(z) N_\varphi(z)\, dA(z) = \int_{r\mathbb{D}} + \int_{\mathbb{D}\setminus r\mathbb{D}} f_n^{\Phi}(z) N_\varphi(z)\, dA(z) = I + II.$$

We first show that the first term above is bounded by a constant multiple of $\epsilon$.

$$\begin{aligned} I &\le \frac{2}{\pi}(\Phi''(\log(1+\epsilon)\epsilon + \Phi'(\log(1+\epsilon)))\ \epsilon \int_{r\mathbb{D}} N_\varphi(z) dA(z) \\ &\le \frac{2}{\pi}(\Phi''(\log(1+\epsilon)\epsilon + \Phi'(\log(1+\epsilon)))\ \epsilon\ (\|\varphi\|_\Phi - |\varphi(0)|^2)) \\ &\le \frac{2}{\pi}(\Phi''(\log(1+\epsilon)\epsilon + \Phi'(\log(1+\epsilon)))\ \epsilon. \end{aligned}$$

Finally, we show that the second term above is bounded by a constant multiple of $\epsilon$.

$$II \le \epsilon \int_{\mathbb{D}\setminus r\mathbb{D}} f_n^{\Phi}(z) \log\frac{1}{|z|} dA(z) \le \epsilon\ (\|f_n\|_\Phi - \log(1 + |f_n(0)|^2)) \le \epsilon \|f_n\|_\Phi \le \epsilon M.$$

To prove the converse direction we assume that $C_\varphi$ is compact on $H^\Phi$. Because of Lemma 3.4, we only need to verify that the pull-back measure $\sigma \circ \varphi^{-1}$ is a vanishing Carleson measure on $\overline{\mathbb{D}}$. To prove this let $a = (1-\delta)\zeta$, where $\zeta \in \partial\mathbb{D}$ and $0 < \delta < 1$. Let

$$g_a(e^{i\theta}) = \frac{1 - |a|^2}{|1 - \overline{a}e^{i\theta}|^2}.$$

Then $g_a$ is non-negative and $g_a \in L^1(d\sigma)$. Let $K(e^{i\theta}) = \Phi^{-1}(g_a(e^{i\theta}))$. Then $K$ is well defined, for $K$ is strictly increasing in the range of $g_a$. Since $\Phi$ is convex, $\Phi^{-1}$ is concave and so there is a constant $C > 0$ such that $\Phi^{-1}(s) \le Cs$ for sufficiently large $s$. Thus $K \in L^1(d\sigma)$. We set

$$h(z) = \exp\Big\{ \int_0^{2\pi} H(z, e^{it}) K(e^{it})\, d\sigma(t) \Big\},$$

where $H(z, e^{it})$ denotes the Herglotz kernel for $\mathbb{D}$; namely,

$$H(z, e^{it}) = \frac{e^{it} + z}{e^{it} - z}, \qquad z \in \mathbb{D}.$$

Then

$$\Phi(\log^+ |h(e^{i\theta})|) = \Phi(K(e^{i\theta})) = g_a(e^{i\theta}) \in L^1(d\sigma).$$

This means that $h \in H^\Phi$. Let

$$f_a(z) = \frac{2(1-|a|)^2}{(1-\overline{a}z)^2} h(z).$$

Then clearly $f_a \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $|a| \to 1$. Moreover,

$$\begin{aligned}
\|f_a\|_\Phi &= \sup_{0<r<1} \int_0^{2\pi} \Phi(\log^+ |f_a(re^{i\theta})|)\, d\sigma(\theta) \\
&\le \sup_{0<r<1} \int_0^{2\pi} \Phi(\log^+(2|h(re^{i\theta})|))\, \frac{d\theta}{2\pi} \\
&\le \sup_{0<r<1} \int_0^{2\pi} \Phi\Big( \log^+ 2 + \log^+ \Big( \exp\Big\{ \int_0^{2\pi} H(re^{i\theta}, e^{it}) K(e^{it})\, d\sigma(t) \Big\}\Big)\Big) \frac{d\theta}{2\pi} \\
&\le \frac{1}{2}\Phi(2\log^+ 2) + \frac{1}{2} C \sup_{0<r<1} \int_0^{2\pi} \frac{1-|a|^2}{|1-\overline{a}re^{i\theta}|^2} \frac{d\theta}{2\pi} = \frac{1}{2}\Phi(2\log^+ 2) + \frac{1}{2}C.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\frac{1-|a|^2}{|1-\overline{a}z|^2} &\ge \Re\Big(\frac{1-|a|^2}{(1-\overline{a}z)^2}\Big) = \frac{1-|a|^2}{(1-|a|)^2} \Re\Big(\frac{1-|a|}{1-\overline{a}z}\Big)^2 \\
&= \frac{1-|a|^2}{(1-|a|)^2} \Re\Big(1 + \frac{|a|(1-z\overline{\zeta})}{(1-|a|)}\Big)^{-2}, \quad \big(\zeta = \frac{a}{|a|}\big) \\
&> \frac{1}{2} \frac{1-|a|^2}{(1-|a|)^2} \ge \frac{1}{2\delta},
\end{aligned}$$

if $\frac{|1-z\overline{\zeta}|}{1-|a|} < \gamma_0$ for some fixed $0 < \gamma_0 < 1/4$, that is, if $z \in S(\gamma_0\delta, \zeta)$. That is,

$$\Phi^{-1}\Big(\frac{1-|a|^2}{|1-\overline{a}z|^2}\Big) \ge \Phi^{-1}\Big(\frac{1}{2\delta}\Big)$$

if $z \in S(\gamma_0\delta, \zeta)$. Thus for $z \in S(\gamma_0\delta, \zeta)$,

$$\Phi(\log^+ |f_a(z)|) = \Phi(\log^+ \frac{2(1-|a|)^2}{(1-\overline{a}z)^2} |h(z)|) \ge \Phi(\log^+(\exp \Phi^{-1}\Big(\frac{1}{2\delta}\Big)) = \frac{1}{2\delta}.$$

Hence for all $\zeta \in \partial\mathbb{D}$ and $0 < \delta < 1$, we have

$$\begin{aligned}
\frac{1}{2\delta}\mu_\varphi(S(\gamma_0\delta, \zeta)) &\le \int_{S(\gamma_0\delta,\zeta)} \Phi(\log^+ |f_a(z)|) d\mu_\varphi(z) \\
&\le \int_{\overline{\mathbb{D}}} \Phi(\log^+ |f_a(z)|) d\mu_\varphi(z) \\
&\le \lim_{r\to 1} \int_0^{2\pi} \Phi(\log^+ |(f_a \circ \varphi)(re^{i\theta})|) \frac{d\theta}{2\pi} = \|f_a \circ \varphi\|_\Phi.
\end{aligned}$$

But the compactness of $C_\varphi$ on $H^\Phi$ forces $\|f_a \circ \varphi\|_\Phi$ to tend to 0 as $|a| \to 1$, which implies that

$$\lim_{\delta\to 0} \frac{\mu_\varphi(S(\gamma_0\delta, \zeta))}{\delta} = 0,$$

uniformly in $\zeta \in \partial D$. Hence $\mu_\varphi$ is a vanishing-Carleson measure on $\mathbb{D}$. ■

REMARK. One can certainly consider $H^{\Phi}$ spaces when either $\Phi$ does not belong to $ST^2(\mathbb{R})$ or $\Phi$ is log-convex but not convex. However, Theorem 3.1 may fail if we consider Hardy-Orlicz space $H^{\Phi}$ induced by an arbitrary convex. For example, if $\Phi$ is a non-negative function on $\mathbb{R}$ such that $\Phi(x) \to 0$, as $x \to -\infty$, and $\Phi$ is non-decreasing but $\Phi(x) > 0$ for some $x \neq 0$, then compactness of $C_\varphi$ on $H^{\Phi}$ is quite different from the compactness of $C_\varphi$ on $H^2$. Here $H^{\Phi}$ is defined as follows;

$$H^{\Phi} = \{f \in H(\mathbb{D}) : \int_0^{2\pi} \Phi(log|\gamma f(re^{i\theta})|)\, d\sigma(\theta) \quad \text{is bounded for } 0 \leq r < 1 \text{ and for some } \gamma > 0.\}$$

In fact, if we take $\Phi(x) = 0$ for $x \leq 1$, and $\Phi(x) = \infty$ for $x > 1$, then $H^{\Phi}$ becomes $H^\infty$ and it is well known that the $C_\varphi$ is compact on $H^\infty$ if and only if $\|\varphi\|_\infty < 1$ [**9**]. Thus $\varphi(z) = 1 - (1-z)^b$ $0 < b < 1$ is a conformal map of $\mathbb{D}$ whose closure intersects the unit circle only at 1 and so $\varphi$ induces a non-compact composition operator on $H^\infty$. However $\varphi$ induces a compact composition operator on $H^2$ [3, p. 131].

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