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A MASS PARTITION PROBLEM IN $\mathbf{R}^4$

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Abstract. The paper considers the existence of the maximal possible hyperplane partition of a continuous probability Borel measure in $\mathbf{R}^4$. The emphases is on the use of the equivariant ideal valued index theory of Fadell and Husseini. The presented result is the tightest positive solution to one of the oldest and most relentless partition problems posed by B. Grünbaum [12].

1. Statement of the main result

A mass/measure partition problem is one of the most interdisciplinary problems in geometric combinatorics with different aspects ranging from convex geometry ([12], [1], [2]), equivariant topology ([3], [4], [5], [14], [15], [9], [6], [8], [19]), to theoretical computer science ([17], [16]). The problem we discuss is the shining beacon of this part of geometric combinatorics. First introduced by B. Grünbaum in 1960, [12], positively answered in dimension $n = 3$ by H. Hadwiger [13] and negatively answered for $n \geq 5$ by D. Avis [2], the problem persisted against all attacks in the dimension 4 ([15], [19]) and remained open.

The general problem considers a mass distribution μ in $\mathbb{R}^n$ and looks for a collection of n-hyperplanes $H_1, \ldots, H_n$ such that each of the 2^n hyper-orthants contains the same amount of measure μ, i. e.

$$(\forall\ (\sigma_1, \ldots, \sigma_n) \in \{0,1\}^n)\ \mu(H_1^{\sigma_1} \cap \cdots \cap H_n^{\sigma_n}) = \tfrac{1}{2^n}\mu(\mathbb{R}^n),$$

where $H_i^{\sigma_i}$ denotes the appropriate closed halfspace determined by H_i.

Here we try to understand how the conditions could be modified, without extra assumptions on the measure, in such a way that instead of a complete equipartition we obtain an almost equipartition.

Theorem 1. *Let μ, ν and η be mass distributions in $\mathbb{R}^4$, assuming that μ, ν, η are finite continuous Borel measures defined by some integrable density functions.*

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Then there exist four different hyperplanes H_1, H_2, H_3, H_4 *and consequently sixteen* 4*-orthants* $H_1^{\sigma_1} \cap H_2^{\sigma_2} \cap H_3^{\sigma_3} \cap H_4^{\sigma_4} = \mathbf{O}_{\sigma_1\sigma_2\sigma_3\sigma_4}$, $(\sigma_1, \dots, \sigma_4) \in \{0,1\}^4$, *such that*

$$\begin{aligned} \mu(\mathbf{O}_{0000}) &= \mu(\mathbf{O}_{0010}) = \mu(\mathbf{O}_{0101}) = \mu(\mathbf{O}_{0111}) \\ &= \mu(\mathbf{O}_{1000}) = \mu(\mathbf{O}_{1010}) = \mu(\mathbf{O}_{1101}) = \mu(\mathbf{O}_{1111}), \qquad (1) \\ \mu(\mathbf{O}_{0001}) &= \mu(\mathbf{O}_{0011}) = \mu(\mathbf{O}_{0100}) = \mu(\mathbf{O}_{0110}) \\ &= \mu(\mathbf{O}_{1001}) = \mu(\mathbf{O}_{1011}) = \mu(\mathbf{O}_{1100}) = \mu(\mathbf{O}_{1110}). \qquad (2) \end{aligned}$$

and the hyperplane H_3 *equiparts the remaining two measures* ν *and* η, *i.e.*

$$\begin{aligned} \sum_{g\in\{0,1\}^2, h\in\{0,1\}} \nu(\mathbf{O}_{g0h}) &= \sum_{g\in\{0,1\}^2, h\in\{0,1\}} \nu(\mathbf{O}_{g1h}), \qquad (3) \\ \sum_{g\in\{0,1\}^2, h\in\{0,1\}} \eta(\mathbf{O}_{g0h}) &= \sum_{g\in\{0,1\}^2, h\in\{0,1\}} \eta(\mathbf{O}_{g1h}). \end{aligned}$$

Remark 2. The result for μ is not a consequence of the fact that in $\mathbb{R}^4$ for every mass there exist three hyperplanes which equipart it. For example, if

(A) $H_1 = H_2$ then $\mu(\mathbf{O}_{01**}) = \mu(\mathbf{O}_{10**}) = 0$; this would imply that all orthants have measure zero providing obvious contradiction;

(B) $H_1 = -H_2$ then $\mu(\mathbf{O}_{00**}) = \mu(\mathbf{O}_{11**}) = 0$; and again all orthants have measure zero providing the same contradiction;

In the similar way the remaining possibilities can be tested.

Remark 3. The result concerning the measure μ is highly relevant to the Grünbaum equipartition problem in dimension 4. Moreover, after it was proved in [19] that a CS/TM scheme fails to provide the existence of an equipartition, this result is the best known approximation (without imposing any additional constrains on the measure μ).

2. History of solution efforts

One of the first attempts of solving similar problems was by E. Ramos [15]. He introduced a more general problem which as a special case contained our problem. Briefly, he wanted to find all triples (d, j, k) such that for every j mass distributions $\mu_1, \dots, \mu_j$ in $\mathbb{R}^d$

$$(\exists\ H_1, \dots, H_k \text{ hyperplanes in } \mathbb{R}^d)(\forall\ (\sigma_1, \dots, \sigma_k) \in \{0,1\}^k) \\ (\forall\ r \in \{1, \dots, j\})\ \mu_r(H_1^{\sigma_1} \cap \dots \cap H_k^{\sigma_k}) = \tfrac{\mu_r(\mathbb{R}^d)}{2^k}.$$

The triples (d, j, k) were traditionally called admissible. As a tool from topology Ramos used a specially modified version of Borsuk-Ulam theorem for "even-odd" maps of the form $f\colon S^{d-1} \times \dots \times S^{d-1} \to \mathbb{R}^n$. The method allowed Ramos to attain very interesting results, for example $(5,1,4)$, $(9,3,3)$, and $(9,5,2)$ were proved to be admissible.

The general problem of Ramos is discussed via CS/TM paradigm. The Configuration Space/Test Map paradigm is a tool for systematic derivation of topological lower bounds for combinatorial problems. The partition problem of Ramos can be reduced to the problem of the existence of a $W_k = (\mathbb{Z}_2)^k \rtimes S_k$ map

$$(S^d)^k \to S(U_k^{\oplus j})$$

where $U_k^{\oplus j}$ is an appropriate W_k-representation.

The paper by R. Živaljević [19] discussed the problem we are interested in. He used the stated reduction to a problem of the existence of a $W_4 = (\mathbb{Z}_2)^4 \rtimes S_4$ map $(S^4)^4 \to S(U_4)$. Using the Koschorke's exact singularity sequence, unfortunately, he proved that a W_4-map $X \to S(U_4)$, for a concrete relevant subspace $X \subset (S^4)^4$, exists. This means that a particular reduction is of no help in solving the partition problem. With an extra assumption of the symmetry on the mass distribution, he obtained the positive answer to the equipartition question.

3. The proof of Theorem 1

The proof of the theorem has two stages. First we use the CS/TM scheme to translate the partition problem to an equivariant one. Second, we use the ideal valued index theory of Fadell-Husseini to solve the associated equivariant problem.

3.1. The CS/TM scheme

The configuration space X. Let X be the space of all collections of four oriented affine hyperplanes in $\mathbb{R}^4$ such that each one equiparts the measure μ. It is not hard to see that for every direction (unit vector) in $\mathbb{R}^4$ there exists a unique oriented affine hyperplane orthogonal to a given direction that equiparts the measure. Therefore, the configuration space is $X = (S^3)^4$.

The test map M. Let us recall that every hyperplane H in $\mathbb{R}^4$ determines two closed halfspaces H^0 and H^1. The orientation of H introduces the order on halfspaces, for example $H^0 < H^1$, such that the change of orientation flips order, $H^1 < H^0$. Therefore, the collection of four oriented hyperplanes H_1, H_2, H_3, H_4 in $\mathbb{R}^4$ defines 16 hyper-orthants. To relax the definition of the test map M, let us assume that coordinates of each copy of $\mathbb{R}^{16}$ are indexed (when it suits us) by the binary words of length four or by the elements of the group $\mathbb{Z}_2^4$. The test map $M : (S^3)^4 \to \mathbb{R}^{16} \oplus \mathbb{R}^{16} \oplus \mathbb{R}^{16}$ is defined by

$$\begin{aligned}
M(H_1,H_2,H_3,H_4)_{(i_1,i_2,i_3,i_4,0)} &= \mu(H_1^{i_1} \cap H_2^{i_2} \cap H_3^{i_3} \cap H_4^{i_4}) - \tfrac{1}{2^4}\mu(\mathbb{R}^4),\\
M(H_1,H_2,H_3,H_4)_{(i_1,i_2,i_3,i_4,1)} &= \nu(H_1^{i_1} \cap H_2^{i_2} \cap H_3^{i_3} \cap H_4^{i_4}) - \tfrac{1}{2^4}\nu(\mathbb{R}^4),\\
M(H_1,H_2,H_3,H_4)_{(i_1,i_2,i_3,i_4,2)} &= \eta(H_1^{i_1} \cap H_2^{i_2} \cap H_3^{i_3} \cap H_4^{i_4}) - \tfrac{1}{2^4}\eta(\mathbb{R}^4),
\end{aligned}$$

where $\mathbb{R}^{16} \oplus \mathbb{R}^{16} \oplus \mathbb{R}^{16}$ is indexed by the elements of the group $\mathbb{Z}_2^4 \oplus \mathbb{Z}_3$. Here the elements of the group $\mathbb{Z}_3$ distinguish between three measures μ, ν and η. The assumption that each hyperplane of a four-tuple (H_1, H_2, H_3, H_4) in X equiparts the measure allows the reduction of the codomain $\mathbb{R}^{16} \oplus \mathbb{R}^{16} \oplus \mathbb{R}^{16}$. The codomain of the map M is a subspace $\mathbb{U}$ of $\mathbb{R}^{16} \oplus \mathbb{R}^{16} \oplus \mathbb{R}^{16}$ defined by the following equalities:

$$\begin{gathered}
\sum x_{i_1 i_2 i_3 00} = \sum x_{i_1 i_2 i_3 10}; \quad \sum x_{i_1 i_2 0 i_3 0} = \sum x_{i_1 i_2 1 i_3 0} \\
\sum x_{i_1 0 i_2 i_3 0} = \sum x_{i_1 1 i_2 i_3 0}; \quad \sum x_{0 i_1 i_2 i_3 0} = \sum x_{1 i_1 i_2 i_3 0} \\
\sum x_{i_1 i_2 i_3 i_4 0} = 0; \quad \sum x_{i_1 i_2 i_3 i_4 1} = 0; \quad \sum x_{i_1 i_2 i_3 i_4 2} = 0;
\end{gathered}$$

where sums are over all $i_1 i_2 i_3 \in \mathbb{Z}_2^3$ and over all $i_1 i_2 i_3 i_4 \in \mathbb{Z}_2^4$. Thus $M((S^3)^4) \subset \mathbb{U}$, where $\mathbb{U}$ is a linear space of dimension $41 = 48 - 7$.

The $\mathbb{Z}_2^4$ action. The group $\mathbb{Z}_2$ acts antipodaly on S^3, i.e. in our interpretation of the sphere S^3 the action presents an orientation change of a hyperplane. Thus the group $\mathbb{Z}_2^4$ acts on the product $(S^3)^4$, and the action is free as a product of free actions. The group $\mathbb{Z}_2^4$ also acts on $\mathbb{R}^{48} = \mathbb{R}^{16} \oplus \mathbb{R}^{16} \oplus \mathbb{R}^{16}$ by permuting the canonical basis $\{e_w \in \mathbb{R}^{16} \oplus \mathbb{R}^{16} \oplus \mathbb{R}^{16} \mid w = i_1 i_2 i_3 i_4 i_5) \in \mathbb{Z}_2^4 \oplus \mathbb{Z}_3\}$ of $\mathbb{R}^{48}$ in the following way

$$(\forall g \in \mathbb{Z}_2^4)(\forall w = u \times i_5 \in \mathbb{Z}_2^4 \times \mathbb{Z}_3)\ g \cdot e_w = e_{g \cdot u \times i_5}.$$

The following statements are obvious considering definitions.

Proposition 4. (1) *The subspace* $\mathbb{U}$ *is a* $\mathbb{Z}_2^4$ *invariant subspace of* $\mathbb{R}^{48}$.

(2) *The test map* $M : (S^3)^4 \to \mathbb{U} \subseteq \mathbb{R}^{48}$ *is a* $\mathbb{Z}_2^4$*-equivariant map.*

The test space. Let T be the minimal $\mathbb{Z}_2^4$-invariant space inside $\mathbb{U}$ containing the linear subspace $\mathbb{U} \cap L$ where L is defined by equalities

$$\begin{gathered}
x_{00000} = x_{00100} = x_{01000} = x_{01100} = x_{10010} = x_{10110} = x_{11010} = x_{11110}, \\
x_{00010} = x_{00110} = x_{01010} = x_{01110} = x_{10000} = x_{10100} = x_{11000} = x_{11100}, \\
\sum_{g \in \{0,1\}^2, h \in \{0,1\}} x_{g0h1} = \sum_{g \in \{0,1\}^2, h \in \{0,1\}} x_{g1h1} \\
\sum_{g \in \{0,1\}^2, h \in \{0,1\}} x_{g0h2} = \sum_{g \in \{0,1\}^2, h \in \{0,1\}} x_{g1h2}.
\end{gathered} \tag{4}$$

Here the coordinates of $\mathbb{R}^{48}$ are indexed by elements of the group $\mathbb{Z}_2^4 \oplus \mathbb{Z}_3$. Since L is a $\mathbb{Z}_2^4$-invariant subspace, the test space is $\mathbb{U} \cap L$. It is not hard to compute that the codimension of $\mathbb{U} \cap L$ inside $\mathbb{U}$ is 12.

We have proved the central proposition of the CS/TM scheme, which relates a partition problem with a problem of the existence of an equivariant map.

Proposition 5. (A) *If there is no* $\mathbb{Z}_2^4$*-equivariant map*

$$(S^3)^4 \to \mathbb{U} \setminus L \subseteq \mathbb{R}^{48}$$

(with already defined actions), then there exists a solution of the mass partition problem stated in Theorem 1.

(B) *If there is no $\mathbb{Z}_2^4$-equivariant map*

$$(S^3)^4 \to S((\mathbb{U} \cap L)^{\perp}) \subseteq \mathbb{R}^{48}$$

(with already defined actions), then the statement of Theorem 1 holds. Here $(\mathbb{U} \cap L)^{\perp}$ *denotes the orthogonal complement of* $\mathbb{U} \cap L$ *inside* $\mathbb{U}$ *and* $S((\mathbb{U} \cap L)^{\perp})$ *the associated unit sphere.*

Remark 6. The statement (B) follows from the fact that there is a $\mathbb{Z}_2^4$ deformation retraction $\mathbb{U} \backslash L \to S((\mathbb{U} \cap L)^{\perp})$. The sphere $S((\mathbb{U} \cap L)^{\perp})$ is 11 dimensional.

3.2. $\mathbb{Z}_2^4$-index of $(S^3)^4$ and $S((U \cap L)^{\perp})$

Corollary 10 implies that $\mathbb{Z}_2^4$-index of the product $(S^3)^4$ is

$$\mathrm{Ind}_{\mathbb{Z}_2^4}((S^3)^4) = \langle t_1^4, t_2^4, t_3^4, t_4^4 \rangle \subseteq \mathbb{F}_2[t_1, t_2, t_3, t_4]. \tag{5}$$

Index of the sphere $S((\mathbb{U} \cap L)^{\perp})$ can be computed using Proposition 12. Let $e_{1,i}$; … ;$e_{16,i}$ denote the vectors of the standard basis of the i-th copy of $\mathbb{R}^{16}$. The first index in the notation of the base vectors $e_{a,i}$ is the decimal value $+1$ of the binary number obtained from an element of $\mathbb{Z}_2^4$ indexing the coordinates of $\mathbb{R}^{16}$. For example, $e_{1,i} = e_{0000,i}$ and $e_{2,i} = e_{0001,i}$. On the other hand, let $v_{1,i}$; … ;$v_{16,i}$ be the vectors of the $\mathbb{Z}_2^4$-invariant basis of the i-th copy of $\mathbb{R}^{16}$ given by (to simplify the notation, we dropped the second index for the moment)

$$\begin{aligned}
v_1 &= e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 + e_8 - e_9 - e_{10} - e_{11} - e_{12} - e_{13} - e_{14} - e_{15} - e_{16},\\
v_2 &= e_1 + e_2 + e_3 + e_4 - e_5 - e_6 - e_7 - e_8 + e_9 + e_{10} + e_{11} + e_{12} - e_{13} - e_{14} - e_{15} - e_{16},\\
v_3 &= e_1 + e_2 - e_3 - e_4 + e_5 + e_6 - e_7 - e_8 + e_9 + e_{10} - e_{11} - e_{12} + e_{13} + e_{14} - e_{15} - e_{16},\\
v_4 &= e_1 - e_2 + e_3 - e_4 + e_5 - e_6 + e_7 - e_8 + e_9 - e_{10} + e_{11} - e_{12} + e_{13} - e_{14} + e_{15} - e_{16},\\
v_5 &= e_1 + e_2 + e_3 + e_4 - e_5 - e_6 - e_7 - e_8 - e_9 - e_{10} - e_{11} - e_{12} + e_{13} + e_{14} + e_{15} + e_{16},\\
v_6 &= e_1 + e_2 - e_3 - e_4 + e_5 + e_6 - e_7 - e_8 - e_9 - e_{10} + e_{11} + e_{12} - e_{13} - e_{14} + e_{15} + e_{16},\\
v_7 &= e_1 - e_2 + e_3 - e_4 + e_5 - e_6 + e_7 - e_8 - e_9 + e_{10} - e_{11} + e_{12} - e_{13} + e_{14} - e_{15} + e_{16},\\
v_8 &= e_1 + e_2 - e_3 - e_4 - e_5 - e_6 + e_7 + e_8 + e_9 + e_{10} - e_{11} - e_{12} - e_{13} - e_{14} + e_{15} + e_{16},\\
v_9 &= e_1 - e_2 + e_3 - e_4 - e_5 + e_6 - e_7 + e_8 + e_9 - e_{10} + e_{11} - e_{12} - e_{13} + e_{14} - e_{15} + e_{16},\\
v_{10} &= e_1 - e_2 - e_3 + e_4 + e_5 - e_6 - e_7 + e_8 + e_9 - e_{10} - e_{11} + e_{12} + e_{13} - e_{14} - e_{15} + e_{16},\\
v_{11} &= e_1 + e_2 - e_3 - e_4 - e_5 - e_6 + e_7 + e_8 - e_9 - e_{10} + e_{11} + e_{12} + e_{13} + e_{14} - e_{15} - e_{16},\\
v_{12} &= e_1 - e_2 + e_3 - e_4 - e_5 + e_6 - e_7 + e_8 - e_9 + e_{10} - e_{11} + e_{12} + e_{13} - e_{14} + e_{15} - e_{16},\\
v_{13} &= e_1 - e_2 - e_3 + e_4 + e_5 - e_6 - e_7 + e_8 - e_9 + e_{10} + e_{11} - e_{12} - e_{13} + e_{14} + e_{15} - e_{16},\\
v_{14} &= e_1 - e_2 - e_3 + e_4 - e_5 + e_6 + e_7 - e_8 + e_9 - e_{10} - e_{11} + e_{12} - e_{13} + e_{14} + e_{15} - e_{16},\\
v_{15} &= e_1 - e_2 - e_3 + e_4 - e_5 + e_6 + e_7 - e_8 - e_9 + e_{10} + e_{11} - e_{12} + e_{13} - e_{14} - e_{15} + e_{16},\\
v_{16} &= e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 + e_8 + e_9 + e_{10} + e_{11} + e_{12} + e_{13} + e_{14} + e_{15} + e_{16}.
\end{aligned}$$

Let $V_i = \mathrm{span}\{v_i\}$, for $i \in \{1, \ldots, 16\}$. Then Proposition 12,A implies that

$$\mathrm{Ind}_{\mathbb{Z}_2^4}(S(V_1)) = \langle t_1 \rangle;\ \mathrm{Ind}_{\mathbb{Z}_2^4}(S(V_2)) = \langle t_2 \rangle;\ \mathrm{Ind}_{\mathbb{Z}_2^4}(S(V_3)) = \langle t_3 \rangle;\ \mathrm{Ind}_{\mathbb{Z}_2^4}(S(V_4)) = \langle t_4 \rangle;$$
$$\mathrm{Ind}_{\mathbb{Z}_2^4}(S(V_5)) = \langle t_1 + t_2 \rangle;\ \mathrm{Ind}_{\mathbb{Z}_2^4}(S(V_6)) = \langle t_1 + t_3 \rangle;\quad \mathrm{Ind}_{\mathbb{Z}_2^4}(S(V_7)) = \langle t_1 + t_4 \rangle;$$

$$\mathrm{Ind}_{\mathbb{Z}_2^4}(S(V_8)) = \langle t_2 + t_3\rangle;\ \mathrm{Ind}_{\mathbb{Z}_2^4}(S(V_9)) = \langle t_2 + t_4\rangle;\quad \mathrm{Ind}_{\mathbb{Z}_2^4}(S(V_{10})) = \langle t_3 + t_4\rangle;$$
$$\mathrm{Ind}_{\mathbb{Z}_2^4}(S(V_{11})) = \langle t_1 + t_2 + t_3\rangle;\quad \mathrm{Ind}_{\mathbb{Z}_2^4}(S(V_{12})) = \langle t_1 + t_2 + t_4\rangle;$$
$$\mathrm{Ind}_{\mathbb{Z}_2^4}(S(V_{13})) = \langle t_1 + t_3 + t_4\rangle;\quad \mathrm{Ind}_{\mathbb{Z}_2^4}(S(V_{14})) = \langle t_2 + t_3 + t_4\rangle;$$
$$\mathrm{Ind}_{\mathbb{Z}_2^4}(S(V_{15})) = \langle t_1 + t_2 + t_3 + t_4\rangle;\quad \mathrm{Ind}_{\mathbb{Z}_2^4}(S(V_{16})) = \langle 0\rangle.$$

A simple computation in some linear algebra package (like `Mathematica` or `Maple`) confirms:

$$(\mathbb{U}\cap L)^{\perp} = \mathrm{span}\{v_{5,0}, v_{6,0}, v_{8,0}, v_{9,0}, v_{10,0}, v_{11,0}, v_{12,0}, v_{13,0}, v_{14,0}, v_{15,0}, v_{3,1}, v_{3,2}\}. \tag{6}$$

Then by the statement (B) of Proposition 12

$$\begin{aligned}\mathrm{Ind}_{\mathbb{Z}_2^4}(S((\mathbb{U}\cap L)^{\perp})) = \langle &(t_1+t_2)(t_1+t_3)(t_2+t_3)(t_2+t_4)(t_3+t_4)\\ &(t_1+t_2+t_3)(t_1+t_2+t_4)(t_1+t_3+t_4)(t_2+t_3+t_4)\\ &(t_1+t_2+t_3+t_4)t_3^2\rangle \subseteq \mathbb{F}_2[t_1,t_2,t_3,t_4].\end{aligned} \tag{7}$$

A direct computation in the polynomial ring $\mathbb{F}_2[t_1,t_2,t_3,t_4]$ implies that

$$\mathrm{Ind}_{\mathbb{Z}_2^4}(S((\mathbb{U}\cap L)^{\perp})) = \langle p(t_1,t_2,t_3,t_4)\rangle \subseteq \mathbb{F}_2[t_1,t_2,t_3,t_4] \tag{8}$$

where

$$\begin{aligned}p(t_1,t_2,t_3,t_4) = {} & t_1^6t_2^3t_3^3 + t_1^2t_2^7t_3^3 + t_1^6t_2t_3^5 + t_2^7t_3^5 + t_1^2t_2t_3^9 + t_2^3t_3^9 + t_1^6t_2^3t_3^2t_4\\ & + t_1^2t_2^7t_3^2t_4 + t_1^5t_2^3t_3^3t_4 + t_1t_2^7t_3^3t_4 + t_1^6t_3^5t_4 + t_1^5t_2t_3^5t_4 + t_1^2t_3^9t_4\\ & + t_1t_2t_3^9t_4 + t_1^5t_2^3t_3^2t_4^2 + t_1t_2^7t_3^2t_4^2 + t_1^4t_2^3t_3^3t_4^2 + t_2^7t_3^3t_4^2\\ & + +t_1^5t_3^5t_4^2 + t_1^4t_2t_3^5t_4^2 + t_1t_3^9t_4^2 + t_2t_3^9t_4^2 + t_1^6t_2t_3^2t_4^3\\ & + t_1^4t_2^3t_3^2t_4^3 + t_1^6t_3^3t_4^3 + \boxed{t_1^3t_2^3t_3^3t_4^3} + t_1^4t_3^5t_4^3 + t_1^3t_2t_3^5t_4^3 + t_1^5t_2t_3^2t_4^4\\ & + t_1^3t_2^3t_3^2t_4^4 + t_1^5t_3^3t_4^4 + t_1^2t_2^3t_3^3t_4^4 + t_1^3t_3^5t_4^4 + t_1^2t_2t_3^5t_4^4 + t_1^4t_2t_3^2t_4^5\\ & + t_1^2t_2^3t_3^2t_4^5 + t_1^4t_3^3t_4^5 + t_1^1t_2^3t_3^3t_4^5 + t_1^2t_3^5t_4^5 + t_1t_2t_3^5t_4^5 + t_1^3t_2t_3^2t_4^6\\ & + t_1t_2^3t_3^2t_4^6 + t_1^3t_3^3t_4^6 + t_2^3t_3^3t_4^6 + t_1t_3^5t_4^6 + t_2t_3^5t_4^6.\end{aligned} \tag{9}$$

Since $p(t_1,t_2,t_3,t_4) - t_1^3t_2^3t_3^3t_4^3 \in \mathrm{Ind}_{\mathbb{Z}_2^4}((S^3)^4)$, it follows that $p(t_1,t_2,t_3,t_4) \notin \mathrm{Ind}_{\mathbb{Z}_2^4}((S^3)^4)$ and consequently

$$\mathrm{Ind}_{\mathbb{Z}_2^4}((S^3)^4) \not\supseteq \mathrm{Ind}_{\mathbb{Z}_2^4}(S((\mathbb{U}\cap L)^{\perp})).$$

Therefore, the basic Proposition 7 of the ideal valued index theory implies that there is no $\mathbb{Z}_2^4$ equivariant map

$$(S^3)^4 \to S((\mathbb{U}\cap L)^{\perp}).$$

Now Proposition 5,(B) provides the final argument for the statement of Theorem 1. ■

4. Appendix: The Fadell-Husseini index theory

For the more complete presentation of the material in the appendix consult following papers [11], [18] and [7].

Ind_G-definition. To every group one can associate a *classifying space* $\mathrm{B}G$ and the *universal G-bundle* $\mathrm{E}G \to \mathrm{B}G$ which has expected natural properties. G-space X induces by Borel construction a G-space $\mathrm{E}G \times_G X$ and a homotopy unique map $\pi_X\colon \mathrm{E}G \times_G X \to \mathrm{B}G$. For a given field $\mathbb{K}$, that map induces a ring homomorphism in cohomology

$$\pi_X^*\colon H^*(\mathrm{B}G, \mathbb{K}) \to H^*(\mathrm{E}G \times_G X, \mathbb{K}).$$

The *cohomology index* of a G-space X is the ker ideal of π_X^*, i.e.,

$$\mathrm{Ind}_G(X) = \ker \pi_X^* \subset H^*(\mathrm{B}G, \mathbb{K}).$$

We state the fundamental index monotonicity property.

Proposition 7. *Let X and Y be G-spaces and $f : X \to Y$ a G-map. Then*

$$\mathrm{Ind}_G(X) \supseteq \mathrm{Ind}_G(Y).$$

Proof. Functoriality of all constructions implies that the following diagrams commute

$$\begin{tikzcd}
\mathrm{E}G \times_G X \arrow[rr, "\hat{f}"] \arrow[dr, "\pi_X"'] & & \mathrm{E}G \times_G Y \arrow[dl, "\pi_Y"] \\
& \mathrm{B}G &
\end{tikzcd}
\qquad
\begin{tikzcd}
H^*(\mathrm{E}G \times_G X, \mathbb{K}) & & H^*(\mathrm{E}G \times_G Y, \mathbb{K}) \arrow[ll, "f^*"'] \\
& H^*(\mathrm{B}G, \mathbb{K}) \arrow[ul, "\pi_X^*"] \arrow[ur, "\pi_Y^*"'] &
\end{tikzcd}$$

i.e., $\pi_X = \hat{f} \circ \pi_Y$ and $\pi_X^* = \pi_Y^* \circ f^*$. Thus $\ker \pi_X^* \supseteq \ker \pi_Y^*$. ■

Example 8. Let the sphere S^n be a $\mathbb{Z}_2$ space with the antipodal action. The cohomology ring $H^*(\mathrm{B}\mathbb{Z}_2, \mathbb{F}_2)$ is the polynomial ring $\mathbb{F}_2[t]$. $\mathbb{Z}_2$-index of S^n is the principal ideal generated by t^{m+1}:

$$\mathrm{Ind}_{\mathbb{Z}_2}(S^n) = \langle t^{n+1} \rangle \subseteq \mathbb{F}_2[t].$$

The Index of a product of two spaces. Let X be a G-space and Y an H-space. Then $X \times Y$ has the natural structure of a $G \times H$ space. The immediate question arises: Is there a relation among the three indexes $\mathrm{Ind}_{G\times H}(X \times Y)$, $\mathrm{Ind}_G(X)$, and $\mathrm{Ind}_H(Y)$? Using Künneth formula one can prove the following proposition.

Proposition 9. *Let X be a G-space and Y an H-space and*

$$H^*(BG, \mathbb{K}) \cong \mathbb{K}[x_1, \dots, x_n], H^*(BH, \mathbb{K}) \cong \mathbb{K}[y_1, \dots, y_n]$$

the cohomology rings of the associated configuration spaces. If

$$\mathrm{Ind}_G(X) = \langle f_1, \dots, f_i \rangle \text{ and } \mathrm{Ind}_H(Y) = \langle g_1, \dots, g_j \rangle,$$

then $\mathrm{Ind}_{G\times H}(X\times Y)=\langle f_1,\dots,f_i,g_1,\dots,g_j\rangle\subseteq\mathbb{K}[x_1,\dots,x_n,y_1,\dots,y_n]$.

Index of a torus can be computed using this proposition and Example 8.

Corollary 10. *Let* $S^{n_1}\times\cdots\times S^{n_k}$ *be a* $\mathbb{Z}_2^k$*-space with the product action. Then*

$$\mathrm{Ind}_{\mathbb{Z}_2^k}(S^{n_1}\times\cdots\times S^{n_k})=\langle t_1^{n_1+1},\dots,t_k^{n_k+1}\rangle\subseteq\mathbb{F}_2[t_1,\dots,t_k].$$

Index of a sphere. We would like to know how to compute the index of a sphere that is not equipped by $\mathbb{Z}_2$ antipodal action only. The following two practical propositions are of significant importance.

Proposition 11. *Let* U *and* V *be two* G *representations and* $S(U)$, $S(V)$ *associated* G *spheres. If* G *is preserving the orientation of the spheres* $S(U)$, $S(V)$ *and*

$$\mathrm{Ind}_G(S(U))=\langle f\rangle\quad \textit{and}\quad \mathrm{Ind}_G(S(V))=\langle g\rangle,$$

then

$$\mathrm{Ind}_G(S(U\oplus V))=\langle f\cdot g\rangle\subseteq H^*(BG,\mathbb{K}).$$

In case of $\mathbb{Z}_2^k$ group it is known that each irreducible representation V is one-dimensional. Every such representation is identified with a group homomorphism $\xi:\mathbb{Z}_2^k\to Z_2$, where $Z_2=\{+1,-1\}$ is a multiplicative group. Thus, it is completely determined by a 0-1 vector $\xi(V)=(\alpha_1,\dots,\alpha_k)\in\mathbb{F}_2^k$ defined by equality

$$\xi(\omega_i)=(-1)^{\alpha_i},\ i\in\{1,\dots,k\}$$

where ω_i is the generator of the i-th $\mathbb{Z}_2$ copy in $\mathbb{Z}_2^k$.

Proposition 12. (A) *Let* V *be an* 1*-dimensional* $\mathbb{Z}_2^k$ *representation with the associated 0-1 vector* $(\alpha_1,\dots,\alpha_k)\in\mathbb{F}_2^k$. *Then*

$$\mathrm{Ind}_{\mathbb{Z}_2^k}(S(V))=\langle\alpha_1t_1+\cdots+\alpha_kt_k\rangle\subseteq\mathbb{F}_2[t_1,\dots,t_k].$$

(B) *Let* U *be an* n*-dimensional* $\mathbb{Z}_2^k$ *representation with a decomposition* $U\cong V_1\oplus\cdots\oplus V_n$ *in* 1*-dimensional* $\mathbb{Z}_2^k$ *representations* $V_1,\dots,V_n$. *If* $(\alpha_{1i},\dots,\alpha_{ki})\in\mathbb{F}_2^k$ *is the associated 0-1 vector of* V_i, *then*

$$\mathrm{Ind}_{\mathbb{Z}_2^k}(S(U))=\langle\prod_{i=1}^{n}(\alpha_{1i}t_1+\cdots+\alpha_{ki}t_k)\rangle\subseteq\mathbb{F}_2[t_1,\dots,t_k].$$

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