МАТЕМАТИЧКИ ВЕСНИК
61 (2009), 25–52

UDK 515.122
оригинални научни рад
research paper

FINITE DIMENSIONS MODULO SIMPLICIAL COMPLEXES AND ANR-COMPACTA

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Abstract. New dimension functions $\mathcal{G}$-dim and $\mathcal{R}$-dim, where $\mathcal{G}$ is a class of finite simplicial complexes and $\mathcal{R}$ is a class of ANR-compacta, are introduced. Their definitions are based on the theorem on partitions and on the theorem on inessential mappings to cubes, respectively. If $\mathcal{R}$ is a class of compact polyhedra, then for its arbitrary triangulation τ, we have $\mathcal{R}_\tau\text{-}\dim X = \mathcal{R}\text{-}\dim X$ for an arbitrary normal space X. To investigate the dimension function $\mathcal{R}$-dim we apply results of extension theory. Internal properties of this dimension function are similar to those of the Lebesgue dimension. The following inequality $\mathcal{R}\text{-}\dim X \le \dim X$ holds for an arbitrary class $\mathcal{R}$. We discuss the following Question: When $\mathcal{R}\text{-}\dim X < \infty \Rightarrow \dim X < \infty$?

Introduction

The following two theorems give us main characterizations of the Lebesgue dimension.

Theorem A. *A normal space X satisfies the inequality $\dim X \le n \ge 0$ if and only if for every sequence (F_1^1, F_2^1), (F_1^2, F_2^2), ..., (F_1^{n+1}, F_2^{n+1}) of $n+1$ pais of disjoint closed subsets of X there exist partitions P_i between F_1^i and F_2^i such that $\bigcap_{i=1}^{n+1} P_i = \varnothing$.*

Theorem B. *A normal space X satisfies the inequality $\dim X \le n \ge 0$ if and only if every continuous mapping $f\colon X \to I^{n+1}$ is inessential.*

Pairs (F_1^i, F_2^i) from Theorem A are families Φ_i of sets such that their nerves $N(\Phi_i)$ coincide with the two point set $\{0,1\}$ which is zero-dimensional simplicial complex. Changing the two point set to arbitrary simplicial complexes G_i we get a definition of a dimension function $\mathcal{G}$-dim (see Definition 3.4).

AMS Subject Classification: 54F45, 55M10

Keywords and phrases: Dimension; simplicial complex; ANR-compactum; extension theory.

The author was supported by the Russian Foundation for Basic Research (Grant 09-01-00741) and the Program "Development of the Scientific Potential of Higher School" of the Ministry for Education of the Russian Federation (Grant 2.1.1. 3704).

Presented at the international conference *Analysis, Topology and Applications 2008 (ATA2008)*, held in Vrnjačka Banja, Serbia, from May 30 to June 4, 2008.

The cube I^{n+1} from Theorem B is homeomorphic to cone S^n and the sphere S^n is homeomorphic to the join $\overset{n+1}{*} S^0$. Changing the sphere S^0 to arbitrary ANR-compacta R_i we get a definition of a dimension function $\mathcal{R}$-dim (see Definition 3.9).

Proposition 3.15 establishes a link between dimension function R-dim, where R is an ANR-compactum, and extension dimension e-dim:

$$R\text{-}\dim X \leq n \iff e\text{-}\dim X \leq \overset{n+1}{*} R. \tag{0.1}$$

According to Proposition 4.5 for homotopy equivalent classes $\mathcal{R}_1$ and $\mathcal{R}_2$ of ANR-compacta we have

$$\mathcal{R}_1\text{-}\dim X = \mathcal{R}_2\text{-}\dim X \quad \text{for every normal space} \quad X. \tag{0.2}$$

Theorem 4.8 states that if $\mathcal{K}$ is a class of compact polyhedra and τ is some of its triangulations, then

$$\mathcal{K}\text{-}\dim X = \mathcal{K}_\tau\text{-}\dim X \tag{0.3}$$

for arbitrary normal space X, where $\mathcal{K}_\tau$ is the class of simplicial complexes defined by the triangulation τ.

In view of (0.2), (0.3), and West's theorem on homotopy types of ANR-compacta, it is sufficient to consider only dimension functions $\mathcal{R}$-dim with $\mathcal{R}$ consisting of compact polyhedra. Property (0.1) allows us to apply results of extension theory introduced by A. Dranishnikov [3].

An internal theory of dimension $\mathcal{R}$-dim is similar to this of the Lebesgue dimension. For example, the following statements hold.

Countable sum theorem 5.1.

Point-finite sum theorem 5.3.

Addition theorem 5.7: $\mathcal{R}\text{-}\dim(X_1 \cup X_2) \leq \mathcal{R}\text{-}\dim X_1 + \mathcal{R}\text{-}\dim X_2 + 1$.

Čech-Stone compactification theorem 5.11.

Universal compact space theorem 5.12.

Decomposition theorem 5.15: *If X is a separable metrizable space with R-$\dim X \leq m + n + 1$, the X can be represented as the union $X = A \cup B$ so that R-$\dim A \leq m$, R-$\dim B \leq n$.*

Completion theorem 5.18.

Inverse system theorems 5.19, 5.20, 5.21.

§ 6 is devoted to comparison of dimensions. Theorem 6.3 states that

$$\mathcal{R}\text{-}\dim X \leq \dim X \tag{0.4}$$

for an arbitrary class $\mathcal{R}$ and every space X. As for the equality

$$\mathcal{R}\text{-}\dim X = \dim X, \tag{0.5}$$

it holds if and only if $\mathcal{R}$ contains a disconnected ANR-compactum. In connection with the inequality (0.4) we study the following problem: When

$$R\text{-}\dim X < \infty \Rightarrow \dim X < \infty \tag{0.6}$$

for every space X?

ANR-compacta R satisfying condition (0.6) are called efd-compacta (notation: $R \in efd\text{-}C$). We give a list of results concerning the class $efd\text{-}C$. In particular, Theorem 6.11 states that if $H_*(R,\mathbb{Q}) = 0$, then $R \notin efd\text{-}C$.

Hypothesis. $R \in efd\text{-}C \iff H_*(R,\mathbb{Q}) \neq 0$.

In § 7 we investigate dimension of products. The inequality

$$R\text{-}\dim(X \times Y) \leq R\text{-}\dim X + R\text{-}\dim Y + 1 \tag{0.7}$$

holds for finite-dimensional compact Hausdorff spaces X and Y and a connected ANR-compactum R (Theorem 7.3).

Inequality (0.7) is not improvable. As an example one can take $X = Y = R = S^1$.

§§ 1,2 have an auxiliary character. There we recall some topological constructions and notions and facts of extension theory. All spaces are assumed to be normal ($+T_1$). All mappings are continuous. *Compacta* stand for metrizable compact spaces. By $\mathrm{Fin}A$ ($\mathrm{Fin}_s A$) we denote the set of all finite subsets of A (finite sequences of elements from A). The symbol $\sqcup$ denotes a union of disjoint sets. For a space X by $\exp X$ we denote the set of all closed subsets of X (including $\varnothing$). The set of all finite indexed open covers of X is denoted by $\mathrm{cov}_\infty(X)$. The symbol $\simeq$ stands for a homotopy equivalence.

The author is grateful to Sasha Karassev for pointing out errors in the first draft of the paper.

1. Simplicial complexes, polyhedra, and ANR-compacta. Cones, joins, and smash products

1.1. We consider only finite simplicial complexes, so that one can identify an *abstract simplicial complex* G with its geometric realization, i.e. with an *Euclidean* complex $\tilde{G}$ with the same vertex scheme. In this context it is clear what is a *simplicial subdivision* of a simplicial complex G.

Recall that a simplicial complex G is said to be *complete* if every face of each simplex from G belongs to G. In what follows *complexes* stand for finite complete simplicial complexes. Hence, geometric realizations of complexes are compact polyhedra.

For a complex G by $v(G)$ we denote the set of all its vertices. Let u be a finite family of sets and let $u_0 = \{U \in u : U \neq \varnothing\}$. The *nerve* of the family u is a complex $N(u)$ such that $v(N(u)) = \{a_U : U \in u_0\}$ and a set $\Delta \subset v(N(u))$ is a simplex of $N(u)$ if and only if $\bigcap\{U : a_U \in \Delta\} \neq \varnothing$.

In what follows *polyhedra* stands for compact polyhedra. Every compact polyhedron is an ANR-space (for normal spaces).

1.2. THEOREM [23]. *Every ANR-compactum is homotopy equivalent to some compact polyhedron.* ■

1.3. The *cone* of a space X is the space cone$X$ which is the quotient space $X \times I/X \times 0$. The set $X \times 1 \subset \operatorname{cone} X$ is called the *base* of the cone$X$. As a rule we shall identify $X \times 1$ and X. Let $q_X : X \times I \to \operatorname{cone} X$ be the quotient mapping. The point $q_X(X \times 0)$ is said to be the *peak* of the cone of X and is usually denoted by a_X.

If Δ is an n-dimensional simplex with vertices $a_0, \dots, a_n$, then coneΔ is an $(n+1)$-dimensional simplex with vertices $a_0, \dots, a_n, a_\Delta$. Hence the cone of a complex is a complex.

The *join* of spaces X and Y is the space $X * Y$ which is the quotient space of $X \times I \times Y$ with respect to the decomposition whose members are sets $x \times 0 \times Y$, $X \times 1 \times y$ $(x \in X,\ y \in Y)$ and singletons of the set $X \times (0;1) \times Y$.

The *boundary join (or* Bd-*join)* of spaces X and Y is the following subset $X \bar{*} Y$ of the product $\operatorname{cone} X \times \operatorname{cone} Y$: $X \bar{*} Y = \operatorname{cone}(X) \times Y \bigcup X \times \operatorname{cone} Y$.

1.4. PROPOSITION ([19]), Lecture 5). *If X and Y are locally compact Hausdorff spaces, then the spaces $X * Y$ and $X \bar{*} Y$ are canonically homeomorphic.* ■

1.5. PROPOSITION ([20], Ch. 1). *If X and Y are compact Hausdorff spaces, then there exists a canonic homeomorphism $h : \operatorname{cone}(X * Y) \to \operatorname{cone} X \times \operatorname{cone} Y$ such that $h(a_{\operatorname{cone}(X*Y)}) = (a_X, a_Y)$ and $h(X * Y) = X \bar{*} Y$.* ■

By induction we define the *iterated join*

$$(\dots((X_1 * X_2) * X_3) \dots) * X_n$$

and the *iterated* Bd-*join*

$$(\dots((X_1 \bar{*} X_2) \bar{*} X_3) \dots) \bar{*} X_n.$$

The operations $*$ and $\bar{*}$ are commutative and associative up to homeomorphism. Thus, for compact Hausdorff spaces $X_1, \dots, X_n$ there defined their *multiple join*

$$X_1 * \dots * X_n \equiv \mathop{*}_{i=1}^{n} X_i$$

and their *multiple* Bd-*join*

$$X_1 \bar{*} \dots \bar{*} X_n \equiv \mathop{\bar{*}}_{i=1}^{n} X_i.$$

There exists a canonical homeomorphism

$$X_1 \bar{*} \dots \bar{*} X_n = \bigcup_{i=1}^{n} \Big(X_i \times \prod_{j \neq i} \operatorname{cone} X_j \Big). \tag{1.1}$$

Proposition 1.5 is generalized as follows.

1.6. Proposition. *If $X_1, \dots, X_n$ are compact Hausdorff spaces, then there is a homeomorphism*

$$g\colon \operatorname{cone} X_1 \times \dots \times \operatorname{cone} X_n \to \operatorname{cone}(X_1 * \dots * X_n)$$

such that

$$g\Big(\bigcup_{i=1}^{n}(X_i \times \prod_{j\neq i} \operatorname{cone} X_j)\Big) = X_1 * \dots * X_n \tag{1.2}$$

and $g(a_1, \dots, a_n) = a_{\operatorname{cone}(X_1*\dots*X_n)}$, where a_i is the peak of $\operatorname{cone} X_i$, $i = 1, \dots, n$. ■

1.7. Remark. In what follows we shall identify the multiple join $X_1 * \dots * X_n$ of compact Hausdorff spaces $X_1, \dots, X_n$ with their multiple Bd-join, i.e. with the set (1.1). Sometimes, we shall use a short notation:

$$B(X_1, \dots, X_n) \equiv \bigcup_{i=1}^{n}\Big(X_i \times \prod_{j\neq i} \operatorname{cone} X_j\Big). \tag{1.3}$$

For mappings $f_i\colon X_i \to Y_i$, let

$$c(f_1, \dots, f_n) = \operatorname{cone} f_1 \times \dots \times \operatorname{cone} f_n : \prod_{i=1}^{n} \operatorname{cone} X_i \to \prod_{i=1}^{n} \operatorname{cone} Y_i.$$

Then

$$c(f_1, \dots, f_n)(B(X_1, \dots, X_n)) \subset B(Y_1, \dots, Y_n); \tag{1.4}$$

$$c(f_1, \dots, f_n)^{-1} B(Y_1, \dots, Y_n) = B(X_1, \dots, X_n). \tag{1.5}$$

Taking into consideration our agreement $X_1 * \dots * X_n = B(X_1, \dots, X_n)$, put

$$f_1 * \dots * f_n = c(f_1, \dots, f_n)|_{B(X_1,\dots,X_n)}. \tag{1.6}$$

From properties of cones and products, and equalities (1.4), (1.5) we get

1.8. Proposition. *The operation of the multiple join*

$$(X_1, \dots, X_n) \to X_1 * \dots * X_n, \qquad (f_1, \dots, f_n) \to f_1 * \dots * f_n$$

is a covariant functor of several variables in the category Comp *of compact Hausdorff spaces. Moreover, it preserves homotopy equivalences of spaces and mappings.* ■

The next statement is also well known.

1.9. Proposition. *If $X_1, \dots, X_n$ are ANR-compacta (polyhedra), then their multiple join $X_1 * \dots * X_n$ is also an ANR-compactum (a polyhedron).* ■

1.10. For pointed spaces (X, x_0) and (Y, y_0) their *wedge* $(X, x_0) \vee (Y, y_0)$ is defined as the quotient space $X \sqcup Y/\{x_0, y_0\}$. The *smash product* $(X, x_0) \wedge (Y, y_0)$ is the quotient space $X \times Y/X \times \{y_0\} \cup \{x_0\} \times Y$.

1.11. Proposition [19]. *If X and Y are connected ANR-compacta, then for arbitrary pairs $(x_i, y_i) \in X, Y$, $i = 0, 1$, the spaces $(X, x_0) \wedge (Y, y_0)$ and $(X, x_1) \wedge (Y, y_1)$ are homotopy equivalent.* ■

In view of Proposition 1.11 we shall denote the smash product $(X, x_0) \wedge (Y, y_0)$ (X, Y are connected ANR-compacta) by $X \wedge Y$.

1.12. Proposition [19]. *If X and Y are connected ANR-compacta, then $\Sigma(X \wedge Y) \simeq X * Y$.* ■

1.13. Proposition. *If X and Y are polyhedra (ANR-compacta), then $X \wedge Y$ is a polyhedron (ANR-compactum).* ■

1.14. Proposition. *If X is an ANR-compactum, then* $\operatorname{cone} X \in AR$. ■

2. Main notions of extension theory

Recall that the *Homotopy Extension Theorem* is fulfilled for a pair (X, Y) of spaces if, for every closed set $F \subset X$, each mapping $f : (X \times 0) \cup (F \times I) \to Y$ extends over $X \times I$.

2.1. Theorem (Borsuk's theorem on extension of homotopy) (see [15], [22]). *Homotopy Extension Theorem is fulfilled for every pair (X, R), where X is a space and R is an ANR-compactum.* ■

2.2. Definition. Let X and Y be spaces and let $Z \subset X$. The property that all partial mappings $f \colon Z \to Y$ extend over X will be denoted by $Y \in AE(X, Z)$. If every mapping $f \colon Z \to Y$ extends over an open set $U_f \supset Z$, then we write $Y \in ANE(X, Z)$. If $Y \in A(N)E(X, Z)$ for every closed $Z \subset X$, then Y is called an *absolute (neighbourhood) extensor of* X (notation: $Y \in A(N)E(X)$). If $Y \in A(N)E(X)$ for all spaces X, then Y is said to be an *absolute (neighbourhood) extensor* (notation: $Y \in A(N)E$).

Brouwer-Tietze-Urysohn theorem on extension of functions yields

2.3. Theorem. *If Y is an $A(N)R$-compactum, then $Y \in A(N)E$.* ■

2.4. Factorization theorem [4]. *Let X be a compact Hausdorff space and let R be an ANR-compactum such that $R \in AE(X)$. Then for every mapping $f : X \to Y$ to a metric space Y there exist a compactum X' and mappings $f' \colon X \to X'$ and $g \colon X' \to Y$ such that $R \in AE(X')$ and $f = g \circ f'$.* ■

2.5. Proposition. *Let X be a space and let Y be a compact Hausdorff space. If $Y \in AE(\beta X)$, then $Y \in AE(X)$.* ■

Recall that a space X is *dominated by a space* Y (notation: $X \leq_h Y$) if there exist mappings $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$.

2.6. Theorem. *Let X be a space and let Y be a compact Hausdorff space. If Y is dominated by an ANR-compactum, then $Y \in AE(X) \iff Y \in AE(\beta X)$.*

Proof. In view of Proposition 2.5 it suffices to check the implication $\Rightarrow$. Let F be a closed subset of βX and let $f \colon F \to Y$ be a mapping. There exist an ANR-compactum R and mappings $\varphi \colon Y \to R$ and $\psi \colon R \to Y$ such that $\psi \circ \varphi \simeq \mathrm{id}_Y$. By Theorem 2.3 the mapping $\varphi \circ f$ extends over some neighbourhood OF.

Consequently, there exist a regular closed set $F_1 \subset \beta X$ and a mapping $f_1\colon F_1 \to R$ such that $F \subset F_1$ and $f_1|_F = \varphi \circ f$ (we recall that a set H is said to be *regular closed* if $H = ClU$, where U is open). Let $F_2 = F_1 \cap X$ and $f_2 = \psi \circ f_1\colon F_2 \to Y$. Since $Y \in AE(X)$, there exists a mapping $f_3 : X \to Y$ such that $f_3|_{F_2} = \psi \circ f_1|_{F_2}$. Put $f_4 = \beta f_3\colon \beta X \to Y$. According to Theorem 2.1 it remains to show that

$$f_4|_F \simeq f. \tag{2.1}$$

Since F_1 is regular closed,

$$\beta F_2 = [F_2]_{\beta X} = F_1. \tag{2.2}$$

We have $f_4|_{F_2} = f_3|_{F_2} = \psi \circ f_1|_{F_2}$. Consequently, from (2.2) we get $f_4|_{F_1} = \psi \circ f_1|_{F_1}$. Then $f_4|_F = \psi \circ f_1|_F = \psi \circ (f_1|_F) = \psi \circ (\varphi \circ f) = (\psi \circ \varphi) \circ f \simeq f$ because $\psi \circ \varphi \simeq \mathrm{id}_Y$. This the equivalence (2.1) is proved. ■

The next statement is well known and based on Theorem 2.1 and Stone-Weierstrass theorem.

2.7. Theorem. *Let R be an ANR-compactum and let X be the limit space of an inverse system $\{X_\alpha, \pi^\alpha_\beta, A\}$ of compact Hausdorff spaces X_α such that $R \in AE(X_\alpha)$. Then $R \in AE(X)$.* ■

Recall that an inverse system $S = \{X_\alpha, \pi^\alpha_\beta, A\}$ is said to be a *σ-spectrum* [21] if

1) all X_α are compacta;

2) the indexing set A is *ω-complete*, i.e. for every countable chain $B \subset A$ there is sup B in A;

3) the system S is continuous, i.e. for each countable chain B in A with $\sup B = \beta$, the diagonal product $\triangle\{\pi^\beta_\alpha : \alpha \in B\}$ maps the space X_β homeomorphically onto the space $\lim(S|B)$.

Applying Theorems 2.4 and 2.7 we get

2.8. Theorem [17]. *Let X be a compact Hausdorff space and let R be an ANR-compactum such that $R \in AE(X)$. Then X is the limit space of a σ-spectrum $S = \{X_\alpha, \pi^\alpha_\beta, A\}$ such that $R \in AE(X_\alpha)$ for every $\alpha \in A$.* ■

Theorem 2.1 yields

2.9. Proposition. *Let R_1 and R_2 be ANR-compacta such that $R_1 \leq_h R_2$. Then $R_2 \in AE(X) \Rightarrow R_1 \in AE(X)$ for every space X.* ■

2.10. Definition. Let $\mathcal{A}$ be a subclass of the class $\mathcal{N}$ of all normal spaces. We define a preorder $\leq_{\mathcal{A}}$ on the class $ANR(\mathcal{MC})$ of all ANR-compacta in the following way: $R_1 \leq_{\mathcal{A}} R_2$ if and only if $R_1 \in AE(X) \Rightarrow R_2 \in AE(X)$ for every space $X \in \mathcal{A}$.

The following statement is an immediate corollary of definitions.

2.11. Proposition. *If $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \mathcal{N}$, then $R_1 \leq_{\mathcal{A}_2} R_2 \Rightarrow R_1 \leq_{\mathcal{A}_1} R_2$ for arbitrary ANR-compacta R_1 and R_2.* ■

Let the symbols $\mathcal{C}$, $\mathcal{MC}$, Sep stand for the classes of all compact Hausdorff spaces, metrizable compacta, separable metrizable spaces. The next statement is well known. We give its proof for convenience of readers.

2.12. Theorem. *For arbitrary ANR-compacta R_1 and R_2 the following conditions are equivalent:*

1) $R_1 \leq_{\mathcal{N}} R_2$; 2) $R_1 \leq_{\mathcal{C}} R_2$; 3) $R_1 \leq_{\mathcal{MC}} R_2$; 4) $R_1 \leq_{\mathrm{Sep}} R_2$.

Proof. Proposition 2.11 implies that 1) $\Rightarrow$ 2) $\Rightarrow$ 3). The implication 2) $\Rightarrow$ 1) is a corollary of Theorem 2.6. Now the implication 3) $\Rightarrow$ 2). Let $R_1 \leq_{\mathcal{MC}} R_2$ and let $R_1 \in AE(X)$ for some compact Hausdorff space X. By Theoren 2.8 X is the limit space of an inverse system $S = \{X_\alpha,\ \pi^\alpha_\beta,\ A\}$ of compacta such that $R_1 \in AE(X_\alpha)$ for all $\alpha \in A$. Since $R_1 \leq_{\mathcal{MC}} R_2$, we have $R_2 \in AE(X_\alpha)$, $\alpha \in A$. Applying Theorem 2.7 we get $R_2 \in AE(X)$, i.e. $R_1 \leq_{\mathcal{C}} R_2$. At last, Proposition 2.11, the equivalence 3) $\iff$ 1), and condition $\mathcal{MC} \subset \mathrm{Sep} \subset \mathcal{N}$ yield the equivalence 4) $\iff$ 3). ■

In what follows we shall denote the equivalent relations $\leq_{\mathcal{N}}$, $\leq_{\mathcal{C}}$, $\leq_{\mathcal{MC}}$, and $\leq_{\mathrm{Sep}}$ simply by $\leq$.

Now we define an equivalence relation $\sim$ on the class $ANR(\mathcal{MC})$. Namely: $R_1 \sim R_2$ if both $R_1 \leq R_2$ and $R_2 \leq R_1$ hold. An equivalence class of an ANR-compactum R under this relation is called an *extension type* of R or $\mathrm{ext}(R)$. Proposition 2.9 yields

2.13. Proposition. *If ANR-compacta R_1 and R_2 are homotopy equivalent, then* $\mathrm{ext}(R_1) = \mathrm{ext}(R_2)$. ■

The set of all extension types is denoted by $\mathbb{E}$. Clearly, the preorder $\leq$ on $ANR(\mathcal{MC})$ implies a partial order on $\mathbb{E}$. If it is not ambiguous, we denote this partial order simply by $\leq$. Proposition 2.9 implies

2.14. Proposition. *If $R_1 \leq_h R_2$, then* $\mathrm{ext}(R_1) \geq \mathrm{ext}(R_2)$. ■

2.15. Definition. Let $R \in ANR(\mathcal{MC})$. Recall that the *extension dimension* of a topological space X is less than or equal to R (notation: $e\text{-}\dim X \leq R$), provided the property $R \in AE(X)$ holds.

If $\mathrm{ext}(R_1) = \mathrm{ext}(R_2)$, then the conditions $e\text{-}\dim X \leq R_1$ and $e\text{-}\dim X \leq R_2$ are obviously equivalent. So sometimes instead of $e\text{-}\dim X \leq R$ we shall write $e\text{-}\dim X \leq \mathrm{ext}(R)$.

Proposition 2.14 yields

2.16. Proposition. *If ANR-compacta R_1 and R_2 are homotopy equivalent, then $e\text{-}\dim X \leq R_1$ if and only if $e\text{-}\dim X \leq R_2$ for an arbitrary space X.* ■

Theorem 2.6 implies

2.17. Proposition. *For an arbitrary topological space X and an arbitrary ANR-compactum R the following conditions are equivalent:*

1) $e\text{-}\dim X \leq R$; 2) $e\text{-}\dim(\beta X) \leq R$. ■

2.18. Definition. Let X and Y be spaces. We write $e\text{-}\dim X \leq e\text{-}\dim Y$ if and only if $e\text{-}\dim Y \leq R$ implies $e\text{-}\dim X \leq R$ for every $R \in ANR(\mathcal{MC})$. We say that $e\text{-}\dim X = e\text{-}\dim Y$ if both $e\text{-}\dim X \leq e\text{-}\dim Y$ and $e\text{-}\dim Y \leq e\text{-}\dim X$ hold.

2.19. Theorem [5]. *Let X and K be spaces. If X can be represented as the union of a sequence $F_1, F_2, \ldots$ of closed subsets, then $K \in AE(X)$ provided $K \in AE(F_n)$ for all n and $K \in ANE(X)$.* ■

By analogy with Theorem 3.1.13 from [9] we get

2.20. Theorem. *Let K be a compact polyhedron and let a space X can be represented as the union of a family $\{F_\alpha : \alpha \in A\}$ of closed subspaces such that $K \in AE(F_\alpha)$ for $\alpha \in A$, and let there exist a point-finite open cover $u = \{U_\alpha : \alpha \in A\}$ of X such that $F_\alpha \subset U_\alpha$ for $\alpha \in A$. Then $K \in AE(X)$.* ■

The proof of the next theorem is similar to that of Theorem 1.2 from [8].

2.21. Theorem. *Let K and L be compact polyhedra and let X be a hereditarily normal space. If $X = A \cup B$ and $K \in AE(A)$, $L \in AE(B)$, then $K * L \in AE(X)$.* ■

Theorem 2.3 yields

2.22. Proposition. *Let R be an ANR-compactum and let X be a space satisfying the following condition:*

there exists a closed set $F \subset X$ such that $R \in AE(F)$ and $R \in AE(C)$ for every closed set $C \subset X$ which does not meet F.

Then $R \in AE(X)$. ■

2.23. Definition [9]. A hereditarily normal space X is said to be *strongly hereditarily normal* if every regular open set $U \subset X$ can be represented as the union of a point-finite family of open F_σ-sets of X.

By analogy with Theorem 3.1.19 from [9] we get

2.24. Theorem. *If X is a strongly hereditarily normal space and K is a compact polyhedron, then $K \in AE(X) \Rightarrow K \in AE(A)$ for any $A \subset X$.* ■

2.25. Factorization theorem for compact Hausdorff spaces [17]. *Let X be a compact Hausdorff space and let R be an ANR-compactum such that $R \in AE(X)$. Then for every mapping $f : X \to Y$ to a compact Hausdorff space Y there exist a compact Hausdorff space X' and mappings $f' : X \to X'$ and $g : X' \to Y$ such that $R \in AE(X')$, $wX' = wY$ and $f = g \circ f'$.* ■

2.26. Theorem ([16], see [7] for a separable case). *Let K, L be countable CW complexes and let X be a metrizable space. If $K * L \in AE(X)$, then $X = A \cup B$, where $K \in AE(A)$, $L \in AE(B)$.* ■

2.27. Theorem [16]. *Let K be a countable CW complex and let λ be an infinite cardinal number. Then there exists a completely metrizable space M^K_λ such that $wM^K_\lambda \le \lambda$, $K \in AE(M^K_\lambda)$, and M^K_λ contains topologically every metrizable space X with $wX \le \lambda$ and $K \in AE(X)$.* ■

For $\lambda = \omega_0$ this theorem was proved by W. Olszewski [18]. A. Chigogidze and V. Valov [2] got a stronger result. Namely, M^K_λ can be chosen so that for any completely metrizable space X of weight $\le \lambda$ and $K \in AE(X)$ the set of closed embeddinds $X \to M^K_\lambda$ is dense in the space $C(X, M^K_\lambda)$ of all continuous mappings from X to M^K_λ endowed with source limitation topology.

2.28. Proposition. *Let R_i, S_i, $i = 1, 2$, be ANR-compacta such that $R_1 \le R_2$, $S_1 \le S_2$. Then $R_1 * S_1 \le R_2 * S_2$.*

Proof. According to Proposition 2.13 we may assume that R_i, S_i are polyhedra. In this case our assertion is proved in ([7], Proposition 3.3) with respect to the order $\le_{\mathrm{Sep}}$. Applying Theorem 2.12 we complete the proof. ■

2.29. Theorem ([6], Theorem 7.10). *Let h^* be a reduced continuous cohomology theory such that $h^*(K) = 0$ for some countable simplicial complex K. Then there exists a strongly infinite-dimensional compactum X having the property $K \in AE(X)$.* ■

2.30. Proposition [12]. *Let $R = R_1 \cup R_2$. If $R_i \in AE(X)$, $i = 1, 2$, and $R_1 \cap R_2 \in AE(X)$, then $R \in AE(X)$.* ■

2.31. Proposition [12]. *Let $R = R_1 \cup R_2$ and let $R \in AE(X)$ and $R_1 \cap R_2 \in AE(X)$. Then $R_i \in AE(X)$, $i = 1, 2$.* ■

2.32. Proposition [12]. $R_1 \times R_2 \in AE(X)$ if and only if $R_i \in AE(X)$, $i = 1, 2$. ■

2.33. Proposition [3]. *Let $R_1 \supset R_2$ and let R_1, R_2 be ANR-compacta. If $R_i \in AE(X)$, $i = 1, 2$, then $R_1/R_2 \in AE(X)$.* ■

2.34. Proposition [3]. Let $f\colon Z \to Y$ be a mapping of ANR-compacta such that $f^{-1}(y) \in ANR$ for all $y \in Y$. Assume that $Y \in AE(X)$ and $f^{-1}(y) \in AE(X)$ for a compactum X and all $y \in Y$. Then $Z \in AE(X)$. ■

Propositions 2.30, 2.32, 2.33, and 2.34 yield

2.35. Proposition. *If $R_i \in AE(X)$, $i = 1, 2$, then $R_1 \wedge R_2 \in AE(X)$ and $R_1 * R_2 \in AE(X)$.* ■

Proposition 2.32 and 2.33 imply

2.36. Proposition. *If $R \in AE(X)$, then $\Sigma(R) \in AE(X)$.* ■

Since $R_1 * R_2 = \Sigma(R_1 \wedge R_2)$, Proposition 2.36 yields

2.37. Proposition. *If $R_1 \wedge R_2 \in AE(X)$, then $R_1 * R_2 \in AE(X)$.* ■

2.38. Theorem [7]. *Let X and Y be metrizable spaces of finite dimension and let Y be compact. If $K \in AE(X)$ and $L \in AE(Y)$ are connected CW complexes, then $K \wedge L \in AE(X \times Y)$.* ■

The next statement is well known.

2.39. Open enlargement lemma. *Let $\Phi = (F_1, \dots, F_m) \in \mathrm{Fin}_s(\exp X)$. Then there exists a family $u = (U_1, \dots, U_m)$ of open subsets of X such that $N(u) = N(\Phi)$ and $F_j \subset U_j$, $j = 1, \dots, m$.* ■

3. Definitions of dimension invariants by means of partitions and essential mappings

Recall that a simplicial complex G is said to be *complete* if every face of each simplex from G belongs to G. In what follows *complexes* stand for finite complete simplicial complexes.

Symbols $\mathcal{G}, \mathcal{H}, \mathcal{G}_1$ and so on denote non-empty classes of complexes.

3.1. Definition [10]. Let X be a space, G be a complex, and $\Phi = (F_1, \dots, F_m) \in \mathrm{Fin}_s(\exp X)$. A family $u = \{U_1, \dots, U_k\}$ $k \geq m$, of open subsets of X is called a *G-neighbourhood* of Φ if $F_j \subset U_j$ and $N(u) \subset G$.

3.2. Definition. A set $P \subset X$ is said to be a *G-partition* of $\Phi \in \mathrm{Fin}_s(\exp X)$ (notation: $P \in \mathrm{Part}(\Phi, G)$) if $P = X \setminus \bigcup u$, where u is a G-neighbourhood of Φ.

Put $\mathrm{Exp}_G(X) = \{\Phi \in \mathrm{Fin}_s(\exp X) : N(\Phi) \subset G\}$.

3.3. Definition. A sequence $\mathcal{G} = (G_1, \dots, G_r)$ of complexes is called *inessential* in X if for every sequence $(\Phi_1, \dots, \Phi_r)$ such that $\Phi_i \in \mathrm{Exp}_{G_i}(X)$ there exist G_i-partitions P_i of Φ_i such that $\bigcap\{P_i : i = 1, \dots, r\} = \varnothing$.

3.4. Definition. Let $\mathcal{G}$ be a class of complexes. To every space X one assigns the dimension $\mathcal{G}$-dim$X$, which is an integer ≥ -1 or ∞. The dimension function $\mathcal{G}$-dim is defined in the following way:

(1) $\mathcal{G}\text{-}\dim X = -1 \iff X = \varnothing$;

(2) $\mathcal{G}\text{-}\dim X \leq n$, where $n = 0, 1, \dots$, if every sequence $(G_1, \dots, G_{n+1})$, $G_i \in \mathcal{G}$, $i = 1, \dots, n+1$, is inessential in X;

(3) $\mathcal{G}\text{-}\dim X = \infty$, if $\mathcal{G}\text{-}\dim X > n$ for all $n = -1, 0, 1, \dots$.

If the class $\mathcal{G}$ contains only one complex G we write $\mathcal{G} = G$ and $\mathcal{G}\text{-}\dim X = G\text{-}\dim X$.

Let $\{0, 1\}$ be a two point set and let Δ^n be an n-dimensional simplex. The next assertion is evident.

3.5. Proposition. *For every space X we have* $\Delta^n\text{-}\dim X \leq 0$. ■

From a characterization of Lebesgue dimension by means of partitions we get

3.6. Theorem. *For every space X we have* $\{0, 1\}\text{-}\dim X = \dim X$. ■

3.7. Symbols $\mathcal{R}$, $\mathcal{R}_1$ and so on denote non-empty classes of metrizable ANR-compacta. If $\mathcal{R}$ contains only one ANR-compactum R we write $\mathcal{R} = R$. Put $C(X, \mathrm{cone}\,\mathcal{R}) = \bigcup\{C(X, \mathrm{cone}\,R) : R \in \mathcal{R}\}$.

3.8. Definition [10]. Let $\sigma = (f_1, \dots, f_n)$ be a finite sequence of mappings

$f_i \colon X \to \mathrm{cone}\,R_i$, $R \in \mathcal{R}$,

$f = f_1 \triangle \dots \triangle f_n : X \to \prod_{i=1}^n \mathrm{cone}\,R_i$,

and let $F = f^{-1}(R_1 * \dots * R_n)$ (see Remark 1.7). The sequence σ is said to be $\mathcal{R}$-*inessential* if the mapping $f|_F : F \to R_1 * \dots * R_n$ extends over X.

3.9. Definition. Let $\mathcal{R}$ be a non-empty class of metrizable ANR-compacta. To every space X one assigns the dimension $\mathcal{R}\text{-}\dim X$ which is an integer ≥ -1 or ∞. The dimension function $\mathcal{R}$-dim is defined in the following way:

(1) $\mathcal{R}\text{-}\dim X = -1 \iff X = \varnothing$;

(2) $\mathcal{R}\text{-}\dim X \leq n$, where $n = 0, 1, \dots$, if every sequence $\sigma = (f_1, \dots, f_{n+1})$, $f_i \in C(X, \mathrm{cone}\,\mathcal{R})$, is $\mathcal{R}$-inessential;

(3) $\mathcal{R}\text{-}\dim X = \infty$, if $\mathcal{R}\text{-}\dim X > n$ for all $n \geq -1$.

If the class $\mathcal{R}$ contains only one compactum R we write $\mathcal{R} = R$ and $\mathcal{R}\text{-}\dim X = R\text{-}\dim X$.

From a characterization of Lebesgue dimension by means of essential mappings we get

3.10. Theorem. *For every space X* $\{0, 1\}\text{-}\dim X = \dim X$. ■

3.11. Remark. At a glance the assertions of Theorems 3.6 and 3.10 coincide. But these theorems deal with different dimension functions: $\mathcal{G}$-dim and $\mathcal{R}$-dim.

For a class $\mathcal{Z}$ of compacta and an integer $m \geq 1$, we put

$$\overset{m}{*}\mathcal{Z} = \{Z_1 * \ldots * Z_m : Z_i \in \mathcal{Z}\}.$$

We shall write $\mathcal{R} \subset AE(X)$ if $R \in AE(X)$ for every $R \in \mathcal{R}$.

3.12. Proposition. *For arbitrary $\mathcal{R}$ and X, we have $\mathcal{R}$-$\dim X \leq n \iff \overset{n+1}{*}\mathcal{R} \subset AE(X)$.*

Proof. Implication $\Rightarrow$. Let $\mathcal{R}$-$\dim X \leq n$, $R_1, \ldots, R_{n+1} \in \mathcal{R}$, and let $f_0\colon F \to R_1 * \ldots * R_{n+1}$ be a mapping of a closed set $F \subset X$. Propositions 1.9, 1.14 and 2.3 imply that $\operatorname{cone}(R_1 * \ldots * R_{n+1}) \equiv R \in AE(X)$. Hence there exists a mapping $f : X \to R$ such that $f|_F = f_0$. By Proposition 1.6, $R = \prod_{i=1}^{n+1} \operatorname{cone} R_i$. Let

$$p_j\colon \prod_{i=1}^{n+1} \operatorname{cone} R_i \to \operatorname{cone} R_j, \qquad j = 1, \ldots, n+1,$$

be projections onto factors. Let $f_j = p_j \circ f$. Then $f = f_1 \triangle \ldots \triangle f_{n+1}$. Since $f|_F = f_0$,

$$F \subset F_1 \equiv f^{-1}(R_1 * \ldots * R_{n+1}). \tag{3.3}$$

From $\mathcal{R}$-$\dim X \leq n$, it follows that the sequence $(f_1, \ldots, f_{n+1})$ is $\mathcal{R}$-inessential and, consequently, there exists a mapping $g : X \to R_1 * \ldots * R_{n+1}$ such that $g|_{F_1} = f$. Hence the equality $f|_F = f_0$ and condition (3.3) imply that $g|_F = f_0$. So g is a required extension of f_0 over X and $R_1 * \ldots * R_{n+1} \in AE(X)$.

Implication $\Leftarrow$. Let $\overset{n+1}{*}\mathcal{R} \subset AE(X)$. We have to prove that an arbitrary sequence

$$f_i\colon X \to \operatorname{cone} R_i, \quad R_i \in \mathcal{R}, \qquad i = 1, \ldots, n+1,$$

is $\mathcal{R}$-inessential. Put

$$f = f_1 \triangle \ldots \triangle f_{n+1}\colon X \to \prod_{i=1}^{n+1} \operatorname{cone} R_i = \operatorname{cone}(R_1 * \ldots * R_{n+1})$$

and $F = f^{-1}(R_1 * \ldots * R_{n+1})$. Since $R_1 * \ldots * R_{n+1} \in AE(X)$, the mapping $f_0 = f|_F : F \to R_1 * \ldots * R_{n+1}$ extends over X. Thus the sequence $(f_1, \ldots, f_{n+1})$ is $\mathcal{R}$-inessential. ∎

3.13. Corollary. *For an arbitrary ANR-compactum R, R-$\dim X \leq n \iff \overset{n+1}{*}R \in AE(X)$. In particular, R-$\dim X \leq 0 \iff R \in AE(X)$.* ∎

Another corollary of Proposition 3.12 is

3.14. Proposition. *If $\mathcal{R}$-$\dim X \leq n$ and F is a closed subset of X, then $\mathcal{R}$-$\dim F \leq n$.* ∎

From Definition 2.15 and Corollary 3.13 we get

3.15. Proposition. *For arbitrary space X, R-$\dim X \leq n \iff e$-$\dim X \leq \overset{n+1}{*}R$. In particular, R-$\dim X \leq 0 \iff e$-$\dim X \leq R$.* ∎

3.16. Remark. Definition 3.4 of dimension function $\mathcal{G}$-dim is based on Definition 3.2 and the definition of the set $\operatorname{Exp}_G(X)$. In these definitions the embeddings

$N(\Phi) \subset G$ and $N(u) \subset G$ do not depend on each other. It is possible to give another definition of dimension function $\mathcal{G}$-dim, where the embedding $N(u) \subset G$ is an extension of the embedding $N(\Phi) \subset G$. We shall show that this new approach gives us the same dimension function.

3.17. Definition. Let G be a complex and let X be a space. Denote by $\operatorname{Exp}^\theta_G(X)$ the set of all triples $T = (\Phi_T, \alpha_T, e_T)$, where:

$\Phi_T = \Phi = (F_1, \dots, F_m) \in \operatorname{Fin}_s(\exp X), m \le |v(G)|$;

$\alpha_T = \alpha : (1, \dots, m) \to v(G)$ is an embedding;

$e_T = e\colon N(\Phi_T) \to G$ is a simplicial embedding

such that $e(F_j) = \alpha(j)$.

3.18. Definition. Let $T \in \operatorname{Exp}^\theta_G(X)$ and let $\Phi_T = (F_1, \dots, F_m)$. A family $u = (U_1, \dots, U_k), k \ge m$, of open subsets of X is said to be a *G-neighbourhood* of T if $F_j \subset U_j$ and there exists an embedding $\alpha'\colon (1, \dots, k) \to v(G)$, $\alpha'|_{(1,\dots,m)} = \alpha$, and a mapping $e'\colon N(u) \to G$, defined by the equality $e'(U_j) = \alpha'(j)$, is a simplicial embedding.

3.19. Definition. A set $P \subset X$ is called a *G-partition* of $T \in \operatorname{Exp}^\theta_G(X)$ (notation: $P \in \operatorname{Part}(T, G)$) if $P = X \setminus \bigcup u$, where u is a G-neighbourhood of T.

3.20. Definition. A sequence $(G_1, \dots, G_r)$ of complexes is called *θ-inessential* in X if for every sequence $(T_1, \dots, T_r)$, $T_i \in \operatorname{Exp}^\theta_{G_i}(X)$, there exist G_i-partitions P_i of T_i such that $\bigcap\{P_i : i = 1, \dots, r\} = \varnothing$.

The inclusion $\operatorname{Part}(T, G) \subset \operatorname{Part}(\Phi_T, G)$ yields

3.21. Proposition. *If a sequence $(G_1, \dots, G_r)$ is θ-inessential in X, then it is inessential in X.* ∎

3.22. Definition. The dimension function $\mathcal{G}$-$\dim_\theta$ is defined as the function $\mathcal{G}$-dim (Definition 3.4). The only difference is that in the item (2) we require a θ-inessentiality of a sequence $(G_1, \dots, G_{n+1})$ instead of its inessentiality.

3.23. Theorem. *For every space X we have $\mathcal{G}$-$\dim X = \mathcal{G}$-$\dim_\theta X$.*

To prove Theorem 3.23 we need an additional information.

3.24. Lemma. *Let $\Phi = (F_1, \dots, F_m) \in \operatorname{Fin}_s(\exp X)$ be a sequence such that the set $X \setminus \bigcup \Phi$ is infinite. Let G be a complex with $v(G) = \{a_1, \dots, a_k\}$, $k \ge m$. Assume that the correspondence $F_j \mapsto a_j$, $j = 1, \dots, m$, generates the embedding $N(\Phi) \to G$. Then there exists a sequence $\Phi_1 = (F^1_1, \dots, F^1_k) \in \operatorname{Fin}_s(\exp X)$ such that $F_j \subset F^1_j$, $j = 1, \dots, m$, and the correspondence $F^1_j \mapsto a_j$, $j = 1, \dots, k$, generates the isomorphism $N(\Phi_1) \to G$.*

Proof. Let $\tilde G$ be a geometric realization of G and let $\tilde a_1, \dots, \tilde a_k$ be vertices of $\tilde G$. Denote by H the set of barycenters of all simplices of $\tilde G$. Let $\beta\colon H \to X \setminus \bigcup \Phi$ be some injection. Put

$$O_j = \beta(H \cap O\tilde a_j), \qquad j = 1, \dots, k,$$

where $O\tilde a_j$ is the star of $\tilde a_j$ in $\tilde G$. Let $\Omega = (O_1, \dots, O_k)$. From the definition of H we get that the correspondence $O_j \to a_j$ generated an isomorphism $N(\Omega) \to G$.

Put

$$F_j^1 = F_j \cup O_j, \quad j = 1, \dots, m; \qquad F_j^1 = O_j, \quad j = m+1, \dots, k.$$

Then $\Phi_1 = (F_1^1, \dots, F_k^1)$ is the required sequence. ■

3.25. Lemma. *Let $\Phi = (F_1, \dots, F_m) \in \mathrm{Fin}_s(\exp X)$ be a sequence such that $X = \bigcup \Phi$. Then*

$$\mathrm{Part}(\Phi, G) = \{\varnothing\} = \mathrm{Part}(T, G),$$

where $T = (\Phi_T, \alpha_T, e_T) \in \mathrm{Exp}_G^\theta(X)$ is an arbitrary triple with $\Phi_T = \Phi$. ■

3.26. Lemma. *Let $\Phi = (F_1, \dots, F_m) \in \mathrm{Fin}_s(\exp X)$ be a sequence such that $N(\Phi) = G$. Then $\mathrm{Part}(\Phi, G) \subset \mathrm{Part}(T, G)$ for an arbitrary triple $T \in \mathrm{Exp}_G^\theta(X)$ with $\Phi_T = \Phi$.*

Proof. Let $P \in \mathrm{Part}(\Phi, G)$. It means that there is a one-to-one correspondence $\alpha_\Phi \colon (1, \dots, m) \to v(G)$ such that the correspondence

$$e_\Phi(F_j) = \alpha_\Phi(j)$$

generates the simplicial isomorphism $e_\Phi \colon N(\Phi) \to G$, and there is a neighbourhood $u = (U_1, \dots, U_m)$ of Φ, $P = X \setminus \bigcup u$, with another one-to-one correspondence $\alpha'_\Phi \colon (1, \dots, m) \to v(G)$ generating an isomorphism $e'_\Phi \colon N(u) \to G$ by means of the correspondence $e'_\Phi(U_j) = \alpha'_\Phi(j)$. Put $T = (\Phi, \alpha_\Phi, e_\Phi)$. Then u becomes a G-neighbourhood of Φ if we put $\alpha' = \alpha$. Since e_Φ is an isomorphism, the mapping $e' : N(u) \to G$, defined by $e'(U_j) = \alpha'(j)$, is an isomorphism as well. ■

Proof of Theorem 3.23. The inequality $\le$ is a consequence of Proposition 3.21. Now let $\mathcal{G}\text{-}\dim X \le n$ and let $(G_1, \dots, G_{n+1})$ be a sequence of complexes from $\mathcal{G}$. We have to prove that this sequence is θ-inessential. Let $T_i \in \mathrm{Exp}_G^\theta(X)$, $i = 1, \dots, n+1$. Let $\Phi_{T_i} = (F_1^i, \dots, F_{m_i}^i)$. We enlarge the sequences Φ_{T_i} to sequences Φ_i^1 in the following way. If $X \setminus \bigcup \Phi_{T_i}$ is finite, then we put $F_1^{i,1} = F_1^i \cup (X \setminus \bigcup \Phi_{T_i})$, $F_j^{i,1} = F_j^i$, $j = 2, \dots, m_i$. If $X \setminus \bigcup \Phi_i$ is infinite, then we take a sequence Φ_i^1 from Lemma 3.24. Since $\mathcal{G}\text{-}\dim X \le n$ there exist partitions $P_i \subset \mathrm{Part}(\Phi_i^1, G_i)$ with $P_1 \cap \ldots \cap P_{n+1} = \varnothing$. Applying Lemmas 3.25 and 3.26 we finish the proof. ■

4. Equality $\mathcal{K}$-$\dim X = \mathcal{K}_\tau$-$\dim X$

4.1. Definition. Let $\mathcal{R}$ be a class of ANR-compacta and let X be a space. According to Definition 2.15 we write $e\text{-}\dim X \le \mathcal{R}$, provided $\mathcal{R} \subset AE(X)$, i.e. the property $R \in AE(X)$ holds for every $R \in \mathcal{R}$.

Proposition 3.12 implies

4.2. Proposition. *For arbitrary $\mathcal{R}$ and X, we have $\mathcal{R}\text{-}\dim X \le n \iff e\text{-}\dim X \le \overset{n+1}{*}\mathcal{R}$.* ■

4.3. Definition. We say that $\mathcal{R}_1$ *is dominated by* $\mathcal{R}_2$ (notation: $\mathcal{R}_1 \le_h \mathcal{R}_2$) if every $R_1 \in \mathcal{R}_1$ is dominated by some $R_2 \in \mathcal{R}_2$. A class $\mathcal{R}_1$ is *homotopy equivalent* to a class $\mathcal{R}_2$ (notation: $\mathcal{R}_1 \simeq \mathcal{R}_2$) if both $\mathcal{R}_1 \le_h \mathcal{R}_2$ and $\mathcal{R}_2 \le_h \mathcal{R}_1$ hold.

4.4. Proposition. *If $\mathcal{R}_1 \le_h \mathcal{R}_2$, then $\mathcal{R}_1\text{-}\dim X \le \mathcal{R}_2\text{-}\dim X$ for an arbitrary space X.*

Proof. The assertion is obvious if $\mathcal{R}_2$-$\dim X = \infty$. Let $\mathcal{R}_2$-$\dim X = n < \infty$. To prove that $\mathcal{R}_1$-$\dim X \le n$ it suffices, according to Proposition 4.2, to show that e-$\dim X \le \overset{n+1}{*}\mathcal{R}_1$. Let $R_1^1, \ldots, R_{n+1}^1 \in \mathcal{R}_1$. Since $\mathcal{R}_1 \le_h \mathcal{R}_2$, there exist $R_j^2 \in \mathcal{R}_2$ such that $R_j^1 \le_h R_j^2$, $j = 1, \ldots, n+1$. From Proposition 1.8 we get $\overset{n+1}{\underset{j=1}{*}} R_j^1 \le_h \overset{n+1}{\underset{j=1}{*}} R_j^2$. The equality $\mathcal{R}_2$-$\dim X = n$ and Proposition 4.2 imply that $R_1^2 * \ldots * R_{n+1}^2 \in AE(X)$.

Consequently, in view of Proposition 2.9, $\overset{n+1}{\underset{j=1}{*}} R_j^1 \in AE(X)$. Applying Proposition 4.2 once more we get $\mathcal{R}_1$-$\dim X \le n$. ∎

As a corollary we have

4.5. Proposition. *If $\mathcal{R}_1 \simeq \mathcal{R}_2$, then $\mathcal{R}_1$-$\dim X = \mathcal{R}_2$-$\dim X$ for every X.* ∎

Theorem 1.2 and Proposition 4.5 yield

4.6. Proposition. *For every class $\mathcal{R}$ of ANR-compacta there exists a class $\mathcal{K} = \mathcal{K}(\mathcal{R})$ of polyhedra such that $\mathcal{R}$-$\dim X = \mathcal{K}$-$\dim X$ for every space X.* ∎

So, when we investigate dimension functions of type $\mathcal{R}$-dim, we can consider only classes $\mathcal{R}$ consisting of compact polyhedra. These classes we shall denote by $\mathcal{K}, \mathcal{L}$ and so on. In what follows all polyhedra are assumed to be compact.

Another corollary of Proposition 4.5 is

4.7. Proposition. *Let K and L be homotopy equivalent polyhedra. Then K-$\dim X = L$-$\dim X$ for every space X.* ∎

Let $\mathcal{K}$ be a class of polyhedra. For each $K \in \mathcal{K}$ we fix a triangulation $t = t(K)$ of K. The pair (K, t) is a simplicial complex which is denoted by K_t. The family $\tau = \{t(K) : K \in \mathcal{K}\}$ is said to be a *triangulation* of the class $\mathcal{K}$. Let $\mathcal{K}_\tau = \{K_t : t \in \tau\}$.

4.8. Theorem. *Let $\mathcal{K}$ be a class of polyhedra and let τ be some its triangulation. Then $\mathcal{K}_\tau$-$\dim X = \mathcal{K}$-$\dim X$ for every space X.*

To prove Theorem 4.8 we need some additional information.

Let $u = (U_1, \ldots, U_m) \in \mathrm{cov}_\infty(X)$. Recall that a mapping $f : X \to N(u)$ is said to be *u-barycentric*, if $f(x) = (\varphi_1(x), \ldots, \varphi_m(x))$, where $(\varphi_1, \ldots, \varphi_m)$ is some partition of unity subordinated to the cover u, and $\varphi_j(x)$ is the barycentric coordinate of $f(x)$ corresponding to the vertex $a_j \equiv U_j \in v(N(u))$.

Let G be a simplicial complex with vertices $a_1, \ldots, a_m$. By $Oa_j \equiv O_j$ we denote the *star* of a_j in G, that is the union of all open simplices σ from G such that $a_j \in v(\sigma)$. Put $\omega = \omega(G) = (Oa_1, \ldots, Oa_m)$.

For $g \in G$, let $\mu_j(g)$ be the barycentric coordinate of the point g corresponding to the vertex a_j. The function $\mu_j : G \to [0; 1]$ is continuous and $\mathrm{supp}\, \mu_j \equiv \mu_j^{-1}(0; 1] = Oa_j$.

4.9. Proposition. *The mapping $\mu : G \to N(\omega(G))$ defined as $\mu(g) = (\mu_1(g), \ldots, \mu_m(g))$ is a simplicial isomorphism and an $\omega(G)$-barycentric mapping.* ∎

If $\Phi = (F_1, \dots, F_m)$ is a sequence of closed subsets of a space X, then a sequence $u = (U_1, \dots, U_m)$ of open subsets of X is called an *open enlargement* of Φ if $F_j \subset U_j$, $j = 1, \dots, m$. Every finite sequence Φ of closed subsets of X has an open enlargement u with $N(u) = N(\Phi)$ (Lemma 2.39).

4.10. Definition. Let $\Phi = (F_1, \dots, F_m)$ be a closed cover of X. A mapping $f : X \to N(\Phi) \equiv G$ is said to be a Φ-*barycentric*, if it is u-barycentric for some open enlargement u of Φ such that $N(u) = N(\Phi)$ and

$$Cl(f(F_j)) \subset Oa_j, \quad j = 1, \dots, m. \tag{4.1}$$

4.11. Proposition. *For every finite closed cover Φ of X there exists a Φ-barycentric mapping $f\colon X \to N(\Phi)$.* ∎

4.12. Definition. Let $\Phi = (F_1, \dots, F_m) \in \mathrm{Fin}_s(\exp X)$ and let $F = F_1 \cup \dots \cup F_m$. Let $u = (U_1, \dots, U_m)$ be an open enlargement of Φ. A mapping $f : F \to N(\Phi) \equiv G$ is said to be (u, Φ)-*barycentric* if it is $(u|F)$-barycentric and satisfies condition (4.1).

4.13. Lemma [10]. *Let G be a simplicial complex with vertices $a_1, \dots, a_k$ and let $u = (U_1, \dots, U_k)$ be a G-neighbourhood of a sequence $\Phi = (F_1, \dots, F_m) \in \mathrm{Fin}_s(\exp X)$, $m \le k$. Put $F = F_1 \cup \dots \cup F_m$, $U = U_1 \cup \dots \cup U_k$, and let B be a closed set such that $F \subset B \subset U$. Then every (u, Φ)-barycentric mapping $f_0\colon F \to N(u|F) \subset N(u) \subset G$ extends to a mapping $f : X \to \mathrm{cone}\, G$ such that*

$$f^{-1}(Oa_j) \subset U_j, \quad Cl(f(F_j)) \subset Oa_j; \tag{4.2}$$

$$f^{-1}(a) \cap B = \varnothing, \tag{4.3}$$

where Oa_j is the star of the vertex a_j in cone G *and a is the peak of* cone G. ∎

4.14. Lemma. *Let $R_1, \dots, R_n$ be ANR-compacta, let $F_1, \dots, F_n$ be closed subsets of a space X, and let $h_i\colon X \to \mathrm{cone}\, R_i$, $i = 1, \dots, n$, be mappings such that*

$$h_i(F_i) \subset R_i. \tag{4.4}$$

Then the mapping $h = h_1 \triangle \dots \triangle h_n$ satisfies the condition

$$h(Y) \subset B(R_1, \dots, R_n), \tag{4.5}$$

where $Y = F_1 \cup \dots \cup F_n$.

Proof. According to (1.3)

$$B \equiv B(R_1, \dots, R_n) = B_1 \cup \dots \cup B_n, \tag{4.6}$$

where $B_i = R_i \times \prod_{j \ne i} \mathrm{cone}\, R_j$. So it suffices to check that

$$h(F_i) \subset B_i. \tag{4.7}$$

Let $p_i\colon \prod_{j=1}^n \mathrm{cone}\, R_j \to \mathrm{cone}\, R_i$ be the projection onto the factor. Then

$$h_i = p_i \circ h; \tag{4.8}$$

$$B_i = p_i^{-1}(R_i). \tag{4.9}$$

From (4.8) and (4.9) we get that (4.7) is equivalent to (4.4). ∎

4.15. LEMMA. *Let $f_0, f_1 \colon X \to R$ be mappings to an AR-compactum and let $f_0|_F = f_1|_F$ for some closed set $F \subset X$. Then there exists a homotopy $f_t \colon X \to R$, $0 \le t \le 1$, such that $f_t|_F = f_0|_F$ for all $t \in I$.* ∎

4.16. LEMMA. *Let $f_i \colon X \to \operatorname{cone} R_i$, $i = 1, \dots, n$, be mappings, where $R_1, \dots, R_n$ are ANR-compacta. Suppose there exist mappings $g_i \colon X \to \operatorname{cone} R_i$ and homotopies $f_i^t : X \to \operatorname{cone} R_i$ such that*

$$f_i^0 = f_i, \qquad f_i^1 = g_i; \tag{4.10}$$

$$(f_i^t)^{-1}(R_i) \supset F_i \equiv f_i^{-1}(R_i); \tag{4.11}$$

$$g(X) \subset \prod_{i=1}^n \operatorname{cone} R_i \setminus \{(a_1, \dots, a_n)\}, \tag{4.12}$$

where $g = g_1 \triangle \dots \triangle g_n$ and a_i is the peak of $\operatorname{cone} R_i$. Then there exists a mapping $\overline{f} \colon X \to B(R_1, \dots, R_n) \equiv B$ such that $\overline{f}|_Y = f$, where $f = f_1 \triangle \dots \triangle f_n$ and $Y = F_1 \cup \dots \cup F_n$.

Proof. According to Proposition 1.6 there exists a retraction

$$r \colon \prod_{i=1}^n \operatorname{cone} R_i \setminus \{(a_1, \dots, a_n)\} \to B.$$

Put $h = r \circ g$ and $h_i = p_i \circ h$. Then

$$h_i|_{g_i^{-1}(R_i)} = g_i|_{g_i^{-1}(R_i)}. \tag{4.13}$$

Indeed, let $x \in g_i^{-1}(R_i)$. Then $R_i \ni g_i(x) = (p_i \circ g)(x)$. Consequently, $g(x) \in p_i^{-1}(R_i) = (\text{by } (4.9)) = B_i \subset B$. Since $r|_B = \mathrm{id}$, we have $h(x) = g(x)$. Hence $h_i(x) = g_i(x)$. Thus equality (4.13) is checked.

The conditions (4.10) and (4.11) imply that $F_i \subset g_i^{-1}(R_i)$. Hence (4.13) yields

$$g_i|_{F_i} = h_i|_{F_i}. \tag{4.14}$$

Lemma 4.15 and the equality (4.14) imply an existence of a homotopy $g_i^t \colon X \to \operatorname{cone} R_i$ such that

$$g_i^0 = g_i, \qquad g_i^1 = h_i; \tag{4.15}$$

$$g_i^t|_{F_i} = g_i|_{F_i} = h_i|_{F_i}. \tag{4.16}$$

From (4.11) and (4.16) we get condition (4.4) for the homotopies g_i^t. Consequently, in accordance with Lemma 4.14 the homotopy $g^t = g_1^t \triangle \dots \triangle g_n^t$ satisfies the condition

$$g^t(Y) \subset B(R_1, \dots, R_n). \tag{4.5$'$}$$

In view of (4.15) the homotopy g^t connects the mappings g and h. On the other hand, according to (4.10) the homotopy $f^t = f_1^t \triangle \dots \triangle f_n^t$ connects the mappings

f and g and satisfies the condition $f^t(Y) \subset B(R_1, \dots, R_n)$ because of (4.11) and Lemma 4.14. Thus we can define a homotopy $h^t : Y \to B$ putting

$$h^t(y) = \begin{cases} f^{2t}(y) & \text{for } \ t \le \frac{1}{2}; \\ g^{2t-1}(y) & \text{for } \ t \ge \frac{1}{2}. \end{cases}$$

This homotopy connects the mappings $f|_Y = h^0$ and $h|_Y = h^1$. By Theorem 2.1 the homotopy h^t can be extended to a homotopy $\overline{h}^t : X \to B$ so that $\overline{h}^1 = h$. Then $\overline{f} = \overline{h}^0$ is the required mapping. ■

Proof of Theorem 4.8. Denote the class $\mathcal{K}_\tau$ by $\mathcal{G} = \mathcal{G}(\mathcal{K})$ and its members K_t by $G = G(K)$. We have to prove the inequalities

$$\mathcal{G}\text{-}\dim X \le \mathcal{K}\text{-}\dim X, \tag{4.17}$$

$$\mathcal{K}\text{-}\dim X \le \mathcal{G}\text{-}\dim X. \tag{4.18}$$

Let $\mathcal{K}\text{-}\dim X \le n$ and $(G_1, \dots, G_{n+1}) \in \mathrm{Fin}_s\mathcal{G}$, $\Phi_i \in \mathrm{Exp}_{G_i}(X)$, $i = 1, \dots, n+1$. To prove inequality (4.17) we have to find G_i-partitions P_i of Φ_i such that $\bigcap_{i=1}^{n+1} P_i = \varnothing$. Let $\Phi_i = (F_1^i, \dots, F_{m_i}^i)$. By definition of $\mathrm{Exp}_{G_i}(X)$ we have $N(\Phi_i) \subset G_i$. Let $v(G_i) = \{a_1^i, \dots, a_{k_i}^i\}$, $m_i \le k_i$. Put $F_i = F_1^i \cup \dots \cup F_{m_i}^i$. According to Proposition 4.11 there exists a Φ_i-barycentric mapping $f_i^0 \colon F_i \to N(\Phi_i) \subset G_i$. This mapping extends to a mapping $f_i : X \to \mathrm{cone}\, G_i$. Since $\mathcal{K}\text{-}\dim X \le n$, the mapping

$$f = f_1 \triangle \dots \triangle f_{n+1} \colon X \to \prod_{i=1}^{n+1} \mathrm{cone}\, G_i = \mathrm{cone}\Big(\mathop{*}_{i=1}^{n+1} G_i\Big)$$

is inessential. Consequently, there exists a mapping $g \colon X \to \mathop{*}\limits_{i=1}^{n+1} G_i = B = B(G_1, \dots, G_{n+1})$ such that

$$g|_Y = f|_Y, \tag{4.19}$$

where $Y = f^{-1}(B)$. In view of Definition 4.10 we have

$$Cl\Big(f_i(F_j^i)\Big) = Cl\Big(f_i^0(F_j^i)\Big) \subset Oa_j^i, \qquad j \le m_i, \tag{4.20}$$

where Oa_j^i is the star of a_j^i in G_i. From (4.20) it follows the existence of closed sets Γ_j^i, $j \le k_i$, such that

$$Cl\Big(f_i(F_j^i)\Big) \subset \Gamma_j^i \subset Oa_j^i, \qquad j \le m_i, \tag{4.21}$$

and the family $\gamma_i = \{\Gamma_1^i, \dots, \Gamma_{k_i}^i\}$ is a cover of G_i. Put ${}^{n+1}\Gamma_j^i = p_i^{-1}(\Gamma_j^i)$, where $p_i \colon \prod_{j=1}^{n+1} \mathrm{cone}\, G_j \to \mathrm{cone}\, G_i$ is the projection onto the factor. Recall that $B = B_1 \cup \dots \cup B_{n+1}$, where

$$B_i = G_i \times \prod\{\mathrm{cone}\, G_j : j \ne i\} = p_i^{-1}(G_i). \tag{4.22}$$

Since γ_i cover G_i, condition (4.22) implies that $B_i = \bigcup\{{}^{n+1}\Gamma_j^i : 1 \le j \le k_i\}$ and hence

$$g^{-1}B_i = \{g^{-1}({}^{n+1}\Gamma_j^i) : 1 \le j \le k_i\}. \tag{4.23}$$

Put

$$^1F^i_j = g^{-1}(^{n+1}\Gamma^i_j) = g^{-1}p_i^{-1}(\Gamma^i_j). \tag{4.24}$$

The mapping $g\colon X \to B \subset \prod_{i=1}^{n+1} \operatorname{cone} G_i$ is the diagonal product of the mappings $g_i = p_i \circ g\colon X \to \operatorname{cone} G_i$. Thus from (4.19) we get

$$g_i|_Y = f_i|_Y, \tag{4.25}$$

Conditions (4.21), (4.24), and (4.25) yield

$$F^i_j \subset {}^1F^i_j, \qquad j \le m_i. \tag{4.26}$$

Since $X = g^{-1}(B)$, from (4.23) and (4.24) we get

$$X = \bigcup\{^1F^i_j : 1 \le j \le k_i,\ 1 \le i \le n+1\}. \tag{4.27}$$

Put $\Phi^1_i = \{^1F^i_1, \dots, {}^1F^i_{k_i}\}$. Proposition 4.9 and conditions (4.21) and (4.24) imply that

$$N(\Phi^1_i) \subset G_i. \tag{4.28}$$

But the closed family Φ^1_i has a neighbourhood $O\Phi^1_i = \{O^1F^1_i, \dots, O^1F^1_{k_i}\}$ with $N(O\Phi^1_i) = N(\Phi^1_i)$. Consequently, (4.26) and (4.28) imply that $O\Phi^1_i$ is a G_i-neighbourhood of Φ_i. Further, (4.27) implies that $\bigcup\{OF^1_i : 1 \le i \le n+1\} \in \operatorname{cov}(X)$. Hence $P_1 \cap \ldots \cap P_{n+1} = \varnothing$, where $P_i = X \setminus \bigcup O\Phi^1_i$. Thus P_i are the required G_i-partitions of Φ_i and inequality (4.17) is proved.

Now let $\mathcal{K}_\tau$-$\dim X \le n$ and let $f_i\colon X \to \operatorname{cone} K_i \in \mathcal{K}$, $i = 1, \dots, n+1$, be mappings. To prove (4.18) we have to show that the family $\sigma = \{f_1, \dots, f_{n+1}\}$ is $\mathcal{K}$-inessential. Denote a simplicial complex $(K_i)_t$ by G_i. Let $v(G_i) = \{a^i_1, \dots, a^i_{m_i}\}$. There exist closed sets $\Gamma^i_j \subset G_i$ such that $\Gamma^i_j \subset Oa^i_j \in \omega(G_i)$, $\gamma_i = \{\Gamma^i_1, \dots, \Gamma^i_{m_i}\}$ is a cover of G_i, and

$$N(\gamma_i) = G_i. \tag{4.29}$$

Put

$$F^i_j = f_i^{-1}(\Gamma^i_j), \tag{4.30}$$

$$\Phi_i = \{F^i_1, \dots, F^i_{m_i}\}. \tag{4.31}$$

If follows from (4.29)–(4.31) that $\Phi_i \in \operatorname{Exp}_{G_i}(X)$ and $N(\Phi_i) \subset G_i$. Since $\mathcal{G}$-$\dim X \le n$, there exist families $u_i = (U^i_1, \dots, U^i_{m_i})$, $i = 1, \dots, n+1$, of open subsets of X such that

$$F^i_j \subset U^i_j; \tag{4.32}$$

$$X = \bigcup_{i,j} U^i_j; \tag{4.33}$$

$$N(u_i) \subset G_i. \tag{4.34}$$

Put $U_i = U^i_1 \cup \ldots \cup U^i_{m_i}$ and $F_i = F^i_1 \cup \ldots \cup F^i_{m_i}$. From (4.33) we get $X = U_1 \cup \ldots \cup U_{n+1}$. There exist closed sets $B_1, \dots, B_{n+1}$ such that

$$F_i \subset B_i \subset U_i, \tag{4.35}$$

$$X = B_1 \cup \ldots \cup B_{n+1}. \tag{4.36}$$

In view of (4.30) and (4.32) f_i is an (u_i, Φ_i)- barycentric mapping, $i = 1, \dots, n+1$. According to Lemma 4.13 there exist mappings $g_i \colon X \to \operatorname{cone} G_i$, $i = 1, \dots, n+1$, such that

$$g_i|_{F_i} = f_i|_{F_i}. \tag{4.37}$$

$$F^i_j \subset g_i^{-1}(Oa^i_j) \subset U^i_j, \tag{4.38}$$

$$g_i^{-1}(a_i) \cap B_i = \varnothing, \tag{4.39}$$

where Oa^i_j is the star of a^i_j in $\operatorname{cone} G_i$ and a_i is the peak of $\operatorname{cone} G_i$. Put $g = g_1 \triangle \dots \triangle g_{n+1} \colon X = \prod_{i=1}^{n+1} \operatorname{cone} G_i = \operatorname{cone}\Big(\mathop{*}\limits_{j=1}^{n+1} G_i\Big)$.

Conditions (4.36) and (4.39) imply that

$$g(X) \subset \operatorname{cone}\Big(\mathop{*}\limits_{j=1}^{n+1} G_i\Big) \setminus \{(a_1, \dots, a_{n+1})\}.$$

Applying Lemma 4.15 to the pair (f_i, g_i) (see (4.37)) we get a homotopy f_i^t connecting $f_i = f_i^0$ and $g_i = f_i^1$ so that $f_i^t|_{F_i} = f_i|_{F_i}$. Condition (4.30) implies that $F_i = f_i^{-1}(G_i)$. Hence we can apply Lemma 4.16 which yields an existence of a mapping $\overline{f} \colon X \to G_1 * \dots * G_{n+1}$ such that $\overline{f}|_{F_1 \cup \dots \cup F_{n+1}} = f_1 \triangle \dots \triangle f_{n+1}$. Hence the family σ is inessential. ∎

4.17. Remark. An analysis of the proof of Theorem 4.8 shows that we actually used Definition 3.22.

4.18. Theorem. *Let $\mathcal{K}$ be class of polyhedra and let $[\mathcal{K}] = \bigcup\{[K] : K \in \mathcal{K}\}$,where $[K]$ be the class of all simplicial complexes which are triangulations of K. Then $\mathcal{K}$-$\dim X = [\mathcal{K}]$-$\dim X$ for every space X.*

Proof. If we consider a triangulation t of a polyhedron K as a pair (G, h),where $h = h(t) \colon G \to K$ is a homeomorphism between a simplicial complex G and K, then the set $T(K)$ of all triangulations of K has cardinality $\leq 2^{\aleph_0}$. Take some set Γ with $\operatorname{card} \Gamma = 2^{\aleph_0}$ and denote by $\mathcal{K}^1$ the class of all indexed polyhedra from $\mathcal{K}$:

$$\mathcal{K}^1 = \{K_\gamma : K \in \mathcal{K},\ \gamma \in \Gamma\}.$$

Clearly, $\mathcal{K} \simeq \mathcal{K}^1$. Hence

$$\mathcal{K}\text{-}\dim X = \mathcal{K}^1\text{-}\dim X, \tag{4.40}$$

because of Proposition 4.5. Let $T(K) = \{(G_\gamma, h_\gamma) : \gamma \in \Gamma\}$. If we consider h_γ as a homeomorphism $h_\gamma \colon G_\gamma \to K_\gamma$, then $\tau = \bigcup\{T(K) : K \in \mathcal{K}\}$ is a triangulation of the class $\mathcal{K}^1$. Hence according to Theorem 4.8 we have

$$\mathcal{K}^1\text{-}\dim X = (\mathcal{K}^1)_\tau\text{-}\dim X. \tag{4.41}$$

On the other hand, one can identify the class $(\mathcal{K}^1)_\tau$ with the class

$$[K] = \{G_\gamma : (G_\gamma, h_\gamma) \in T(K),\ K \in \mathcal{K}\}.$$

Consequently, $[K]$-$\dim X = (\mathcal{K}^1)_\tau$-$\dim X = (4.41) = \mathcal{K}^1$-$\dim X = (4.40) = \mathcal{K}$-$\dim X$. ∎

Let $\mathcal{G}$ be a class of complexes. For each $G \in \mathcal{G}$ we fix a simplicial subdivision $s = s(G)$ of G. This subdivision can be considered as a triangulation of an Euclidean complex $\tilde{G}$ which is a geometric realization of G. The pair $(\tilde{G}, s)$ is a simplicial complex which is denoted by G_s. The family $\sigma = \{s(G) : G \in \mathcal{G}\}$ is said to be a *simplicial subdivision* of the class $\mathcal{G}$. Let $\mathcal{G}_\sigma = \{G_s : s \in \sigma\}$.

Theorem 4.8 yields

4.19. Theorem. *Let $\mathcal{G}$ be class of a complexes and let $\mathcal{G}_\sigma$ be some its simplicial subdivision. Then $\mathcal{G}$-$\dim X = \mathcal{G}_\sigma$-$\dim X$ for every space X.* ∎

5. Dimension $\mathcal{R}$-dim

Dimension functions $\mathcal{R}$-dim have intrinsic properties similar to those of the classical Lebesgue dimension dim. In what follows X is a space and $\mathcal{R}$ is a class of ANR-compacta.

5.1. Countable sum theorem. *If X can be represented as the union of a sequence $F_1, F_2, \dots$ of closed subsets with $\mathcal{R}$-$\dim F_i \le n$ for all i, then $\mathcal{R}$-$\dim X \le n$.*

Proof. Let $R_1, \dots, R_{n+1} \in \mathcal{R}$. Since $\mathcal{R}$-$\dim F_i \le n$, we have $\mathop{*}\limits_{j=1}^{n+1} R_j \in AE(F_i)$ in view of Proposition 3.12. According to Proposition 1.9 and Theorem 2.3, $\mathop{*}\limits_{j=1}^{n+1} R_j \in ANE(X)$. Consequently, $\mathop{*}\limits_{j=1}^{n+1} R_j \in AE(X)$ in accordance with Theorem 2.19. Applying Proposition 3.12 once more we get $\mathcal{R}$-$\dim X \le n$. ∎

Theorem 5.1 yields

5.2. σ-discrete sum theorem. *Let $\varphi_i = \{F^i_\alpha : \alpha \in A_i\}$, $i \in \mathbb{N}$, be discrete families of closed subsets of X such that $\mathcal{R}$-$\dim F^i_\alpha \le n$ and $X = \bigcup_{i,\alpha} F^i_\alpha$. Then $\mathcal{R}$-$\dim X \le n$.* ∎

5.3. Point-finite sum theorem. *If a space X can be represented as the union of a family $\{F_\alpha : \alpha \in A\}$ of closed subsets such that $\mathcal{R}$-$\dim F_\alpha \le n$ for $\alpha \in A$, and if there exists a point-finite open cover $\{U_\alpha : \alpha \in A\}$ of X such that $F_\alpha \subset U_\alpha$ for $\alpha \in A$, then $\mathcal{R}$-$\dim X \le n$.*

Proof. By Proposition 4.6 there exists a class $\mathcal{K}$ of polyhedra such that

$$\mathcal{R}\text{-}\dim Y = \mathcal{K}\text{-}\dim Y \text{ for every space } Y, \tag{5.1}$$

in particular,

$$\mathcal{K}\text{-}\dim F_\alpha \le n,\ \alpha \in A. \tag{5.2}$$

Let $K_1, \dots, K_{n+1} \in \mathcal{K}$. From (5.2) and Corollary 3.13 we get $\mathop{*}\limits_{i=1}^{n+1} K_i \in AE(F_\alpha)$ for all $\alpha \in A$. Theorem 2.20 implies that $\mathop{*}\limits_{i=1}^{n+1} K_i \in AE(X)$. Applying Corollary 3.13 we get $\mathcal{K}$-$\dim X \le n$. Hence $\mathcal{R}$-$\dim X \le n$, because of (5.1). ∎

5.4. Definition. We say that loc-$\mathcal{R}$-$\dim X \le n$ if for every point $x \in X$ there is a neighbourhood Ox such that $\mathcal{R}$-$\dim Cl(Ox) \le n$.

Theorem 5.3 yields

5.5 Theorem. *If X is a weakly paracompact space, then* loc-$\mathcal{R}$-dim X = $\mathcal{R}$-dim X. ■

5.6. Remark. For the Lebesgue dimension $\dim(\mathcal{R} = \{0,1\})$ Theorems 5.3 and 5.5 were proved by A. Zarelua [24].

5.7. Addition (Urysohn-Menger) theorem. *If a hereditarily normal space X is the union of its subsets A and B such that $\mathcal{R}\text{-}\dim A \leq m$ and $\mathcal{R}\text{-}\dim B \leq n$, then $\mathcal{R}\text{-}\dim X \leq m+n+1$.*

Proof. According to Proposition 4.6 we can assume that $\mathcal{R}$ consists of polyhedra. Let $R_1, \dots, R_{m+n+2} \in \mathcal{R}$. Proposition 3.12 yields

$$R_1 * \dots * R_{m+1} \in AE(A), \quad R_{m+2} * \dots * R_{m+n+2} \in AE(B). \tag{5.3}$$

It follows from Theorem 2.21 and (5.3) that $(R_1 * \dots * R_{m+1}) * (R_{m+2} * \dots * R_{m+n+2}) \in AE(X)$, i.e. $\mathop{*}\limits_{i=1}^{m+n+2} R_i \in AE(X)$. Consequently, Proposition 3.12 implies that $\mathcal{R}\text{-}\dim X \leq m+n+1$. ■

5.8. Definition. Let A be a subset of a space X. We say that rd-$\mathcal{R}$-dim $A \leq n$ if $\mathcal{R}\text{-}\dim F \leq n$ for every $F \subset A$ and F is closed X.

Propositions 2.22, 3.12, and 4.6 yield

5.9. Dowker's type theorem. *Let F be a closed subset of X such that $\mathcal{R}\text{-}\dim F \leq n$ and rd-$\mathcal{R}$-dim$(X \setminus F) \leq n$. Then $\mathcal{R}\text{-}\dim X \leq n$.*

Theorem 2.24 and Propositions 3.12 and 4.6 imply

5.10. Subspase theorem. *If X is strongly hereditarily normal, then $\mathcal{R}\text{-}\dim A \leq \mathcal{R}\text{-}\dim X$ for any $A \subset X$.*

Theorem 2.6 and Proposition 3.12 yield

5.11. Theorem. $\mathcal{R}\text{-}\dim X = \mathcal{R}\text{-}\dim \beta X$. ■

From Theorem 2.25, Corollary 3.13, and Theorem 5.11 we get

5.12. Theorem [17]. *Let λ be an infinite cardinal, number, n be a non-negative integer, and let R be an ANR-compactum. Then there is a compact Hausdorff space $\Pi_\lambda^{R,n}$ such that $w\Pi_\lambda^{R,n} = \lambda$, $R\text{-}\dim \Pi_\lambda^{R,n} = n$, and $\Pi_\lambda^{R,n}$ contains topologically every space X with $wX \leq \lambda$ and $R\text{-}\dim X \leq n$.*

An immediate corollary of Theorem 5.12 is

5.13. Theorem. *For every space X with $R\text{-}\dim X \leq n$ there exists a compactification bX such that $wbX = wX$ and $R\text{-}\dim bX \leq n$.* ■

5.14. Remark. For $\lambda = \omega_0$ Theorems 5.12 and 5.13 were proved by J. Dydak [8].

Theorem 2.26 and Propositions 3.12 and 4.6 imply

5.15. Decomposition theorem. *Let X be a metrizable space such that $R\text{-}\dim X \leq m+n+1$. Then X can be represented as the union $X = A \cup B$ so that $R\text{-}\dim A \leq m$ and $R\text{-}\dim B \leq n$.* ■

5.16. Corollary. *Let X be a metrizable space with R-$\dim X \le n$. Then X can be represented as the union $X = X_1 \cup \ldots \cup X_{n+1}$ so that R-$\dim X_i \le 0$, $i = 1, \ldots, n+1$.* ■

Theorem 2.27 and Propositions 3.12 and 4.6 yield

5.17. Theorem. *Let R be an ANR-compactum, λ be an infinite cardinal number, n be a non-negative integer. Then there exists a completely metrizable space $M_\lambda^{R,n}$ such that $wM_\lambda^{R,n} = \lambda$, R-$\dim M_\lambda^{R,n} = n$, and $M_\lambda^{R,n}$ contains topologically every metrizable space X^λ with $wX \le \lambda$ and R-$\dim X \le n$.* ■

As a corollary we get

5.18. Completion theorem. *Let X be a metrizable space with R-$\dim X \le n$. Then there is a completely metrizable space $\tilde{X}$ containing X with R-$\dim \tilde{X} \le n$.* ■

Theorem 2.7 and Proposition 3.12 imply

5.19. The first inverse system theorem. *Let X be the limit space of an inverse system $\{X_\alpha, \pi_\beta^\alpha, A\}$ of compact Hausdorff spaces X_α such that $\mathcal{R}$-$\dim X_\alpha \le n$. Then $\mathcal{R}$-$\dim X \le n$.* ■

Theorem 2.8 and Corollary 3.13 yield

5.20. The second inverse system theorem. *Let X be a compact Hausdorff space such that R-$\dim X \le n$. Then X is the limit space of a σ-spectrum $S = \{X_\alpha, \pi_\beta^\alpha, A\}$ such that R-$\dim X_\alpha \le n$ for every $\alpha \in A$.* ■

Theorem 5.20 and Shchepin's spectral theorem [21] imply

5.21. The third inverse system theorem. *Let X be the limit space of a σ-spectrum $S = \{X_\alpha, \pi_\beta^\alpha, A\}$ such that R-$\dim X_\alpha \ge n$ for every $\alpha \in A$. Then R-$\dim X \ge n$.* ■

6. Comparison of dimensions

6.1. Definition. Let $\mathcal{R}_1$, $\mathcal{R}_2$ be classes of ANR-compacta. We say that $\mathcal{R}_1 \le \mathcal{R}_2$ if for every $R_2 \in \mathcal{R}_2$ there is $R_1 \in \mathcal{R}_1$ such that $R_1 \le R_2$.

6.2. Proposition. *If $\mathcal{R}_1 \le \mathcal{R}_2$, then $\mathcal{R}_2$-$\dim X \le \mathcal{R}_1$-$\dim X$ for every space X.*

Proof. Let $\mathcal{R}_1$-$\dim X \le n$ and $R_1^2, \ldots, R_{n+1}^2 \in \mathcal{R}_2$. Since $\mathcal{R}_1 \le \mathcal{R}_2$, there are $R_i^1 \in \mathcal{R}_1$, $i = 1, \ldots, n+1$, such that $R_i^1 \le R_i^2$. According to Proposition 2.28 we have

$$R_1^1 * \ldots * R_{n+1}^1 \le R_1^2 * \ldots * R_{n+1}^2. \tag{6.1}$$

From $\mathcal{R}_1$-$\dim X \le n$ and Proposition 3.12 we get

$$R_1^1 * \ldots * R_{n+1}^1 \in AE(X). \tag{6.2}$$

Conditions (6.1) and (6.2) yield the condition $R_1^2 * \ldots * R_{n+1}^2 \in AE(X)$. Hence $\mathcal{R}_2$-$\dim X \le n$ in view of Proposition 3.12. ■

6.3. Theorem. *For an arbitrary class $\mathcal{R}$ and for every space X we have*

$$\mathcal{R}\text{-}\dim X \le \dim X. \tag{6.3}$$

Proof. In [4] it was noticed that $\{0,1\} \leq R$ for every ANE-space R. Hence $\{0,1\} \leq R$ for every ANR-compactum R by Theorem 2.3. Consequently $\{0,1\} \leq \mathcal{R}$. Applying Theorem 3.10 and Proposition 6.2 we complete the proof. ∎

Proposition 3.12 (or 6.2) yields

6.4. Proposition. *If* $\mathcal{R}_1 \subset \mathcal{R}_2$, *then* $\mathcal{R}_1$-$\dim X \leq \mathcal{R}_2$-$\dim X$. ∎

In connection with inequality (6.3) two problems arise.

Problem 1. When

$$\mathcal{R}\text{-}\dim X = \dim X \tag{6.4}$$

for every space X?

Problem 2. When

$$\mathcal{R}\text{-}\dim X < \infty \Rightarrow \dim X < \infty \tag{6.5}$$

for every space?

We start with the first problem.

6.5. Theorem. *Equality* (6.4) *holds for every space* X *if and only if* $\mathcal{R}$ *contains a disconnected* ANR*-compactum* R.

Proof. $\Rightarrow$. Our condition implies

$$\mathcal{R}\text{-}\dim X \leq 0 \Rightarrow \dim X \leq 0. \tag{6.6}$$

Assume that all $R \in \mathcal{R}$ are connected. Take an arbitrary metric space X with $\dim X = 1$. Then $\mathcal{R} \in AE(X)$ by Kuratowski-Dugundji theorem (see [1], Theorem 9.1). So $\mathcal{R}$-$\dim X \leq 0 < 1 = \dim X$. This contradicts to (6.6). Thus the implication $\Rightarrow$ is checked.

$\Leftarrow$. Let $\mathcal{R}$ contains a disconnected ANR-compactum R. Then $\{0,1\} = S^0 \leq_h R$. Let X be an arbitrary space. We have

$$\dim X = S^0\text{-}\dim X \leq \text{(by Proposition 4.4)} \leq R\text{-}\dim X \leq \text{(in accordance with Proposition 6.4)} \leq \mathcal{R}\text{-}\dim X \leq \text{(in view of Theorem 6.3)} \leq \dim X.$$

Thus $\mathcal{R}$-$\dim X = \dim X$. ∎

As for the second problem, it reduces to the zero-dimensional case.

6.6. Theorem. *The condition*

$$\mathcal{R}\text{-}\dim X < \infty \Rightarrow \dim X < \infty \tag{6.7}$$

holds if and only if

$$\mathcal{R}\text{-}\dim X \leq 0 \Rightarrow \dim X < \infty \tag{6.8}$$

for every space X.

Proof. It suffices to check that (6.8) $\Rightarrow$ (6.7). Moreover, in view of Proposition 6.4 we can assume that $\mathcal{R} = R$. Let R-$\dim X \leq n$. We start with compact metrizable spaces X. According to Corollary 5.16, $X = X_1 \cup \ldots \cup X_{n+1}$, where R-$\dim X_i \leq 0$. In view of (6.8) $\dim X_i < \infty$ and, consequently, $\dim X < \infty$.

Now let X be a compact Hausdorff space. By Theorem 5.20 X is the limit space of a σ-spectrum $S = \{X_\alpha, \pi^\alpha_\beta, A\}$ such that $R\text{-}\dim X_\alpha \leq n$ for all $\alpha \in A$. Consequently, condition (6.8) for compacta implies that $\dim X_\alpha < \infty$. Let $B_m = \{\alpha \in A : \dim X_\alpha \leq m\}$. Clearly,

$$A = \bigcup\{B_m : m \in \omega\}. \tag{6.9}$$

Since A is ω-complete, (6.9) implies that B_m is cofinal for some $m \in \omega$. Hence $X = \lim(S|B_m)$ and $\dim X \leq m$. Thus implication (6.8) $\Rightarrow$ (6.7) is proved for compact Hausdorff space X.

Now let condition (6.8) holds for every space X and let Y be a space with R-$\dim Y < \infty$. By Theorem 5.11 we have R-$\dim \beta Y < \infty$. Then condition (6.8) for compact spaces implies that $\dim \beta Y < \infty$. The equality $\dim Y = \dim \beta Y$ completes the proof. ■

6.7. Definition. An ANR-compactum R is said to be an *extensionally finite-dimensional compactum* or *efd-compactum* (notation: $R \in efd\text{-}C$) if

$$R\text{-}\dim X \leq 0 \Rightarrow \dim X < \infty \tag{6.10}$$

for every space X.

From the proof of Theorem 6.6 we get

6.8. Proposition. *If* (6.10) *holds for every compactum X, then $R \in efd\text{-}C$.* ■

Theorem 2.29 implies

6.9. Theorem. *Let $H_*(R, \mathbb{Q}) = 0$. Then $R \notin efd\text{-}C$.* ■

6.10. Corollary. *All Moore complexes $M(\mathbb{Z}_p, n)$, in particular the real projective plane $\mathbb{R}P^2$, are not efd-compacta.* ■

6.11. Hypothesis. *If $H_*(R, \mathbb{Q}) \neq 0$, then $R \in efd\text{-}C$.*

Theorem B yields

6.12. Proposition. *$S^n \in efd\text{-}C$ for all $n \geq 0$.* ■

Proposition 6.2 implies

6.13. Proposition. *If $R_1 \leq R_2$ and $R_2 \in efd\text{-}C$, then $R_1 \in efd\text{-}C$.* ■

From Proposition 4.4 we get

6.14. Proposition. *If $R_1 \leq_h R_2$ and $R_1 \in efd\text{-}C$, then $R_2 \in efd\text{-}C$.* ■

6.15. Proposition. *If S is a classical compact surface, then $S \in efd\text{-}C \iff S \neq \mathbb{R}P^2$.*

Proof. Corollary 6.10 yields the implication $\Rightarrow$. On the other hand, it is well known that if $S \neq \mathbb{R}P^2$, then $S^1 \leq S$. So applying Propositions 6.12 and 6.14 we complete the proof. ■

An immediate corollary of Proposition 6.14 is

6.16. Proposition. *If $R \in efd\text{-}C$, then $R \times S \in efd\text{-}C$, $R \vee S \in efd\text{-}C$ for an arbitrary ANR-compactum S.* ■

6.17. Proposition. *For an arbitrary ANR-compactum R the following conditions are equivalent*:

1) $R \in efd\text{-}C$; 2) $R \vee R \in efd\text{-}C$; 3) $R * R \in efd\text{-}C$.

Proof. According to Propositions 2.35 and 2.37 we have $R \leq R \vee R \leq R * R$. Consequently, Proposition 6.15 implies that 3) $\Rightarrow$ 2) $\Rightarrow$ 1). It remains to check the implication 1) $\Rightarrow$ 3). Let $R \in efd\text{-}C$ and let X be a space such that $R*R\text{-}\dim X \leq n$. Hence, $\overset{n+1}{*}(R*R) \in AE(X)$ in view of Corollary 3.13. But $\overset{n+1}{*}(R*R) = \overset{2n+2}{*}R$. Applying Corollary 3.13 once again we get $R\text{-}\dim X \leq 2n+1$. Thus $\dim X < \infty$, because $R \in efd\text{-}C$. ■

A partial case of Hypothesis 6.11 is

6.18. Question. Let M be an orientable closed manifold. Is it true that $M \in efd\text{-}C$?

7. Dimension of products

7.1. Theorem. *Let X and Y be finite-dimensional metrizable spaces. Then*

$$R\text{-}\dim(X \times Y) \leq R\text{-}\dim X + R\text{-}\dim Y + 1. \tag{7.1}$$

To prove this theorem we need an auxiliary information.

7.2. Definition [11]. A mapping $f\colon X \to Y$ from a metric space X to a space Y is said to be *strongly 0-dimensional* if for every $\epsilon > 0$ and every $y \in f(X)$ there exists an open neighbourhood V of y such that $f^{-1}V$ splits into the union of disjoint open sets of diam $< \epsilon$.

The next statement is rather obvious.

7.3. Lemma. *Let $f_i\colon X_i \to Y_i$ be strongly 0-dimensional mappings of metric spaces $X_i = (X_i, \rho_i)$, $i = 1, 2$. Then the mapping $f = f_1 \times f_2\colon X_1 \times X_2 \to Y_1 \times Y_2$ is strongly 0-dimensional with respect to the metric ρ in $X_1 \times X_2$ which is the l_2-product of the metrics ρ_1 and ρ_2, i.e. $\rho^2((x_1^1, x_2^1), (x_1^2, x_2^2)) = \rho_1^2(x_1^1, x_1^2) + \rho_2^2(x_2^1, x_2^2)$.* ■

7.4. Theorem [16]. *Let K be a countable CW-complex and let X be a metric space. Then $e\text{-}\dim X \leq K$ if and only if there exists a strongly 0-dimensional mapping $f\colon X \to Y$ to a separable metrizable space Y of $e\text{-}\dim Y \leq K$.* ■

Proof of Theorem 7.1. Assume that X and Y are compact spaces with $R\text{-}\dim X = m$, $R\text{-}\dim Y = n$. If R is disconnected, then $R\text{-}\dim = \dim$ by Theorem 6.7. Hence inequality (7.1) is a corollary of the logarithmic law

$$\dim(X \times Y) \leq \dim X + \dim Y$$

proved by M. Katětov [11] and K. Morita [14]. If R is connected, then in view of Corollary 3.13 we have $\overset{m+1}{*}R \in AE(X)$, $\overset{n+1}{*}R \in AE(Y)$.

Put $R_1 = (\overset{m+1}{*}R) \wedge (\overset{n+1}{*}R)$. Theorem 2.38 implies that $R_1 \in AE(X \times Y)$. Hence, in view of Proposition 2.37, we have $\overset{m+n+2}{*}R \in AE(X \times Y)$. Applying Corollary 3.13 once again we complete the proof.

Consider now a general case. We may again assume that R-$\dim X = m$, R-$\dim Y = n$; $m, n < \infty$. Fix metrics in X and Y. According to Theorem 7.4 and Proposition 3.15, 4.5 there exist strongly 0-dimensional mappings $f : X \to X_0$ and $g : Y \to Y_0$ to separable metrizable spaces X_0 and Y_0 of R-$\dim X_0 \leq m$ and R-$\dim Y_0 \leq n$. In view of Theorem 5.13 there exist metrizable compactifications bX_0 and bY_0 of R-$\dim bX_0 \leq m$ and R-$\dim bY_0 \leq n$. In accordance with (7.1) for compact spaces we have

$$R\text{-}\dim(bX_0 \times bY_0) \leq m + n + 1. \tag{7.2}$$

Theorem 5.10 and (7.2) yield

$$R\text{-}\dim X_0 \times Y_0 \leq m + n + 1. \tag{7.3}$$

Then

$$R\text{-}\dim X \times Y \leq m + n + 1$$

because of Theorem 7.4, Propositions 3.15, 4.5, Lemma 7.3, and condition (7.3). ■

7.5. Remark. Inequality (7.1) is not improvable. In fact, S^1-$\dim X = 0$ for every one-dimensional compactum X. But S^1-$\dim(X \times X) = 1 = 0 + 0 + 1$.

Theorems 5.19, 5.20, and 7.1 imply

7.6. Theorem. *Let X and Y be finite-dimensional compact Hausdorff spaces. Then R-$\dim(X \times Y) \leq R$-$\dim X + R$-$\dim Y + 1$.* ■

7.7. Proposition. *If X is a metrizable space of finite dimension, then*

$$R\text{-}\dim(X \times I) \leq R\text{-}\dim X + 1 \tag{7.4}$$

for an arbitrary ANR-compactum R.

Proof. If R is disconnected then R-dim $=$ dim according to Theorem 6.7. Thus (7.4) is a usual inequality of the Lebesgue dimension. If R is connected, then $R \in AE(I)$, i.e. R-$\dim I = 0$. Applying Theorem 7.1 we complete the proof. ■

Acknowledgements. The author would like to thank the referee for helpful comments and suggestions which led to many improvements of the original version of this article.

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(received 20.7.2008; in revised form 5.1.2009)

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