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ASYMPTOTIC BEHAVIOUR OF DIFFERENTIATED BERNSTEIN POLYNOMIALS

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Abstract. In the present note we give a full quantitative version of a theorem of Floater dealing with the asymptotic behaviour of differentiated Bernstein polynomials. While Floater's result is a generalization of the classical Voronovskaya theorem, ours generalizes a hardly known quantitative version of this theorem due to Videnskiĭ, among others.

1. Introduction

In a recent article Floater [2] proved the following

THEOREM 1. *If $f \in C^{k+2}[0,1]$ for some $k \geq 0$, then*

$$\lim_{n\to\infty} n\left\{(B_n f)^{(k)}(x) - f^{(k)}(x)\right\} = \frac{1}{2}\cdot\frac{d^k}{dx^k}\{x(1-x)f''(x)\},$$

uniformly for $x \in [0,1]$.

Here B_n is the Bernstein operator defined for a function $f\colon [0,1] \to \mathbf{R}$ and $x \in [0,1]$ by

$$B_n f(x) = \sum_{i=0}^{n} f\left(\frac{i}{n}\right) p_{n,i}(x),$$

where

$$p_{n,i}(x) = \binom{n}{i} x^i (1-x)^{n-i}, \ \ i = 0, \dots, n.$$

In the sequel we will also use the abbreviations $X = x(1-x)$ and $e_i(x) = x^i$ for $i = 0, 1, 2, \dots$. Floater's result is a generalization of the classical Voronovskaya theorem (see [10]) which is obtained for $k = 0$. In a recent paper [3] the latter

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theorem was given in quantitative form as follows, improving an earlier estimate by Videnskiĭ (see [9]).

Theorem 2. *For $f \in C^2[0,1]$, $x \in [0,1]$ and $n \in \mathbb{N}$ one has*

$$\left| n \cdot [B_n(f;x) - f(x)] - \frac{x(1-x)}{2} \cdot f''(x) \right| \leq \frac{x(1-x)}{2} \cdot \tilde{\omega}\left(f''; \sqrt{\frac{1}{n^2} + \frac{x(1-x)}{n}} \right).$$

Here $\tilde{\omega}$ is the least concave majorant of ω, the first order modulus of continuity, satisfying

$$\omega(f;\epsilon) \leq \tilde{\omega}(f;\epsilon) \leq 2\omega(f;\epsilon), \qquad \epsilon \geq 0.$$

The above inequality follows from a more general asymptotic statement which was inspired by results of Bernstein [1] and Mamedov [7]. This is given in

Theorem 3. *Let $q \in \mathbb{N}_0$, $f \in C^q[0,1]$ and $L\colon C[0,1] \to C[0,1]$ be a positive linear operator. Then*

$$\left| L(f;x) - \sum_{r=0}^{q} L((e_1 - x)^r; x) \cdot \frac{f^{(r)}(x)}{r!} \right|$$
$$\leq \frac{L(|e_1 - x|^q; x)}{q!} \tilde{\omega}\left(f^{(q)}; \frac{L(|e_1 - x|^{q+1}; x)}{(q+1)L(|e_1 - x|^q; x)} \right).$$

It is the aim of this note to prove a quantitative version of Floater's result. In doing so we will make essential use of a corollary of Theorem 3 for the case $q = 2$.

Corollary 1. *Under the assumptions of Theorem 3 one has, for $q = 2$, the inequality*

$$\left| L(f;x) - f(x) \cdot L(e_0;x) - f'(x) \cdot L(e_1 - x; x) - \frac{1}{2} f''(x) \cdot L((e_1 - x)^2; x) \right|$$
$$\leq \frac{1}{2} \cdot L((e_1 - x)^2; x) \cdot \tilde{\omega}\left(f''; \frac{1}{3} \cdot \sqrt{\frac{L((e_1 - x)^4; x)}{L((e_1 - x)^2; x)}} \right).$$

The square root is obtained by using the Cauchy-Schwarz inequality for positive linear functionals.

2. An auxiliary result

An operator $L\colon C[0,1] \to C^k[0,1]$ is said to be convex of order $k-1$ if it preserves convexity of order $k-1$, $k \in \mathbb{N} \cup \{0\}$. This means that any function f with divided differences

$$[x_0, \ldots, x_k; f] \geq 0 \text{ for any } x_0 < \cdots < x_k \in [0,1]$$

is mapped to a function Lf having the same property.

The Bernstein operator is an example of a mapping which is convex of all orders $k \in \mathbf{N} \cup \{0\}$.

For an operator L being convex of order $k-1$ and satisfying $L(\Pi_{k-1}) \subseteq \Pi_{k-1}$ consider

$$I_k : C[0,1] \to C[0,1] \quad \text{given by} \quad (I_k f)(x) = \int_0^x \frac{(x-t)^{k-1}}{(k-1)!} f(t)\,dt.$$

Let $Q^k := D^k \circ L \circ I_k$ where $D^k = \frac{d^k}{dx^k}$.

Q^k may be considered as a k-th order Kantorovich modification of L. Since L is convex of order $k-1$, it follows that Q^k is a linear and positive (convex of order -1) operator. Since $I_k D^k f - f \in \Pi_{k-1}$ and $L(\Pi_{k-1}) \subseteq \Pi_{k-1}$, we have $L(I_k D^k f - f) \in \Pi_{k-1}$. It follows $D^k L I_k D^k f = D^k L f$, hence $Q^k D^k f = D^k L f$, for all $f \in C^k[0,1]$.

To our knowledge the latter construction is due to Sendov and Popov [8].

3. Main result

In this section we will prove the main result of this note by providing the following quantitative version of Floater's convergence result.

THEOREM 4. *If $f \in C^{k+2}[0,1]$ for some $k \geq 0$, then*

$$\left| n[(B_n f)^{(k)}(x) - f^{(k)}(x)] - \frac{1}{2} \cdot \frac{d^k}{dx^k}\{x(1-x)f''(x)\} \right| \\ \leq O\left(\frac{1}{n}\right) \cdot \max_{k \leq \kappa \leq k+2} \{|f^{(\kappa)}(x)|\} + O(1) \cdot \tilde{\omega}\left(f^{(k+2)}; \frac{1}{\sqrt{n}}\right).$$

Here $O\left(\frac{1}{n}\right)$ and $O(1)$ represent sequences of order $O\left(\frac{1}{n}\right)$ and $O(1)$, respectively, which depend on the fixed k.

Proof. Put $Q_n^k := D^k B_n I_k$. For this positive linear operator we apply Corollary 1 and write the left hand side of the inequality for $f \in C^{k+2}[0,1]$ as

$$\begin{aligned} &|Q_n^k(f^{(k)}; x) - f^{(k)}(x) \cdot Q_n^k(e_0; x) - f^{(k+1)}(x) \cdot Q_n^k(e_1 - x; x) \\ &\qquad - \frac{1}{2} \cdot f^{(k+2)}(x) \cdot Q_n^k((e_1 - x)^2; x)| \\ &= |D^k B_n(f; x) - f^{(k)}(x) + f^{(k)}(x)(1 - Q_n^k(e_0; x)) - f^{(k+1)}(x) \cdot Q_n^k(e_1 - x; x) \\ &\quad - \frac{1}{2} f^{(k+2)}(x) \cdot Q_n^k((e_1 - x)^2; x) - \frac{1}{2n} \cdot \frac{d^k}{dx^k}\{x(1-x) \cdot f''(x)\} \\ &\quad + \frac{1}{2n} \cdot \frac{d^k}{dx^k}\{x(1-x) \cdot f''(x)\}| \\ &= |D^k B_n(f; x) - f^{(k)}(x) - \frac{1}{2n} \cdot \frac{d^k}{dx^k}\{x(1-x) f''(x)\} \\ &\quad - \{(Q_n^k(e_0; x) - 1) \cdot f^{(k)}(x) + f^{(k+1)}(x) \cdot Q_n^k(e_1 - x; x) \\ &\quad + \frac{1}{2} f^{(k+2)}(x) \cdot Q_n^k((e_1 - x)^2; x) - \frac{1}{2n} \cdot \frac{d^k}{dx^k}\{x(1-x) \cdot f''(x)\}|. \end{aligned}$$

Multiplying the inequality of Corollary 1 by n and using the (second) triangular inequality yields

$$\begin{aligned}
&\left| n \cdot \{D^k B_n(f;x) - f^{(k)}(x)\} - \frac{1}{2} \cdot \frac{d^k}{dx^k}\{x(1-x)\cdot f''(x)\}\right| \\
&\quad \le \Big| n \cdot (Q_n^k(e_0;x) - 1)\cdot f^{(k)}(x) + f^{(k+1)}(x)\cdot n \cdot Q_n^k(e_1 - x;x) \\
&\qquad + \frac{1}{2} f^{(k+2)}(x)\cdot n \cdot Q_n^k((e_1-x)^2;x) - \frac{1}{2}\cdot\frac{d^k}{dx^k}\{x(1-x)\cdot f''(x)\}\Big| \\
&\qquad + \frac{n}{2}\cdot Q_n^k((e_1-x)^2;x)\cdot \tilde{\omega}\left(f^{(k+2)}; \frac{1}{3}\sqrt{\frac{Q_n^k((e_1-x)^4;x)}{Q_n^k((e_1-x)^2;x)}}\right).
\end{aligned}$$

Both summands of the r.h.s. will now be inspected separately. In order to do so, first observe that by Leibniz' rule one has

$$\frac{1}{2}\cdot\frac{d^k}{dx^k}\{Xf''(x)\} = \frac{1}{2}\left\{f^{(k+2)}(x)\cdot X + k\cdot f^{(k+1)}(x)\cdot X' + \frac{k(k-1)}{2} f^{(k)}(x)(-2)\right\}.$$

Note that this is correct also for $k \in \{0,1\}$. So the first summand can be estimated from above by

$$\begin{aligned}
&\left| n\cdot (Q_n^k(e_0;x) - 1) + \frac{k(k-1)}{2}\right| \cdot |f^{(k)}(x)| + \left| n \cdot Q_n^k(e_1 - x;x) - \frac{k}{2}X'\right| \cdot |f^{(k+1)}(x)| \\
&\qquad + \left|\frac{n}{2}\cdot Q_n^k((e_1-x)^2;x) - \frac{1}{2}X\right| \cdot |f^{(k+2)}(x)| \\
&\qquad := A_n^k \cdot |f^{(k)}(x)| + B_n^k \cdot |f^{(k+1)}(x)| + C_n^k \cdot |f^{(k+2)}(x)|.
\end{aligned}$$

Now for $n \ge 1$ and $k \ge 0$, also noting that $(n)_0 = 1$, we have

$$\begin{aligned}
A_n^k &= \left| n \cdot \left(\frac{(n)_k}{n^k} - 1\right) + \frac{k(k-1)}{2}\right| \\
&= \left| n \cdot \frac{\overbrace{n(n-1)\dots(n-k+1)}^{k \text{ terms}} - n^k}{n^k} + \frac{k(k-1)}{2}\right| \\
&= \left| \frac{(n)_k - n^k}{n^{k-1}} + \frac{k(k-1)}{2}\right| \le \left| -\frac{k(k-1)}{2} + \frac{k(k-1)}{2}\right| + O\left(\frac{1}{n}\right).
\end{aligned}$$

In order to verify the last inequality it is only necessary to observe that

$$\frac{1}{n^{k-1}}\{(n)_k - n^k\} = \frac{1}{n^{k-1}}\left\{n^k - \frac{k(k-1)}{2} n^{k-1} + \text{ (lower order terms in } n) - n^k\right\}.$$

Note that $A_n^k = 0$ for $k \in \{0,1\}$.

Moreover (see [5], pp. 44–45), for $n \geq 1$ and $k \geq 0$ we get

$$B_n^k = \left| n \cdot Q_n^k(e_1 - x; x) - \frac{k}{2} \cdot X' \right| \left| n \cdot \frac{(n)_k}{n^k} \cdot \frac{k}{2n} X' - \frac{k}{2} \cdot X' \right|$$
$$= \frac{k}{2}|X'| \cdot \left| \frac{(n)_k}{n^k} - 1 \right| \leq \frac{k}{2}|X'| \cdot \frac{k(k-1)}{2k} = O\left(\frac{1}{n}\right).$$

We also have $B_n^k = 0$ for $k \in \{0, 1\}$. For the last factor we have (see [6], p. 26) for $n \geq k + 2$

$$C_n^k = \left| \frac{n}{2} \cdot Q_n^k((e_1 - x)^2; x) - \frac{1}{2} X \right|$$
$$= \left| \frac{n}{2} \cdot \frac{(n)_k}{n^{k+2}} \left[(n - k(k+1)) \cdot X + \frac{1}{12} k(3k+1) \right] - \frac{1}{2} X \right|$$
$$= \left| \frac{1}{2} X \left\{ \frac{(n)_k}{n^{k+1}} (n - k(k+1)) - 1 \right\} + \frac{1}{24} \cdot \frac{(n)_k}{n^{k+1}} k(3k+1) \right|$$
$$\leq \frac{1}{2} X \left| \frac{(n)_k}{n^{k+1}} (n - k(k+1)) - 1 \right| + O\left(\frac{1}{n}\right).$$

It remains to consider the quantity inside $|\dots|$. The latter is equal to

$$\frac{1}{n^{k+1}} \left(n^k - \frac{k(k-1)}{2} n^{k-1} + O(n^{k-2})(n - k(k+1)) \right) - 1$$
$$= 1 - \frac{k(k-1)}{2n} + O\left(\frac{1}{n^2}\right) - O\left(\frac{1}{n}\right) + O\left(\frac{1}{n^2}\right) - O\left(\frac{1}{n^3}\right) - 1 = O\left(\frac{1}{n}\right).$$

Note that $C_n^k = 0$ for $k = 0$. So we know that

$$\left| n \cdot \{D^k B_n(f; x) - f^{(k)}(x)\} - \frac{1}{2} \cdot \frac{d^k}{dx^k} \{x(1-x) \cdot f''(x)\} \right|$$
$$\leq O\left(\frac{1}{n}\right) \cdot \max\{|f^{(k)}(x)|, |f^{(k+1)}(x)|, |f^{(k+2)}(x)|\}$$
$$+ \frac{n}{2} \cdot Q_n^k((e_1 - x)^2; x) \cdot \tilde{\omega} \left(f^{(k+2)}; \frac{1}{3} \sqrt{\frac{Q_n^k((e_1 - x)^4; x)}{Q_n^k((e_1 - x)^2; x)}} \right),$$

where O depends only on k.

For the factor in front of $\tilde{\omega}(f^{(k+2)}; \dots)$ we have already observed that

$$\frac{n}{2} \cdot Q_n^k((e_1 - x)^2; x) = \frac{n}{2} \cdot \frac{(n)_k}{n^{k+2}} \left[(n - k(k+1)) \cdot X + \frac{1}{12} k(3k+1) \right] = O(1).$$

Hence it remains to consider the square root in $\tilde{\omega}(f^{(k+2)}; \dots)$. This is done in the following lemmas dealing with the moments of Q_n^k. ■

Lemma 1. *Suppose that $L_n \colon \Pi \to \Pi$, $n \geq 1$, is a linear operator mapping polynomials into polynomials and such that $L_n(\Pi_j) \subset \Pi_j$, $n \geq 1, j \geq 0$. If we* *define*

$$M_{n,m}(x) := \frac{1}{m!} L_n((e_1 - x)^m; x), n \geq 1, m \geq 0,$$
$$R_{n,p}^k(x) := \frac{1}{p!} Q_n^k((e_1 - x)^p; x), n \geq 1, k \geq 0, p \geq 0,$$

then

$$R_{n,p}^k(x) = \sum_{j=p}^{p+k} \binom{k}{p+k-j} \cdot M_{n,j}^{(j-p)}(x).$$

Proof. We will use the notation ${}^{(k)}f = I_k f$ to denote a k-th antiderivative of $f, f \in C[0,1]$.

First observe that

$$M_{n,m} \in \Pi_m \ \text{ for } \ n \geq 1 \ \text{ and } \ m \geq 0.$$

Now let $k \geq 0$, $p \geq 0$ be fixed, $f \in \Pi_p$, $x \in [0,1]$. Then

$${}^{(k)}f(t) = \sum_{j=0}^{p+k} ({}^{(k)}f)^{(j)}(x) \frac{1}{j!} (t-x)^j,$$

and hence

$$L_n({}^{(k)}f)(x) = \sum_{j=0}^{p+k} ({}^{(k)}f)^{(j)}(x) M_{n,j}(x).$$

Thus

$$Q_n^k f = \sum_{j=0}^{p+k} (({}^{(k)}f)^{(j)} M_{n,j})^{(k)} = \sum_{j=0}^{p+k} \sum_{i=0}^{k} \binom{k}{i} ({}^{(k)}f)^{(j+i)} \cdot M_{n,j}^{(k-i)}.$$

Noting that $M_{n,j}^{(k-i)} \equiv 0$ if $0 \leq i < k-j$, we may write

$$Q_n^k f = \sum_{j=0}^{p+k} \sum_{i=\max\{0,k-j\}}^{k} \binom{k}{i} f^{(i+j-k)} M_{n,j}^{(k-i)}.$$

Substituting $i + j - j = \ell$, $i = \ell + k - j$, the latter becomes

$$= \sum_{j=0}^{p+k} \sum_{\ell=\max\{j-k,0\}}^{j} \binom{k}{\ell+k-j} f^{(\ell)} M_{n,j}^{(j-\ell)}$$
$$= \sum_{\ell=0}^{p+k} \sum_{j=\ell}^{\min\{\ell+k,p+k\}} \binom{k}{\ell+k-j} f^{(\ell)} M_{n,j}^{(j-\ell)}, f \in \Pi_p.$$

This is correct in view of $(a_{j,\ell} \in \mathbb{R})$

$$\sum_{j=0}^{p+k} \sum_{\ell=\max\{j-k,0\}}^{j} a_{j,\ell} = \sum_{\ell=0}^{p+k} \left\{ \begin{array}{ll} \sum_{j=\ell}^{k+\ell} a_{j,\ell}, & \text{if } \ k+\ell \leq k+p, \\ \sum_{j=\ell}^{k+p} a_{j,\ell}, & \text{if } \ k+\ell > k+p. \end{array} \right\}$$
$$= \sum_{\ell=0}^{p+k} \sum_{j=\ell}^{\min\{\ell+k,p+k\}} a_{j,\ell}.$$

Note that the l.h.s. corresponds to "horizontal summation first, then vertical", while the r.h.s. corresponds to the opposite.

We obtain $R^k_{n,p}(x)$ if we put $f = \frac{1}{p!}(e_1 - x)^p$ in $Q^k_n f$, also observing that $f^{(\ell)}(x) = 0$ for $\ell \in \{0, \dots, p+k\} \setminus \{p\}$ and $f^{(p)}(x) = 1$. Hence

$$R^k_{n,p}(x) = \sum_{j=p}^{p+k} \binom{k}{p+k-j} M^{(j-p)}_{n,j}(x), \quad n \geq 1, k \geq 0, p \geq 0. \quad \blacksquare$$

In order to come up with a description of the asymptotic behavior of the ratio in question, we investigate the quantities $R^k_{n,p}(x)$ further in case that $L_n = B_n$. We have the following

Lemma 2. *For the Bernstein operators B_n we have*

$$n \frac{R^k_{n,4}}{R^k_{n,2}} \leq A, \quad n \geq 1,$$

for some positive constant A.

Proof. For B_n we have

$$M_{n,2j}(x) = \frac{X}{n^{2j-1}} \sum_{i=0}^{j-1} a_{ji}(n) X^i,$$
$$M_{n,2j+1}(x) = \frac{XX'}{n^{2j}} \sum_{i=0}^{j-1} b_{ji}(n) X^i,$$

where $a_{ji}(n)$ and $b_{ji}(n)$ are polynomials in n, of degree i.

By the previous lemma,

$$R^k_{n,4} = \sum_{j=4}^{k+4} \binom{k}{k+4-j} M^{(j-4)}_{n,j} = \frac{1}{n^2}(X^2 u_{k+1}(n) + X v_k(n) + w_{k-1}(n)),$$

where u_{k+1}, v_k and w_{k-1} are polynomials of degrees indicated by the corresponding indices. Analogously,

$$R^k_{n,2} = \frac{1}{n}(X q_{k+1}(n) + r_k(n)).$$

Now the claim of the lemma is a consequence of the latter two representations. $\blacksquare$

Continuation of the proof of Theorem 4. All we have to observe is that

$$\sqrt{\frac{Q^k_n((e_1 - x)^4; x}{Q^k_n((e_1 - x)^2; x)}} = \sqrt{\frac{4! \cdot R^k_{n,4}(x)}{2! \cdot R^k_{n,2}(x)}} \leq \sqrt{6A \cdot \frac{1}{n}},$$

where A is the uniform bound from Lemma 2. The final statement follows from the inequality

$$\tilde{\omega}(f; c\epsilon) \leq (c+1) \cdot \tilde{\omega}(f; \epsilon), \quad c, \epsilon \geq 0. \quad \blacksquare$$

REMARK 1. We noted earlier that $A_n^k = B_n^k = C_n^k = 0$ for $k = 0$. In this case the inequality of Theorem 4 can be replaced by

$$|n[B_n f(x) - f(x)] - \frac{1}{2}x(1-x)\cdot f''(x)| \le O(1)\cdot \tilde{\omega}\left(f''; \frac{1}{\sqrt{n}}\right).$$

In fact, looking at the proof again shows that we even get the right hand side

$$\frac{n}{2}\cdot B_n((e_1-x)^2;x)\cdot \tilde{\omega}\left(f''; \frac{1}{3}\sqrt{\frac{B_n((e_1-x)^4;x)}{B_n((e_1-x)^2;x)}}\right) \le \frac{x(1-x)}{2}\cdot \tilde{\omega}\left(f''; \frac{1}{3\sqrt{n}}\right).$$

This is the quantitative version of the classical Voronovskaya theorem given first in [4].

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