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SZÁSZ-MIRAKJAN TYPE OPERATORS OF TWO VARIABLES PROVIDING A BETTER ESTIMATION ON $[0,1]\times[0,1]$

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Abstract. This paper deals with a modification of the classical Szász-Mirakjan type operators of two variables. It introduces a new sequence of non-polynomial linear operators which hold fixed the polynomials $x^2+\alpha x$ and $y^2+\beta y$ with $\alpha,\beta\in[0,\infty)$ and we study the convergence properties of the new approximation process. Also, we compare it with Szász-Mirakjan type operators and show an improvement of the error of convergence in $[0,1]\times[0,1]$. Finally, we study statistical convergence of this modification.

1. Introduction

Most of the approximating operators, L_n, preserve $e_i(x)=x^i$, $(i=0,1)$, i.e., $L_n(e_i;x)=e_i(x)$, $n\in\mathbb{N}$, $i=0,1$, but $L_n(e_2;x)\neq e_2(x)=x^2$. Especially, these conditions hold for the operators given by Agratini [1], the Bernstein polynomials [4, 5] and the Szász-Mirakjan type operators [3, 14]. Agratini [2] has investigated a general technique to construct operators which preserve e_2. Recently, King [13] presented a non-trivial sequence of positive linear operators defined on the space of all real-valued continuous functions on $[0,1]$ while preserving the functions e_0 and e_2. Duman and Orhan [7] have studied King's results using the concept of statistical convergence. Recently, Duman and Özarslan [8] have investigated some approximation results on the Szász-Mirakjan type operators preserving $e_2(x)=x^2$.

The functions $f_0(x,y)=1$, $f_1(x,y)=x$ and $f_2(x,y)=y$ are preserved by most of approximating operators of two variables, $L_{m,n}$, i.e., $L_{m,n}(f_0;x,y)=f_0(x,y)$, $L_{m,n}(f_1;x,y)=f_1(x,y)$ and $L_{m,n}(f_2;x,y)=f_2(x,y)$, $m,n\in\mathbb{N}$, but $L_{m,n}(f_3;x,y)\neq f_3(x,y)=x^2+y^2$. These conditions hold, specifically, for the Bernstein polynomials of two variables, the Szász-Mirakjan type operators of two variables. In this paper, we give a modification of the well-known Szász-Mirakjan type operators of two variables and show that this modification holds fixed some polynomials different from $f_i(x,y)$. The resulting approximation processes turn out to have an order of approximation at least as good as the one of Szász-Mirakjan

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type operators of two variables in certain subsets of $[0,\infty)\times[0,\infty)$. Finally, we study A-statistical convergence of this modification.

We first recall the concept of A-statistical convergence for double sequences.

Let $A=(a_{j,k,m,n})$ be a four-dimensional summability matrix. For a given double sequence $x=(x_{m,n})$, the A-transform of x, denoted by $Ax:=((Ax)_{j,k})$, is given by

$$(Ax)_{j,k}=\sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}x_{m,n}$$

provided the double series converges in Pringsheim's sense for every $(j,k)\in\mathbb{N}^2$.

A two-dimensional matrix transformation is said to be regular if it maps every convergent sequence into a convergent sequence with the same limit. The well-known characterization for two-dimensional matrix transformations is known as Silverman-Toeplitz conditions (see, for instance, [12]). In 1926, Robison [18] presented a four-dimensional analog of the regularity by considering an additional assumption of boundedness. This assumption was made because a double Pringsheim convergent (P-convergent) sequence is not necessarily bounded. The definition and the characterization of regularity for four-dimensional matrices is known as Robison-Hamilton conditions, or briefly, RH-regularity (see [11, 18]).

Recall that a four-dimensional matrix $A=(a_{j,k,m,n})$ is said to be RH-regular if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit. The Robison-Hamilton conditions state that a four-dimensional matrix $A=(a_{j,k,m,n})$ is RH-regular if and only if

(i) $P-\lim_{j,k} a_{j,k,m,n}=0$ for each $(m,n)\in\mathbb{N}^2$,

(ii) $P-\lim_{j,k} \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}=1$,

(iii) $P-\lim_{j,k} \sum_{m\in\mathbb{N}} |a_{j,k,m,n}|=0$ for each $n\in\mathbb{N}$,

(iv) $P-\lim_{j,k} \sum_{n\in\mathbb{N}} |a_{j,k,m,n}|=0$ for each $m\in\mathbb{N}$,

(v) $\sum_{(m,n)\in\mathbb{N}^2} |a_{j,k,m,n}|$ is P-convergent for each $j,k\in\mathbb{N}$,

(vi) there exist finite positive integers A and B such that $\sum_{m,n>B} |a_{j,k,m,n}|<A$ holds for every $(j,k)\in\mathbb{N}^2$.

Now let $A=(a_{j,k,m,n})$ be a non-negative RH-regular summability matrix, and let $K\subset\mathbb{N}^2$. Then A-density of K is given by

$$\delta_A^{(2)}\{K\}:=P-\lim_{j,k}\sum_{(m,n)\in K} a_{j,k,m,n}$$

provided that the limit on the right-hand side exists in Pringsheim's sense. A real double sequence $x=(x_{m,n})$ is said to be A-statistically convergent to a number L if, for every $\varepsilon>0$,

$$\delta_A^{(2)}\{(m,n)\in\mathbb{N}^2:|x_{m,n}-L|\geq\varepsilon\}=0.$$

In this case, we write $st^2_{(A)} - \lim_{m,n} x_{m,n} = L$. Clearly, a P-convergent double sequence is A-statistically convergent to the same value but its converse is not always true. Also, note that an A-statistically convergent double sequence need not to be bounded. For example, consider the double sequence $x = (x_{m,n})$ given by

$$x_{m,n} = \begin{cases} mn, & \text{if } m \text{ and } n \text{ are squares,} \\ 1, & \text{otherwise.} \end{cases}$$

We should note that if we take $A = C(1,1) := [c_{j,k,m,n}]$, the double Cesáro matrix, defined by

$$c_{j,k,m,n} = \begin{cases} \frac{1}{jk}, & \text{if } 1 \leq m \leq j \text{ and } 1 \leq n \leq k, \\ 0, & \text{otherwise,} \end{cases}$$

then $C(1,1)$-statistical convergence coincides with the notion of statistical convergence for double sequence, which was introduced in [15, 16]. Finally, if we replace the matrix A by the identity matrix for four dimensional matrices, then A-statistical convergence reduces to the Pringsheim convergence, which was introduced in [17].

By $C(D)$, we denote the space of all continuous real valued functions on D where $D = [0,\infty) \times [0,\infty)$. By E_2, we denote the space of all real valued functions of exponential type on D. More precisely, $f \in E_2$ if and only if there are three positive finite constants c, d and α with the property $|f(x,y)| \leq \alpha e^{cx+dy}$. Let L be a linear operator from $C(D) \cap E_2$ into $C(D) \cap E_2$. Then, as usual, we say that L is a positive linear operator provided that $f \geq 0$ implies $L(f) \geq 0$. Also, we denote the value of $L(f)$ at a point $(x,y) \in D$ by $L(f;x,y)$.

Now fix $a, b > 0$. For the proof of the our approximation results we use the lattice homomorphism $H_{a,b}$, which maps $C(D) \cap E_2$ into $C(E) \cap E_2$, defined by $H_{a,b}(f) = f|_E$, where $E = [0,a] \times [0,b]$ and $f|_E$ denotes the restriction of the domain f to the rectangle E. The space $C(E)$ is equipped with the supremum norm

$$\|f\| = \sup_{(x,y)\in E} |f(x,y)|, \quad (f \in C(E)).$$

Hence, from the Korovkin-type approximation theorem for double sequences of positive linear operators of two variables which is introduced by Dirik and Demirci [6] the following results follow.

Theorem 1. [6] *Let $A = (a_{j,k,m,n})$ be a non-negative RH-regular summability matrix. Let $\{L_{m,n}\}$ be a double sequence of positive linear operators acting from $C(D) \cap E_2$ into itself. Assume that the following conditions hold:*

$$st^2_{(A)} - \lim_{m,n} L_{m,n}(f_i; x,y) = f_i(x,y), \text{ uniformly on } E, \ (i = 0,1,2,3),$$

where $f_0(x,y) = 1$, $f_1(x,y) = x$, $f_2(x,y) = y$ and $f_3(x,y) = x^2 + y^2$. Then, for all $f \in C(D) \cap E_2$, we have

$$st^2_{(A)} - \lim_{m,n} L_{m,n}(f; x,y) = f(x,y), \text{ uniformly on } E.$$

2. Construction of the operators

Szász-Mirakjan type operators introduced by Favard [9] is the following:

$$S_{m,n}(f;x,y) = e^{-mx}e^{-ny}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty} f\left(\frac{s}{m},\frac{t}{n}\right)\frac{(mx)^s}{s!}\frac{(ny)^t}{t!}, \tag{2.1}$$

where $(x,y) \in D$ and $f \in C(D) \cap E_2$. It is clear that

$$\begin{aligned} S_{m,n}(f_0;x,y) &= f_0(x,y),\\ S_{m,n}(f_1;x,y) &= f_1(x,y),\\ S_{m,n}(f_2;x,y) &= f_2(x,y),\\ S_{m,n}(f_3;x,y) &= f_3(x,y) + \frac{x}{m} + \frac{y}{n}, \end{aligned}$$

where $f_0(x,y) = 1$, $f_1(x,y) = x$, $f_2(x,y) = y$ and $f_3(x,y) = x^2 + y^2$. Then, we observe that $P - \lim_{m,n} S_{m,n}(f_i;x.y) = f_i(x,y)$, uniformly on E, where $i = 0,1,2,3$. If we replace the matrix A by double identity matrix in Theorem 1, then we immediately get the classical result. Hence, for the $S_{m,n}$ operators given by (2.1), we have, for all $f \in C(D) \cap E_2$,

$$P - \lim_{m,n} S_{m,n}(f;x,y) = f(x,y), \text{ uniformly on } E.$$

For each integer $k \in \mathbb{N}$, let $r_k\colon [0,\infty) \times X \to \mathbb{R}$ be the function defined by

$$r_k(\gamma,z) := \frac{-(k\gamma+1) + \sqrt{(k\gamma+1)^2 + 4k^2(z^2+\gamma z)}}{2k} \tag{2.2}$$

where if z is the first variable of the following operator, then $X = [0,a]$ and if z is the second variable of the following operator, then $X = [0,b]$. Let

$$\begin{aligned} H_{m,n}^{\alpha,\beta}(f;x,y) &= S_{m,n}(f;r_m(\alpha,x),r_n(\beta,y))\\ &= e^{-mr_m(\alpha,x)}e^{-nr_n(\beta,y)}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty} f\left(\frac{s}{m},\frac{t}{n}\right)\frac{(mr_m(\alpha,x))^s}{s!}\frac{(nr_n(\beta,y))^t}{t!} \end{aligned} \tag{2.3}$$

where $\alpha,\beta \in [0,\infty)$, for $f \in C(D) \cap E_2$.

Hence, in the special case $\lim_{\alpha\to\infty} r_m(\alpha,x) = x$ and $\lim_{\alpha\to\infty} r_n(\beta,y) = y$, the operator $H_{m,n}^{\alpha,\beta}$ becomes the classical Szász-Mirakjan type operators which is given by (2.1).

It is clear that $H_{m,n}^{\alpha,\beta}$ are positive and linear. It is easy to see that

$$\begin{aligned} H_{m,n}^{\alpha,\beta}(f_0;x,y) &= f_0(x,y),\\ H_{m,n}^{\alpha,\beta}(f_1;x,y) &= r_m(\alpha,x),\\ H_{m,n}^{\alpha,\beta}(f_2;x,y) &= r_n(\beta,y),\\ H_{m,n}^{\alpha,\beta}(f_1^2;x,y) &= r_m^2(\alpha,x) + \frac{r_m(\alpha,x)}{m},\\ H_{m,n}^{\alpha,\beta}(f_2^2;x,y) &= r_n^2(\beta,y) + \frac{r_n(\beta,y)}{n}. \end{aligned} \tag{2.4}$$

From the definition of r_k one can check the validity of the following.

Proposition 1. *The operators* $H^{\alpha,\beta}_{m,n}$ *hold fixed the polynomials* $f_1^2+\alpha f_1$ *and* $f_2^2+\beta f_2$, *i.e.*

$$H^{\alpha,\beta}_{m,n}(f_1^2+\alpha f_1;x,y)=x^2+\alpha x \text{ and } H^{\alpha,\beta}_{m,n}(f_2^2+\beta f_2;x,y)=y^2+\beta y.$$

Now, we give the following result using Theorem 1 for $A=I$, which is the double identity matrix.

Theorem 2. *Let* $H^{\alpha,\beta}_{m,n}$ *denote the sequence of positive linear operators given by* (2.3). *If*

$$P-\lim_{m,n} H^{\alpha,\beta}_{m,n}(f_1;x,y)=x,\quad P-\lim_{m,n} H^{\alpha,\beta}_{m,n}(f_2;x,y)=y, \text{ uniformly on } E,$$

then, for all $f\in C(D)\cap E_2$,

$$P-\lim_{m,n} H^{\alpha,\beta}_{m,n}(f;x,y)=f(x,y), \text{ uniformly on } E,$$

where $\alpha,\beta\in[0,\infty)$.

Proof. For $\alpha,\beta\in[0,\infty)$, $H^{\alpha,\beta}_{m,n}(f_1;x,y)$ converges to x as m,n (in any manner) tends to infinity. Also, we get

$$\begin{aligned} r_{m,n}(\alpha) &= \sup_{(x,y)\in E} |x-H^{\alpha,\beta}_{m,n}(f_1;x,y)| \\ &= a-\frac{-(m\alpha+1)+\sqrt{(m\alpha+1)^2+4m^2(a^2+\alpha a)}}{2m}. \end{aligned}$$

Since $r_{m,n}(\alpha)$ and $r_{m,n}(\beta)$ converge to 0 as $m,n\to\infty$, the convergence is uniform on E. From (2.4), Proposition 1 and Theorem 1 for $A=I$, which is the double identity matrix, the proof is completed. ■

3. Comparison with Szász-Mirakjan type operators

In this section, we estimate the rates of convergence of the operators $H^{\alpha,\beta}_{m,n}(f;x,y)$ to $f(x,y)$ by means of the modulus of continuity. Thus, we show that our estimations are more powerful than those obtained by the operators given by (2.1) on D.

By $C_B(D)$ we denote the space of all continuous and bounded functions on D. For $f\in C_B(D)\cap E_2$, the modulus of continuity of f, denoted by $\omega(f;\delta)$, is defined as

$$\omega(f;\delta)=\sup\{|f(u,v)-f(x,y)|:\sqrt{(u-x)^2+(v-y)^2}<\delta,\ (u,v),(x,y)\in D\}.$$

Then it is clear that for any $\delta>0$ and each $(x,y)\in D$

$$|f(u,v)-f(x,y)|\le\omega(f;\delta)\left(\frac{\sqrt{(u-x)^2+(v-y)^2}}{\delta}+1\right).$$

After some simple calculations, for any double sequence $\{L_{m,n}\}$ of positive linear operators on $C_B(D)\cap E_2$, we can write, for $f\in C_B(D)\cap E_2$,

$$\begin{aligned} |L_{m,n}(f;x,y)-f(x,y)| &\le \omega(f;\delta)\Big\{L_{m,n}(f_0;x,y)+ \\ &+\frac{1}{\delta^2}L_{m,n}((u-x)^2+(v-y)^2;x,y)\Big\}+|f(x,y)||L_{m,n}(f_0;x,y)-f_0(x,y)|. \qquad (3.1) \end{aligned}$$

Now we have the following:

THEOREM 3. *If $H_{m,n}^{\alpha,\beta}$ is defined by (2.1), then for every $f \in C_B(D) \cap E_2$, $(x,y) \in D$ and any $\delta > 0$, we have*

$$|H_{m,n}^{\alpha,\beta}(f;x,y) - f(x,y)| \leq \omega(f,\delta)\Big\{1 + \frac{1}{\delta^2}(2x^2 + \alpha x - H_{m,n}^{\alpha,\beta}(f_1;x,y)(\alpha + 2x)) + \\ + \frac{1}{\delta^2}(2y^2 + \beta y - H_{m,n}^{\alpha,\beta}(f_2;x,y)(\beta + 2y))\Big\}. \quad (3.2)$$

Furthermore, when (3.2) holds,

$$2x^2 + \alpha x - H_{m,n}^{\alpha,\beta}(f_1;x,y)(\alpha + 2x) + 2y^2 + \beta y - H_{m,n}^{\alpha,\beta}(f_2;x,y)(\beta + 2y) \geq 0$$

for $(x,y) \in D$.

REMARK 1. For the Szász-Mirakjan type operators given by (2.1), we may write from (3.1) that for every $f \in C_B(D) \cap E_2$, $m,n \in \mathbb{N}$,

$$|S_{m,n}(f;x,y) - f(x,y)| \leq \omega(f,\delta)\{1 + \frac{1}{\delta^2}(\frac{x}{m} + \frac{y}{n})\}. \quad (3.3)$$

The estimate (3.2) is better than the estimate (3.3) if and only if

$$2x^2+\alpha x-H_{m,n}^{\alpha,\beta}(f_1;x,y)(\alpha+2x)+2y^2+\beta y-H_{m,n}^{\alpha,\beta}(f_2;x,y)(\beta+2y) \leq \frac{x}{m}+\frac{y}{n}, \quad (3.4)$$

$(x,y) \in D$. Thus, the order of approximation towards a given function $f \in C_B(D) \cap E_2$ by the sequence $H_{m,n}^{\alpha,\beta}$ will be at least as good as that of $S_{m,n}$ whenever the following function $\phi_{m,n}^{\alpha,\beta}(x,y)$ is non-negative:

$$\phi_{m,n}^{\alpha,\beta}(x,y) = \\ = \frac{x}{m} + \frac{y}{n} - 2x^2 - \alpha x + H_{m,n}^{\alpha,\beta}(f_1;x,y)(\alpha+2x) - 2y^2 - \beta y + H_{m,n}^{\alpha,\beta}(f_2;x,y)(\beta+2y).$$

The non-negativity of $\phi_{m,n}^{\alpha,\beta}(x,y)$ is obviously fulfilled at those points (x,y) where simultaneously

$$H_{m,n}^{\alpha,\beta}(f_1;x,y)(\alpha + 2x) - 2x^2 - \alpha x + \frac{x}{m} \geq 0$$

and

$$H_{m,n}^{\alpha,\beta}(f_2;x,y)(\beta + 2y) - 2y^2 - \beta y + \frac{y}{n} \geq 0.$$

Some calculations state the validity of these inequalities when and only when (x,y) lies in the subset of D given by the rectangle

$$\left[0, \frac{2\alpha m + \alpha + 2}{2\alpha m + 1}\right] \times \left[0, \frac{2\beta n + \beta + 2}{2\beta n + 1}\right].$$

As $m,n \to \infty$, the endpoints of these intervals decrease to 1 and 1, respectively. As a consequence the order of approximation of $H_{m,n}^{\alpha,\beta} f$ towards f is at least as good as the order of approximation to f given by $S_{m,n}$ whenever (x,y) lies in $[0,1] \times [0,1]$.

4. *A*-statistical convergence

Gadjiev and Orhan [10] have investigated the Korovkin-type approximation theory via statistical convergence. In this section, using the concept of A-statistical convergence for double sequence, we give the Korovkin-type approximation theorem for the $H_{m,n}^{\alpha,\beta}$ operators given by (2.3). The Korovkin-type approximation theorem is given by Theorem 1 and Proposition 1 as follows:

Theorem 4. *Let $A = (a_{j,k,m,n})$ be a non-negative RH-regular summability matrix. Let $H_{m,n}^{\alpha,\beta}$ be the double sequence of positive linear operators given by* (2.3)*. If*

$$st^2_{(A)} - \lim_{m,n} H_{m,n}^{\alpha,\beta}(f_1;x,y) = x,\ st^2_{(A)} - \lim_{m,n} H_{m,n}^{\alpha,\beta}(f_2;x,y) = y,\ \text{uniformly on } E,$$

then, for all $f \in C(D) \cap E_2$,

$$st^2_{(A)} - \lim_{m,n} H_{m,n}^{\alpha,\beta}(f;x,y) = f(x,y),\ \text{uniformly on } E.$$

Now, we choose a subset K of $\mathbb{N}^2$ such that $\delta_A^{(2)}(K) = 1$. Define function sequences $\{r_m^*(\alpha,x)\}$ and $\{r_n^*(\beta,y)\}$ by

$$\begin{aligned} r_m^*(\alpha,x) &= \begin{cases} 0, & (m,n) \notin K \\ \frac{-(m\alpha+1)+\sqrt{(m\alpha+1)^2+4m^2(x^2+\alpha x)}}{2m}, & (m,n) \in K \end{cases} \\ r_n^*(\beta,y) &= \begin{cases} 0, & (m,n) \notin K \\ \frac{-(n\beta+1)+\sqrt{(n\beta+1)^2+4n^2(y^2+\beta y)}}{2n}, & (m,n) \in K \end{cases} \end{aligned} \tag{4.1}$$

It is clear that $r_m^*(\alpha,x)$ and $r_n^*(\beta,y)$ are continuous and exponential-type on $[0,\infty)$. We now turn our attention to $\{H_{m,n}^{\alpha,\beta}\}$ given by (2.3) with $\{r_m(\alpha,x)\}$ and $\{r_n(\beta,y)\}$ replaced by $\{r_m^*(\alpha,x)\}$ and $\{r_n^*(\beta,y)\}$ where $r_m^*(\alpha,x)$ and $r_n^*(\beta,y)$ are defined by (4.1). Observe that $\{H_{m,n}^{\alpha,\beta}\}$ is a positive linear operator and

$$H_{m,n}^{\alpha,\beta}(f_1;x,y) = r_m^*(\alpha,x), \quad H_{m,n}^{\alpha,\beta}(f_2;x,y) = r_n^*(\beta,y), \tag{4.2}$$

and

$$\begin{aligned} H_{m,n}^{\alpha,\beta}(f_1^2;x,y) &= \begin{cases} r_m^2(\alpha,x) + \frac{r_m(\alpha,x)}{m}, & (m,n) \in K \\ 0, & \text{otherwise} \end{cases} \\ H_{m,n}^{\alpha,\beta}(f_2^2;x,y) &= \begin{cases} r_n^2(\beta,y) + \frac{r_n(\beta,y)}{n}, & (m,n) \in K \\ 0, & \text{otherwise} \end{cases} \end{aligned} \tag{4.3}$$

Since $\delta_A^{(2)}(K) = 1$, we obtain

$$st^2_{(A)} - \lim_{m,n} H_{m,n}^{\alpha,\beta}(f_1;x,y) = x,\ \ st^2_{(A)} - \lim_{m,n} H_{m,n}^{\alpha,\beta}(f_2;x,y) = y,\ \text{uniformly on } E \tag{4.4}$$

and

$$st^2_{(A)} - \lim_{m,n} H_{m,n}^{\alpha,\beta}(f_1^2 + f_2^2;x,y) = x^2 + y^2,\ \text{uniformly on } E. \tag{4.5}$$

The relations (4.2)–(4.5) and Theorem 1 yield the following:

THEOREM 5. *Let $A = (a_{j,k,m,n})$ be a non-negative RH-regular summability matrix and let $\{H^{\alpha,\beta}_{m,n}\}$ denote the double sequence of positive linear operators given by* (2.3) *with $\{r_m(\alpha,x)\}$ and $\{r_n(\beta,y)\}$ replaced by $\{r^*_m(\alpha,x)\}$ and $\{r^*_n(\beta,y)\}$ where $r^*_m(\alpha,x)$ and $r^*_n(\beta,y)$ are defined by* (4.1)*. Then, for all $f \in C(D) \cap E_2$, we have*

$$st^2_{(A)} - \lim_{m,n} H^{\alpha,\beta}_{m,n}(f;x,y) = f(x,y), \text{ uniformly on } E.$$

We note that $r^*_m(\alpha,x)$ and $r^*_n(\beta,y)$ in Theorem 5 do not satisfy the conditions of Theorem 2.

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