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ON PRIME FUZZY BI-IDEALS IN TERNARY SEMIGROUPS

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Abstract. In this research we concentrate on the analytical study and concept of fuzzification on prime bi-ideals in ternary semigroups and look for some of their related characteristics. Strongly prime and semiprime fuzzy bi-ideals are initiated and traits are discussed. Besides irreducible and strongly irreducible fuzzy bi-ideals in ternary semigroups have been researched. Employing the fuzzy bi-ideals of ternary semigroups, parity statements for a regular ternary semigroup have been collaborated. Furthermore, it has observed that, the set of all strongly prime proper fuzzy bi-ideals in a ternary semigroup form a topology. Conclusively, it has been proved that in case of the totally ordered set of fuzzy bi-ideals of a semigroup S, the concept of irreducible prime and strongly irreducible prime coincides.

1. Introduction and historical background

A fuzzy set theory was conceptualized by Professor L. A. Zadeh at the University of California in 1965 in [15] as a generalization of abstract set theory. Zadeh's initiation is virtually a complete paradigm shift that initially gained popularity in the Far East and its successful applications has gained further ground almost round the globe.

A paradigm is a concept encompassing rules and regulations which define boundaries and suggest standards as to how to successfully solve problems within these limits. For example the use of transistors in place of vacuum tubes is a paradigm shift. Similarly, the development of fuzzy set theory from conventional bivalent set theory is a paradigm shift.

In the ending years of the decade of 1980's a variety of user-friendly tools for fuzzy control, fuzzy expert systems and fuzzy data analysis have come to force. This has absolutely transferred the character of this area and initiated the new area of fuzzy technology. The next major advance in the development appeared in 1992 when concurrently in Europe, Japan and the USA; The three areas of fuzzy technology: artificial neural nets and genetic algorithms; joined forces under the title of computational intelligence or soft computing. The synergism among these three

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areas have been creatively made use of very successfully. Practical applications of fuzzy sets focuses on model and real applications of fuzzy sets and is divided in to four main parts: engineering and natural sciences, medicine, management and behavioural cum social sciences.

In [7] J. N. Mordeson, D.S. Malik and N. Kuroki launched fuzzy bi-ideals in semigroup. In [9] M. Shabir, Y. B. June and M. Bano iniatiated prime bi-ideals in ternary semigroup. In this study we analyze the prime, semiprime, strongly prime, irreducible and strongly irreducible fuzzy bi-ideals in ternary semigroup.

2. Preliminaries

Definition 1. [6] A ternary semigroup S is a non-empty set whose elements are closed under the ternary operation [ ] of multiplication and satisfy the associative law defined as

$$[[abc]\,de] = [a\,[bcd]\,e] = [ab\,[cde]] \qquad \text{for all } a, b, c, d, e \in S.$$

For simplicity we shall write $[abc]$ as abc.

If a ternary semigroup S contains an element 0 such that $0ab = a0b = ab0 = 0$ for all $a, b \in S$, then 0 is called the zero element of S. A ternary semigroup with zero means a ternary semigroup having 0 element. In this paper we will assume that S has zero element. An element a of a ternary semigroup S is said to be regular if there exists an element x in S such that $a = axa$. A ternary semigroup S is called regular if every element of S is regular. A non-empty subset B of a ternary semigroup S is called ternary subsemigroup of S if $B^3 \subseteq B$, and is called idempotent if $B.B.B = B^3 = B$. A subsemigroup B of a ternary semigroup S is called bi-ideal of S if $BSBSB \subseteq B$ [3].

A function f from a non-empty set S to the unit interval $[0, 1]$ of real numbers is called a *fuzzy subset* of S, that is $f : S \to [0, 1]$. A fuzzy subset $f : S \to [0, 1]$ is non-empty if f is not the constant map which assumes the value 0. For fuzzy subsets f and g of S, $f \le g$ means that for all $a \in S$, $f\,(a) \le g\,(a)$. The characteristic function f_S of S is a function which gives $f_S(x) = 1$ for all $x \in S$. The symbols $f \wedge g \wedge h$ and $f \vee g \vee h$ will mean the following fuzzy subsets of S:

$$(f \wedge g \wedge h)\,(a) = f\,(a) \wedge g\,(a) \wedge h\,(a)$$
$$\text{and} \quad (f \vee g \vee h)\,(a) = f\,(a) \vee g\,(a) \vee h\,(a) \quad \text{for all } a \in S,$$

where $\wedge$ denotes min or infimum and $\vee$ denotes max or supremum [7].

Definition 2. [14] If f, g and h are fuzzy subsets of a ternary semigroup S and x be an element of S, then

$$(f \circ g \circ h)\,(x) = \begin{cases} \bigvee_{x=abc} [f\,(a) \wedge g\,(b) \wedge h\,(c)]\,, & \text{if } x \text{ is expressible as } x = abc \\ 0, & \text{otherwise.} \end{cases}$$

The operation '$\circ$' is associative.

Lemma 1. [4] *For any non-empty subsets X, Y and Z of a ternary semigroup S, we have*

(1) $f_X \circ f_Y \circ f_Z = f_{XYZ}$.

(2) $f_X \wedge f_Y \wedge f_Z = f_{X \cap Y \cap Z}$.

Definition 3. [7] Let f be a fuzzy subset of X. Let $t \in [0,1]$. Define $f_t = \{x \in X : f(x) \geq t\}$. We call f_t a *t-cut* or a *level set*.

Definition 4. [14] A fuzzy subset f of a ternary semigroup S is a *fuzzy ternary subsemigroup* of S if $f(abc) \geq f(a) \wedge f(b) \wedge f(c)$ for all $a, b, c \in S$.

Example 1. [14] Let Z be the set of integers and $S = Z_0^- \subset Z$ be the set of all negative integers with zero. Then $\left(Z_0^-, [\,]\right)$ forms a ternary semigroup S with zero. Define a fuzzy subset $f : Z \to [0,1]$ as $f(x) = \begin{cases} 1, & \text{if } x \in S \\ 0, & \text{otherwise.} \end{cases}$ Then f is a fuzzy ternary subsemigroup of S.

Definition 5. [14] A fuzzy subset f of a ternary semigroup S is a *fuzzy left* (*right, lateral*) *ideal* of S if $f(abc) \geq f(c)$ $[f(abc) \geq f(a), f(abc) \geq f(b)]$ for all $a, b, c \in S$.

Example 2. [14] Consider the set $Z_5^- = \{0, -1, -2, -3, -4\}$. Then $\left(Z_5^-, \cdot\right)$ is a ternary semigroup where ternary multiplication $'\cdot'$ is defined as

$\cdot$	0	-1	-2	-3	-4
0	0	0	0	0	0
-1	0	1	2	3	4
-2	0	2	4	1	3
-3	0	3	1	4	2
-4	0	4	3	2	1

$\cdot$	0	-1	-2	-3	-4
0	0	0	0	0	0
1	0	-1	-2	-3	-4
2	0	-2	-4	-1	-3
3	0	-3	-1	-4	-2
4	0	-4	-3	-2	-1

Define a fuzzy subset $f : Z_5^- \to [0,1]$ as $f(0) = t_0$ and $f(-1) = f(-2) = f(-3) = f(-4) = t_1$ where $t_0, t_1 \in [0,1]$ such that $t_0 \geq t_1$. Then f is a fuzzy ideal of Z_5^-.

Lemma 2. [14] *Let A be a non-empty subset of a ternary semigroup S, then A is ternary subsemigroup of S if and only if f_A is a fuzzy ternary subsemigroup of S.*

Definition 6. [14] A fuzzy ideal f of a ternary semigroup S is called *prime* if for any three fuzzy ideals f_1, f_2, f_3 of S, $f_1 \circ f_2 \circ f_3 \leqslant f$ implies $f_1 \leqslant f$ or $f_2 \leqslant f$ or $f_3 \leqslant f$.

Definition 7. [14] A proper fuzzy ideal ξ of a ternary semigroup S is called a *fuzzy semiprime ideal* of S if there exists a fuzzy ideal λ such that $\lambda^3 \leq \xi$ implies $\lambda \leq \xi$.

Definition 8. [14] A proper fuzzy ideal ξ of a ternary semigroup S is said to be fuzzy irreducible if for fuzzy ideals λ and μ of S, $\lambda \wedge \mu = \xi$ implies that $\lambda = \xi$ or $\mu = \xi$. This condition is equivalent to $\lambda \wedge \mu \leq \xi$ implies that $\lambda \leq \xi$ or $\mu \leq \xi$.

3. Fuzzy bi-ideals in ternary semigroups

Definition 9. A fuzzy subset f of a ternary semigroup S is called a *fuzzy bi-ideal* of S if

(1) $f(abc) \geq f(a) \wedge f(b) \wedge f(c)$ and

(2) $f(abcde) \geq f(a) \wedge f(c) \wedge f(e)$ for all $a, b, c, d, e \in S$.

Proposition 1. *A fuzzy subset f of a ternary semigroup S is a fuzzy bi-ideal of S if and only if $f \circ f \circ f \leq f$ and $f \circ f_S \circ f \circ f_S \circ f \leq f$.*

Proof. Assume that f is a fuzzy bi-ideal of a ternary semigroup S. In the case when $(f \circ f_S \circ f \circ f_S \circ f)(a) = 0$, it is clear that $f \circ f_S \circ f \circ f_S \circ f \leq f$. Otherwise there exist elements x, y, z and p, q, r of S such that $a = xyz$ and $x = pqr$. Since f is a fuzzy bi-ideal of S, we have $f(pqryz) \geq f(p) \wedge f(r) \wedge f(z)$. Therefore

$$\begin{aligned}
(f \circ f_S \circ f \circ f_S \circ f)(a) &= \bigvee_{a=xyz} \{(f \circ f_S \circ f)(x) \wedge f_S(y) \circ f(z)\} \\
&= \bigvee_{a=xyz} \{ \bigvee_{x=pqr} (f(p) \wedge f_S(q) \wedge f(r)) \wedge f_S(y) \wedge f(z)\} \\
&= \bigvee_{a=xyz} \{ \bigvee_{x=pqr} (f(p) \wedge 1 \wedge f(r)) \wedge 1 \wedge f(z)\} \\
&= \bigvee_{a=xyz} \{ \bigvee_{x=pqr} (f(p) \wedge f(r)) \wedge f(z)\} \\
&= \bigvee_{a=(pqr)yz} \{(f(p) \wedge f(r)) \wedge f(z)\} \\
&\leq \bigvee_{a=(pqr)yz} \{(f(pqryz)\} = f(a)
\end{aligned}$$

and so we have $f \circ f_S \circ f \circ f_S \circ f \leq f$.

Conversely, assume that $f \circ f_S \circ f \circ f_S \circ f \leq f$. Let a, b, c, d and e be any elements of S such that $x = abcde$, then we have

$$\begin{aligned}
f(abcde) &= f(x) \\
&\geq (f \circ f_S \circ f \circ f_S \circ f)(x) \\
&= \bigvee_{x=a'b'c'} \{(f \circ f_S \circ f)(a') \wedge f_S(b') \wedge f(c')\} \\
&\geq \{(f \circ f_S \circ f)(abc) \wedge f_S(d) \wedge f(e)\} \\
&= \bigvee_{abc=x_1y_1z_1} \{f(x_1) \wedge f_S(y_1) \wedge f(z_1)\} \wedge f_S(d) \wedge f(e)\} \\
&\geq \{(f(a) \wedge f_S(b) \wedge f(c)\} \wedge f_S(d) \wedge f(e)
\end{aligned}$$

$$= (f(a) \wedge 1 \wedge f(c)) \wedge 1 \wedge f(e)$$
$$= f(a) \wedge f(c) \wedge f(e)$$

Thus f is a fuzzy bi-ideal of S. ∎

Lemma 3. *A non-empty subset B of a ternary semigroup S is a bi-ideal of S if and only if the characteristic function f_B of B is a fuzzy bi-ideal of S.*

Proof. Suppose B is a bi-ideal of S. Then B is a subsemigroup of S and $BSBSB \subseteq B$. By lemma 2 f_B is a fuzzy subsemigroup of S. Let a, b, c, d and e be any elements of S. If $a, c, e \in B$, then $f_B(a) = f_B(c) = f_B(e) = 1$. Since $abcde \in BSBSB \subseteq B$, we have $f_B(abcde) = 1 = f_B(a) \wedge f_B(c) \wedge f_B(e) \ldots \ldots$ (**i**). If $a \notin B$ or $c \notin B$ or $e \notin B$, then $f_B(a) = 0$ or $f_B(c) = 0$ or $f_B(e) = 0$ and so we have $f_B(abcde) \geq 0 = f_B(a) \wedge f_B(c) \wedge f_B(e) \ldots \ldots$ (**ii**). Combining (**i**) and (**ii**) we get $f_B(abcde) \geq f_B(a) \wedge f_B(c) \wedge f_B(e)$.

Conversely, suppose that f_B is a fuzzy bi-ideal of S, then it follows from lemma 2, B is a subsemigroup of S. Let $x = abcde$ be any element of $BSBSB(a, c, e \in B)$, then $f_B(x) = f_B(abcde) \geq f_B(a) \wedge f_B(c) \wedge f_B(e) = 1 \wedge 1 \wedge 1 = 1$. Implies $f_B(x) = 1$, so $x \in B$. Thus $BSBSB \subseteq B$. Hence B is a bi-ideal of S. ∎

Lemma 4. *Let A, B and C are three subsets of a ternary semigroup S. Then $f_A o f_B o f_C = f_{ABC}$.*

Theorem 1. *Let μ be a fuzzy subset of S. Then μ is a fuzzy bi-ideal of S if and only if μ_t is a bi-ideal of S, for all $t \in [0, 1]$.*

Proof. Let μ be a fuzzy bi-ideal of S. Let $t \in [0, 1]$. Suppose $x, y, z \in S$ such that $x, y, z \in \mu_t$. Implies $\mu(x) \geq t$, $\mu(y) \geq t$ and $\mu(z) \geq t$. Thus $\mu(xyz) \geq \mu(x) \wedge \mu(y) \wedge \mu(z) \geq t \wedge t \wedge t = t$. Implies $\mu(xyz) \geq t$, so $xyz \in \mu_t$. Hence μ_t is a ternary subsemigroup of S. Suppose $a \in \mu_t S \mu_t S \mu_t$ then $a = bs_1cs_2d$ where $b, c, d \in \mu_t, s_1, s_2 \in S$. Implies $\mu(b) \geq t$, $\mu(c) \geq t, \mu(d) \geq t$. Now consider

$$\begin{aligned}(\mu \circ f_S \circ \mu \circ f_S \circ \mu)(a) = \mu(a) &= \mu(bs_1cs_2d) \\ &\geq \mu(b) \wedge \mu(c) \wedge \mu(d) \geq t \wedge t \wedge t = t\end{aligned}$$

implies $(\mu \circ f_S \circ \mu \circ f_S \circ \mu)(a) \geq t$.

Hence $a \in (\mu \circ f_S \circ \mu \circ f_S \circ \mu)_t \subseteq \mu_t$ so $a \in \mu_t$. Thus $\mu_t S \mu_t S \mu_t \subseteq \mu_t$. Thus μ_t is a bi-ideal of S. Conversely, suppose that μ_t is a bi-ideal of S. Let $t \in [0, 1]$ and $p \in S$. Also suppose $p = xs_1ys_2z$

$$\begin{aligned}(\mu \circ f_S \circ \mu \circ f_S \circ \mu)(p) &= \bigvee_{p=p_1s_2z} \{(\mu \circ f_S \circ \mu)(p_1) \wedge f_S(s_2) \wedge \mu(z)\} \\ &= \bigvee_{p=p_1s_2z} \left\{ \bigvee_{p_1=xs_1y} (\mu(x) \wedge f_S(s_1)) \wedge (\mu)(y) \wedge f_S(s_2) \wedge \mu(z) \right\} \\ &= \bigvee_{p=(xs_1y)s_2z} \{(\mu(x) \wedge 1) \wedge (\mu)(y) \wedge 1 \wedge \mu(z)\}\end{aligned}$$

$$= \bigvee_{p=(xs_1y)s_2z} \{\mu(x) \wedge (\mu)(y) \wedge \mu(z)\}. \tag{i}$$

Let $\mu(x) = t_1 < \mu(y) = t_2 < \mu(z) = t_3$. So (i) becomes $(\mu \circ f_S \circ \mu \circ f_S \circ \mu)(p) = t_1$. Implies $x \in \mu_{t_1}, y \in \mu_{t_2}, z \in \mu_{t_3}$ and $\mu_{t_1} \supseteq \mu_{t_2} \supseteq \mu_{t_3}$. Thus $x, y, z \in \mu_{t_1}$ and $p = xs_1ys_2z \in \mu_{t_1}S\mu_{t_1}S\mu_{t_1} \subseteq \mu_{t_1}$ implies $p \in \mu_{t_1}$. So $\mu(p) \geq t_1$. Thus $(\mu \circ f_S \circ \mu \circ f_S \circ \mu)(p) = t_1 \leq \mu(p)$. Hence $\mu \circ f_s \circ \mu \circ f_s \circ \mu \leq \mu$. ∎

Example 3. Let $\mathbf{Z}^- = S$ be the set of all negative integers. Then $\mathbf{Z}^-$ is a ternary semigroup. Let $B = 3S$. Then $BSBSB = 3SS3SS3S = 27(SSS)SS = 27(SSS) = 27S \subseteq 3S = B$. Hence B is a bi-ideal of $\mathbf{Z}^-$.

Define $\mu : S \longrightarrow [0,1]$ by $\mu(x) = \begin{cases} t, & \text{if } t \in 3S \\ 0, & \text{otherwise} \end{cases}$. For any $t \in [0,1]$, $\mu_t = \{x \in S \mid \mu(x) \geq t\} = \{3S\}$, since $\{3S\}$ is a bi-ideal in $\mathbf{Z}^-$, μ_t is the bi-ideal in $\mathbf{Z}^-$ for all t. Hence μ is a fuzzy bi-ideal of $\mathbf{Z}^-$.

Lemma 5. *Intersection of any collection of fuzzy bi-ideals of a ternary semigroup S is a fuzzy bi-ideal of S.*

Proof. Let a, b and c be any element of S.Then

$$\begin{aligned}
\text{(i)} \qquad (f \cap g)(abc) &= f(abc) \cap g(abc) \\
&\geq (f(a) \wedge f(b) \wedge f(c)) \wedge (g(a) \wedge g(b) \wedge g(c)) \\
&= (f(a) \wedge g(a)) \wedge (f(b) \wedge g(b)) \wedge (f(c) \wedge g(c)) \\
&= (f \cap g)(a) \wedge (f \cap g)(b) \wedge (f \cap g)(c) \\
(f \cap g)(abc) &\geq (f \cap g)(a) \wedge (f \cap g)(b) \wedge (f \cap g)(c) \\
\text{(ii)} \qquad (f \cap g)(abcde) &= f(abcde) \cap g(abcde) \\
&= f(abcde) \wedge g(abcde) \\
&\geq (f(a) \wedge f(c) \wedge f(e)) \wedge (g(a) \wedge g(c) \wedge g(e)) \\
&= (f(a) \wedge g(a)) \wedge (f(c) \wedge g(c)) \wedge (f(e) \wedge g(e)) \\
&= (f \cap g)(a) \wedge (f \cap g)(e) \wedge (f \cap g)(e).
\end{aligned}$$

Thus, $(f \cap g)(abcde) \geq (f \cap g)(a) \wedge (f \cap g)(e) \wedge (f \cap g)(e)$. Hence $f \cap g$ is a fuzzy bi-ideal of a ternary semigroup S. ∎

Remark 1. The union of any collection of fuzzy bi-ideals of a ternary semigroup S may not be a fuzzy bi-ideal of S.

Proof. Proof follows from the example bellow and Lemma 3. ∎

Example 4. [11] Let $S = \{0, 1, 2, 3, 4, 5\}$ and $abc = (a * b) * c$ for all $a, b, c \in S$, where $*$ is defined by the table

$*$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	1	1	1	1
2	0	1	2	3	1	1
3	0	1	1	1	2	3
4	0	1	4	5	1	1
5	0	1	1	1	4	5

Then S is a ternary semigroup with bi-ideals: $\{0\}, \{0,1\}, \{0,1,2\}, \{0,1,3\}$, $\{0,1,4\}, \{0,1,5\}$, $\{0,1,2,4\}, \{0,1,3,5\}, \{0,1,2,3\}, \{0,1,4,5\}$ and S. Here $\{0,1,2\}$ and $\{0,1,5\}$ are fuzzy bi-ideals but $\{0,1,2\} \cup \{0,1,5\} = \{0,1,2,5\}$ is not a fuzzy bi-ideal of S as $2.2.2.3.5 = 3$ is not in S. Hence union of two fuzzy bi-ideals of a ternary semigroup S may not be a fuzzy bi-ideal of S.

Proposition 2. *The following assertions on a ternary semigroup S are equivalent:*

(1) *S is regular,*

(2) *$u = u * f_S * u$ for every fuzzy bi-ideal u of S.*

Proof. Suppose S is a regular ternary semigroup and u be any fuzzy bi-ideals of S. Let $a \in S$, since S is regular so there exists $x \in S$ such that $a = axa$. Then we have

$$\begin{aligned}(u * f_S * u)(a) &= \bigvee_{a=bcd} \{u(b) \wedge f_S(c) \wedge u(d)\} \\ &\geq u(a) \wedge f_S(x) \wedge u(a) \\ &= u(a) \wedge 1 \wedge u(a) = u(a).\end{aligned}$$

This implies $u * f_S * u \geq u$ ... ... (**i**). Now since $u \leq u * f_S * u$, we have $u * f_S * u \leq (u * f_S * u) * f_S * u \leq u$, as u is a fuzzy bi-ideal of S. Thus $u * f_S * u \leq u$ ... ... (**ii**). From (**i**) and (**ii**) we get $u = u * f_S * u$.

Conversely, suppose that B is any bi-ideal of S. Let a be any element of B, then by Lemma 2 f_B is a fuzzy bi-ideal of S. Thus we have

$$f_B = f_B * f_S * f_B \ \text{ by supposition}$$
$$C_{BSB}(a) = (f_B * f_S * f_B(a) = f_B(a) = 1$$

So $a \in BSB$. Thus $B \subseteq BSB$. Also $BSB \subseteq (BSB)SB \subseteq B$, as B is a bi-ideal of S. Thus $B = BSB$. Hence S is regular ternary semigroup by [11]. ∎

Lemma 6. *Let f and g be any fuzzy subsets of a ternary semigroup S and h be a fuzzy bi-ideal of S. Then the product $f \circ g \circ h$, $g \circ f \circ h$ and $g \circ h \circ f$ are fuzzy bi-ideals of S.*

Proof.

$$(f \circ g \circ h)^3 = (f \circ g \circ h) \circ (f \circ g \circ h) \circ (f \circ g \circ h)$$

$$
\begin{aligned}
&= f \circ (g \circ h \circ f) \circ (g \circ h \circ f) \circ g \circ h \\
&\leq f \circ (f_S \circ f_S \circ f) \circ (f_S \circ f_S \circ f) \circ g \circ h \\
&= f \circ (f_S \circ f_S \circ f \circ f_S \circ f_S) \circ (f \circ g \circ h) \\
&\leq f \circ (f_S \circ f_S \circ f_S \circ f_S \circ f_S) \circ f \circ g \circ h \\
&\leq (f \circ f_S \circ f) \circ g \circ h \\
&\leq f \circ g \circ h \quad \text{since } f \circ f_S \circ f = f \\
(f \circ g \circ h)^3 &\leq f \circ g \circ h
\end{aligned}
$$

Hence $f \circ g \circ h$ is a fuzzy subsemigroup of S. Now we prove that $f \circ g \circ h$ is a fuzzy bi-ideal of S.

$$
\begin{aligned}
&(f \circ g \circ h) \circ f_S \circ (f \circ g \circ h) \circ f_S \circ (f \circ g \circ h) \\
&\qquad \leq f \circ (f_S \circ f_S \circ f_S) \circ f \circ (f_S \circ f_S \circ f_S) \circ (f \circ g \circ h) \\
&\qquad \leq (f \circ f_S \circ f \circ f_S \circ f) \circ g \circ h \\
&\qquad \leq f \circ g \circ h \qquad \text{since } f \text{ is a fuzzy bi-ideal of } S \\
&(f \circ g \circ h) \circ f_S \circ (f \circ g \circ h) \circ f_S \circ (f \circ g \circ h) \leq f \circ g \circ h.
\end{aligned}
$$

Similarly we can show that $g \circ f \circ h$ and $g \circ h \circ f$ are fuzzy bi-ideals of S. ■

Corollary 1. *Product of any three fuzzy bi-ideals of a ternary semigroup S is a fuzzy bi-ideal of S.*

Definition 10. Let $a \in S$ and t$\in (0,1]$, then the fuzzy subset a_t of S, defined by $a_t(x) = \begin{cases} t, & \text{if } x = a \\ 0, & \text{otherwise} \end{cases}$ for all $x \in S$, is called a *fuzzy point*. For any fuzzy subset f of S, it is clear that, $f = V_{a_t \leq f} a_t$. Let a_t, b_r and c_s be fuzzy points of S, then $a_t \circ b_r \circ c_s = (abc)_{t \wedge r \wedge s}$.

Definition 11. Let f be a fuzzy subset of a ternary semigroup S. Let $B(f)$ denote the intersection of all fuzzy bi-ideals of S which contain f. Then $B(f)$ is a bi-ideal of S, called the *fuzzy bi-ideal generated by f*.

Definition 12. Let a_t be a fuzzy point of a ternary semigroup S. Then the fuzzy bi-ideal generated by a_t denoted by $B(a_t)$ is defined at each $x \in S$ by $B(a_t)(x) = \begin{cases} t, & \text{if } x \in B(a) \\ 0, & \text{otherwise} \end{cases}$ where $B(a) = \{a\} \cup aS^1a$, and $S^1 = \begin{cases} S, & \text{if } S \text{ has identity} \\ S \cup \{1\}, & \text{otherwise.} \end{cases}$

4. Prime, strongly prime and semiprime fuzzy bi-ideals in ternary semigroups

Definition 13. A fuzzy bi-ideal f of a ternary semigroup S is called *prime* if $f_1 \circ f_2 \circ f_3 \leq f$ implies $f_1 \leq f$ or $f_2 \leq f$ or $f_3 \leq f$ for any fuzzy bi-ideals f_1, f_2, f_3 of S.

Definition 14. A fuzzy bi-ideal f of a ternary semigroup S is called *strongly prime* if $f_1 \circ f_2 \circ f_3 \wedge f_2 \circ f_3 \circ f_1 \wedge f_3 \circ f_1 \circ f_2 \leq f$ implies $f_1 \leq f$ or $f_2 \leq f$ or $f_3 \leq f$ for any fuzzy bi-ideals f_1, f_2, f_3 of S.

Definition 15. A fuzzy bi-ideal f of a ternary semigroup S is called *semiprime* if $f_1^3 \leq f$ implies $f_1 \leq f$ for any fuzzy bi-ideal f_1 of S.

Definition 16. A fuzzy bi-ideal f of a ternary semigroup S is called an *idempotent* if, $f.f.f = f^3 = f$.

Lemma 7. *Let A be a subset of a ternary semigroup S. Then A is a prime bi-ideal of S if and only if f_A (the characteristic function of A) is a prime fuzzy bi-ideal of S.*

Proof. Let A be a bi-ideal of S which is prime then by $lemma 3$ f_A is a fuzzy bi-ideal of S and let $f_{1,} f_2, f_3$, be fuzzy bi-ideals of S such that, $f_1 o f_2 o f_3 \leq f_A$. If $f_1 \nleq f_A$ and $f_2 \nleq f_A$, then there exist fuzzy points $\alpha_t \leq f_1$ $(t > 0)$ and $\beta_s \leq f_2$ $(s > 0)$ such that $\alpha_t \nleq f_A$ and $\beta_s \nleq f_A$. For any $\gamma_r \leq f_3$ (r$\neq 0$), since $B(\alpha_t) o B(\beta_s) o B(\gamma_r) \leq f_1 o f_2 o f_3 \leq f_A$, then for all $x \in S$ we have

$$B(\alpha_t) o B(\beta_s) o B(\gamma_r)(x) = \begin{cases} t \wedge s \wedge r > 0, & \text{if } x \in B(\alpha).B(\beta).B(\gamma) \\ 0, & \text{otherwise} \end{cases}$$

$\forall x \in S$, which implies $B(\alpha).B(\beta).B(\gamma) \subseteq A$ but A is a prime fuzzy bi-ideal of S, implies $B(\alpha) \subseteq A$ or $B(\beta) \subseteq A$ or $B(\gamma) \subseteq A$ but $\alpha_t \nleq f_A$ and $\beta_s \nleq f_A$ thus we have $f_3 = V_{\gamma_t \leq f_3} \gamma_t \leq f_{A,}$ shows that f_A is a prime fuzzy bi-ideal of S. Conversely, let B_1, B_2, and B_3 be bi-ideals of S and $B_1 B_2 B_3 \subseteq A$, then by Lemma 3, f_{B_1}, f_{B_2} and f_{B_3} are fuzzy bi-ideals of S and $f_{B_1} o f_{B_2} \circ f_{B_3} = f_{B_1 B_2 B_3} \leq f_A$ and f_A is prime fuzzy bi-ideal of S implies $f_{B_1} \leq f_A$ or $f_{B_2} \leq f_A$ or $f_{B_3} \leq f_A$ implies $B_1 \subseteq A$ or $B_2 \subseteq A$ or $B_3 \subseteq A$, thus A is prime bi-ideal of S. ∎

Lemma 8. *Let A be a subset of a ternary semigroup S.Then A is a strongly prime bi-ideal of S if and only if f_A(the characteristic function of A) is a strongly prime fuzzy bi-ideal of S.*

Proof. Straightforward from Lemma 7. ∎

Lemma 9. *Let A be a subset of a ternary semigroup S. Then A is a semiprime bi-ideal of S if and only if f_A(the characteristic function of A) is a semiprime fuzzy bi-ideal of S.*

Proof. Straightforward from Lemma 7. ∎

Proposition 3. *Every strongly prime fuzzy bi-ideal of a ternary semigroup S is a prime fuzzy bi-ideal of S.*

Proof. Let f be a strongly prime fuzzy bi-ideal of a ternary semigroup S. Now let f_1, f_2, f_3 be three fuzzy bi-ideals of S such that $f_1 \circ f_2 \circ f_3 \leq f$ implies

$f_1 \circ f_2 \circ f_3 \wedge f_2 \circ f_3 \circ f_1 \wedge f_3 \circ f_1 \circ f_2 \leq f$ implies $f_1 \leq f$ or $f_2 \leq f$ or $f_3 \leq f$ Thus f is a prime fuzzy bi-ideal of S. ■

PROPOSITION 4. *Every prime fuzzy bi-ideal of a ternary semigroup S is a semiprime fuzzy bi-ideal of S.*

Proof. Let f be a prime fuzzy bi-ideal of a ternary semigroup S. Now let f_1 be any fuzzy bi-ideal of S such that $f_1^3 \leq f$ implies $f_1 \leq f$. Thus f is a semiprime fuzzy bi-ideal of S. ■

REMARK 2. The converse to above two propositions is not true.

EXAMPLE 5. Consider $S = \{0, 1, 2\}$. Define ternary multiplication on S as

$\cdot$	0	1	2
0	0	0	0
1	0	1	1
2	0	2	2

Then S is a ternary semigroup. $\{0\}, \{0,1\}, \{0,2\}$ and $\{0,\ 1,\ 2\}$ are bi-ideals of S and these all are prime ideals of S. Here $\{0\}$ is a prime bi-ideal of S but it is not strongly prime as $(\{0,1\}\{0,2\}\{0,1,2\}) \cap (\{0,2\}\{0,1,2\}\{0,1\}) \cap (\{0,1,2\}\{0,1\}\{0,2\}) = \{0\} \subseteq \{0\}$ but $\{0,1\} \nsubseteq \{0\}$, and $\{0,2\} \nsubseteq \{0\}$, and $\{0,1,2\} \nsubseteq \{0\}$. Hence by Lemma 7, $f_{\{0\}}$ is a prime fuzzy bi-ideal of S but not a strongly prime fuzzy bi-ideal of S.

EXAMPLE 6. Let $0 \in S$ and $|S| > 3$. Then S is a ternary semigroup with zero, under the ternary operation defined by

$$abc = \begin{cases} a, & \text{if } a = b = c, \\ 0, & \text{otherwise,} \end{cases}$$

Since for all subsets A, B, C of S containing $\{0\}$ we have $ASASA = A$ and $ABC = A \cap B \cap C$, all these subsets are semiprime bi-ideal. Here a semiprime bi-ideal I of S such that $|S \backslash I| \geq 3$ is not a prime bi-ideal as for distinct $a, b, c \in S \backslash I$, we have $(I \cup \{a\})(I \cup \{b\})(I \cup \{c\}) = (I \cup \{a\}) \cap (I \cup \{b\}) \cap (I \cup \{c\}) = I$, but $(I \cup \{a\}) \nsubseteq I$ and $(I \cup \{b\}) \nsubseteq I$ and $(I \cup \{c\}) \nsubseteq I$. In particularly, $\{0\}$ is a semiprime bi-ideal but not a prime bi-ideal of S. Hence by Lemma 8 f_I (and $\mathrm{f}_{\{0\}}$)is a semiprime fuzzy bi-ideal of S but not a prime fuzzy bi-ideal of S.

LEMMA 10. *Minimum of any family of prime fuzzy bi-ideals of a ternary semigroup S is a semiprime fuzzy bi-ideal of S.*

Proof. Let $\{f_i : i \in I\}$ be a collection of prime fuzzy bi-ideals of a ternary semigroup S. Then $\bigwedge_{i \in I} f_i$ is a fuzzy bi-ideal of S, by Lemma 4. Let f be any fuzzy bi-ideal of S such that $f^3 \leq \bigwedge_{i \in I} f_i$. This implies $f^3 \leq f_i$ for all $i \in I$. So $f \leq f_i$ for all $i \in I$, because each f_i is a prime fuzzy bi-ideal of S. Thus $f \leq \bigwedge_{i \in I} f_i$. Hence $\bigwedge_{i \in I} f_i$ is a semiprime fuzzy bi-ideal of S. ■

5. Irreducible and strongly irreducible fuzzy bi-ideals

DEFINITION 17. A fuzzy bi-ideal f of a ternary semigroup S is called an *irreducible fuzzy bi-ideal* of S if $f_1 \wedge f_2 \wedge f_3 = f$ implies either $f_1 = f$ or $f_2 = f$ or $f_3 = f$ for any fuzzy bi-ideals f_1, f_2, f_3 of S.

DEFINITION 18. A fuzzy bi-ideal f of a ternary semigroup S is called a *strongly irreducible fuzzy bi-ideal* of S if $f_1 \wedge f_2 \wedge f_3 \leq f$ implies either $f_1 \leq f$ or $f_2 \leq f$ or $f_3 \leq f$ for any fuzzy bi-ideals f_1, f_2, f_3 of S.

REMARK 3. Every strongly irreducible bi-ideal of a ternary semigroup S is an irreducible bi-ideal but converse is not true in general.

EXAMPLE 7. Let $S = \{0,1,2,3,4,5\}$ and $abc = (a*b)*c$ for all $a, b, c \in S$, where $*$ is defined by the table in Example 4. Then S is a ternary semigroup with bi-ideals: $\{0\}$, $\{0,1\}$, $\{0,1,2\}$, $\{0,1,3\}$, $\{0,1,4\}$, $\{0,1,5\}$, $\{0,1,2,4\}$, $\{0,1,3,5\}$, $\{0,1,2,3\}$, $\{0,1,4,5\}$ and S. Here $\{0\}, \{0,1,2,4\}, \{0,1,3,5\}, \{0,1,2,3\}, \{0,1,4,5\}$ and S are irreducible but only $\{0\}$ and S are strongly irreducible and hence their characteristic functions gives same for irreducible and strongly irreducible fuzzy bi-ideals.

PROPOSITION 5. *Every strongly irreducible semiprime fuzzy bi-ideal of a ternary semigroup S is a strongly prime fuzzy bi-ideal of S.*

Proof. Let f be a strongly irreducible semiprime fuzzy bi-ideal of a ternary semigroup S. Let f_1, f_2, f_3 be any three fuzzy bi-ideals of S such that

$$f_1 \circ f_2 \circ f_3 \wedge f_2 \circ f_3 \circ f_1 \wedge f_3 \circ f_1 \circ f_2 \leq f \tag{i}$$

Then we have to show that either $f_1 \leq f$ or $f_2 \leq f$ or $f_3 \leq f$. As $f_1 \wedge f_2 \wedge f_3 \leq f_1$, $f_1 \wedge f_2 \wedge f_3 \leq f_2$ and $f_1 \wedge f_2 \wedge f_3 \leq f_3$ implies $(f_1 \wedge f_2 \wedge f_3)^3 \leq f_1 \circ f_2 \circ f_3$ and $(f_1 \wedge f_2 \wedge f_3)^3 \leq f_2 \circ f_3 \circ f_1$ and $(f_1 \wedge f_2 \wedge f_3)^3 \leq f_3 \circ f_1 \circ f_2$. Thus $(f_1 \wedge f_2 \wedge f_3)^3 \leq f_1 \circ f_2 \circ f_3 \wedge f_2 \circ f_3 \circ f_1 \wedge f_3 \circ f_1 \circ f_2 \leq f$ (using (i)). Implies $f_1 \wedge f_2 \wedge f_3 \leq f$, because f is a semiprime fuzzy bi-ideal of S. Thus $f_1 \leq f$ or $f_2 \leq f$ or $f_3 \leq f$, because f is a strongly irreducible fuzzy bi-ideal of S. Hence f is a strongly prime fuzzy bi-ideal of S. ∎

Note that the converse of above hold only if each fuzzy bi-ideal of S is idempotent as in Proposition 9.

PROPOSITION 6. *Let f be a fuzzy bi-ideal of a ternary semigroup S with $f(a) = \alpha$ where $a \in S$ and $\alpha \in [0,1]$. Then there exists an irreducible fuzzy bi-ideal g of S such that $f \leq g$ and $g(a) = \alpha$.*

Proof. Let $X = \{h : h \text{ is a fuzzy bi-ideal of } S,\ h(a) = \alpha \text{ and } f \leq h\}$, then $X \neq \Phi$, as $f \in X$. The collection X is a partially ordered set under inclusion. If $Y = \{h_i : h_i \text{ is a fuzzy bi-ideal of } S,\ h_i(a) = \alpha \text{ and } f \leq h_i \text{ for all } i \in I\}$ is any totally ordered subset of X, then $\bigvee_{i \in I} h_i$ is a fuzzy bi-ideal of S such that $f \leq \bigvee_{i \in I} h_i$. Indeed, if a, b, c, x, $y \in S$ then

$$\left(\bigvee_{i \in I} h_i\right)(abc) = \bigvee_{i \in I} \left(h_i\,(abc)\right)$$

$$\geq \bigvee_{i\in I}\left(h_i(a)\wedge h_i(b)\wedge h_i(c)\right) \quad \text{as each } h_i \text{ is a fuzzy bi-ideal of } S.$$
$$=\left(\bigvee_{i\in I} h_i(a)\right)\wedge\left(\bigvee_{i\in I} h_i(b)\right)\wedge\left(\bigvee_{i\in I} h_i(c)\right)$$
$$=\left(\bigvee_{i\in I} h_i\right)(a)\wedge\left(\bigvee_{i\in I} h_i\right)(b)\wedge\left(\bigvee_{i\in I} h_i\right)(c)$$

Now

$$\begin{aligned}\left(\bigvee_{i\in I} h_i\right)(axbyc) &= \bigvee_{i\in I}\left(h_i(axbyc)\right)\\ &\geq \bigvee_{i\in I}\left(h_i(a)\wedge h_i(b)\wedge h_i(c)\right) \quad \text{as each } h_i \text{ is a fuzzy bi-ideal of } S.\\ &=\left(\bigvee_{i\in I} h_i(a)\right)\wedge\left(\bigvee_{i\in I} h_i(b)\right)\wedge\left(\bigvee_{i\in I} h_i(c)\right)\\ &=\left(\bigvee_{i\in I} h_i\right)(a)\wedge\left(\bigvee_{i\in I} h_i\right)(b)\wedge\left(\bigvee_{i\in I} h_i\right)(c).\end{aligned}$$

Hence $\bigvee_{i\in I} h_i$ is a fuzzy bi-ideal of S. As $f \leq h_i$ for all $i \in I$. This implies $f \leq \bigvee_{i\in I} h_i$. Also $\left(\bigvee_{i\in I} h_i\right)(a) = \bigvee_{i\in I} h_i(a) = \alpha$. Thus $\bigvee_{i\in I} h_i \in X$ and $\bigvee_{i\in I} h_i$ is an upper bound of Y. Hence by Zorn's lemma, there exists a fuzzy bi-ideal g of S which is maximal with the property $f \leq g$ and $g(a) = \alpha$. Now we show that g is an irreducible fuzzy bi-ideal of S. For this, suppose that for any fuzzy bi-ideals g_1, g_2, g_3 of S, we have $g_1 \wedge g_2 \wedge g_3 = g$. This implies $g \leq g_1$ and $g \leq g_2$ and $g \leq g_3$. We claim that $g = g_1$ or $g = g_2$ or $g = g_3$. On contrary, suppose that $g \neq g_1$, $g \neq g_2$ and $g \neq g_3$. This implies $g < g_1$, $g < g_2$ and $g < g_3$. So $g_1(a) \neq \alpha$, $g_2(a) \neq \alpha$ and $g_3(a) \neq \alpha$, as $g(a) = \alpha$. Hence $(g_1 \wedge g_2 \wedge g_3)(a) = g_1(a) \wedge g_2(a) \wedge g_3(a) \neq \alpha$, which is a contradiction to the fact that $g_1(a) \wedge g_2(a) \wedge g_3(a) = g(a) = \alpha$. Hence either $g = g_1$ or $g = g_2$ or $g = g_3$. Thus g is an irreducible fuzzy bi-ideal of S. ∎

THEOREM 2. *For a regular ternary semigroup S, the following assertions are equivalent:*

(1) *Every fuzzy bi-ideal of S is idempotent.*

(2) $f_1 \circ f_2 \circ f_3 \wedge f_2 \circ f_3 \circ f_1 \wedge f_3 \circ f_1 \circ f_2 = f_1 \wedge f_2 \wedge f_3$ *for any fuzzy bi-ideals f_1, f_2, f_3 of S.*

(3) *Each fuzzy bi-ideal of S is semiprime.*

(4) *Each proper fuzzy bi-ideal of S is the intersection of irreducible semiprime fuzzy bi-ideals of S which contain it.*

Proof. (1) $\Rightarrow$ (2) Let f_1, f_2 and f_3 be three fuzzy bi-ideals of S. Then by Lemma 4, $f_1 \wedge f_2 \wedge f_3$ is also a fuzzy bi-ideal of S. Thus by hypothesis, we have $f_1 \wedge f_2 \wedge f_3 = (f_1 \wedge f_2 \wedge f_3) \circ (f_1 \wedge f_2 \wedge f_3) \circ (f_1 \wedge f_2 \wedge f_3) \leq f_1 \circ f_2 \circ f_3$. Similarly $f_1 \wedge f_2 \wedge f_3 \leq f_2 \circ f_3 \circ f_1$ and $f_1 \wedge f_2 \wedge f_3 \leq f_3 \circ f_1 \circ f_2$. Implies

$f_1 \wedge f_2 \wedge f_3 \leq f_1 \circ f_2 \circ f_3 \wedge f_2 \circ f_3 \circ f_1 \wedge f_3 \circ f_1 \circ f_2$. Now $f_1 \circ f_2 \circ f_3$, $f_2 \circ f_3 \circ f_1$ and $f_3 \circ f_1 \circ f_2$ being the products of three fuzzy bi-ideals of S, are fuzzy bi-ideals of S, by Lemma 5. Also $f_1 \circ f_2 \circ f_3 \wedge f_2 \circ f_3 \circ f_1 \wedge f_3 \circ f_1 \circ f_2$ is a fuzzy bi-ideal of S by Lemma 4. Thus by hypothesis we have

$$\begin{aligned} f_1 \circ f_2 \circ f_3 \wedge f_2 \circ f_3 \circ f_1 \wedge f_3 \circ f_1 \circ f_2 &= (f_1 \circ f_2 \circ f_3 \wedge f_2 \circ f_3 \circ f_1 \wedge f_3 \circ f_1 \circ f_2)^3 \\ &\leq (f_1 \circ f_2 \circ f_3) \circ (f_3 \circ f_1 \circ f_2) \circ (f_2 \circ f_3 \circ f_1) \\ &\leq f_1 \circ (f_S \circ f_S \circ f_S) \circ f_1 \circ (f_S \circ f_S \circ f_S) \circ f_1 \\ &\leq f_1 \circ f_S \circ f_1 \circ f_S \circ f_1 \leq f_1 \end{aligned}$$

Similarly $f_1 \circ f_2 \circ f_3 \wedge f_2 \circ f_3 \circ f_1 \wedge f_3 \circ f_1 \circ f_2 \leq f_2$ and $f_1 \circ f_2 \circ f_3 \wedge f_2 \circ f_3 \circ f_1 \wedge f_3 \circ f_1 \circ f_2 \leq f_3$. Thus $f_1 \circ f_2 \circ f_3 \wedge f_2 \circ f_3 \circ f_1 \wedge f_3 \circ f_1 \circ f_2 \leq f_1 \wedge f_2 \wedge f_3$. Hence $f_1 \circ f_2 \circ f_3 \wedge f_2 \circ f_3 \circ f_1 \wedge f_3 \circ f_1 \circ f_2 = f_1 \wedge f_2 \wedge f_3$.

(2) $\Rightarrow$ (1) Let f be a fuzzy bi-ideal of S, then by hypothesis $f = f \wedge f \wedge f = f \circ f \circ f \wedge f \circ f \circ f \wedge f \circ f \circ f = f \circ f \circ f$.

(1) $\Rightarrow$ (3) Let f be a fuzzy bi-ideal of S such that $f_1^3 \leq f$ for any fuzzy bi-ideal f_1 of S. Then by hypothesis, we have $f_1 = f_1^3 \leq f$. Hence every fuzzy bi-ideal of S is a semiprime fuzzy bi-ideal of S.

(3) $\Rightarrow$ (4) Let f be a proper fuzzy bi-ideal of S and $\{f_i : i \in I\}$ be the collection of all irreducible fuzzy bi-ideals of S such that $f \leq f_i$ for all $i \in I$. This implies $f \leq \bigwedge_{i \in I} f_i$.

Let $a \in S$, then by Proposition 6, there exists an irreducible fuzzy bi-ideal f_α of S such that $f \leq f_\alpha$ and $f(a) = f_\alpha(a)$. This implies $f_\alpha \in \{f_i : i \in I\}$. Thus $\bigwedge_{i \in I} f_i \leq f_\alpha$. So $\bigwedge_{i \in I} f_i(a) \leq f_\alpha(a) = f(a)$ for all $a \in S$. This implies $\bigwedge_{i \in I} f_i \leq f$. Hence $\bigwedge_{i \in I} f_i = f$. By hypothesis, each fuzzy bi-ideal of S is semiprime. Thus each fuzzy bi-ideal of S is the minimum of all irreducible semiprime fuzzy bi-ideals of S which contain it.

(4) $\Rightarrow$ (1) Let f be a fuzzy bi-ideal of S. Then by the definition of fuzzy bi-ideal we have, $f^3 = f \circ f \circ f \leq f$. Also $f^3 = f \circ f \circ f$, being the product of three fuzzy bi-ideals of S is a fuzzy bi-ideal of S, by Lemma 5. Then by hypothesis $f^3 = \bigwedge_{i \in I} f_i$, where each f_i is an irreducible semiprime fuzzy bi-ideal of S such that $f^3 \leq f_i$ for all $i \in I$. This implies $f \leq f_i$ for all $i \in I$, because each f_i is a semiprime fuzzy bi-ideal of S. Thus $f \leq \bigwedge_{i \in I} f_i = f^3$. Hence $f^3 = f$. ∎

PROPOSITION 7. *If each fuzzy bi-ideal of a ternary semigroup S is idempotent, then a fuzzy bi-ideal f of S is strongly irreducible if and only if f is strongly prime.*

Proof. Let f is a strongly irreducible fuzzy bi-ideal of S and let f_1, f_2, f_3 be any three fuzzy bi-ideals of S such that $f_1 \circ f_2 \circ f_3 \wedge f_2 \circ f_3 \circ f_1 \wedge f_3 \circ f_1 \circ f_2 \leq f$. By Theorem 2, we have $f_1 \wedge f_2 \wedge f_3 = f_1 \circ f_2 \circ f_3 \wedge f_2 \circ f_3 \circ f_1 \wedge f_3 \circ f_1 \circ f_2 \leq f$. But f is a strongly irreducible fuzzy bi-ideal of S. Thus we have $f_1 \leq f$ or $f_2 \leq f$ or $f_3 \leq f$. Hence f is a strongly prime fuzzy bi-ideal of S. Conversely suppose that f is a strongly prime fuzzy bi-ideal of S and let f_1, f_2, f_3 be any fuzzy bi-ideals of S such that $f_1 \wedge f_2 \wedge f_3 \leq f$. By Theorem 2, we have $f_1 \circ f_2 \circ f_3 \wedge f_2 \circ f_3 \circ f_1 \wedge f_3 \circ f_1 \circ f_2 =$

$f_1 \wedge f_2 \wedge f_3 \leq f$. But f is a strongly prime fuzzy bi-ideal of S. Thus we have $f_1 \leq f$ or $f_2 \leq f$ or $f_3 \leq f$. Hence f is a strongly irreducible fuzzy bi-ideal of S. ∎

Next we characterize those ternary semigroups in which each fuzzy bi-ideal is strongly prime and also those ternary semigroups in which each fuzzy bi-ideal is strongly irreducible.

THEOREM 3. *Each fuzzy bi-ideal of a regular ternary semigroup S is strongly prime if and only if each fuzzy bi-ideal of S is idempotent and the set of fuzzy bi-ideals of S is totally ordered by inclusion.*

Proof. Suppose each fuzzy bi-ideal of S is strongly prime. This implies that each fuzzy bi-ideal of S is semiprime. By Theorem 2, each fuzzy bi-ideal of S is idempotent. Now we show that the set of fuzzy bi-ideals of S is totally ordered by inclusion. For this let f_1, f_2 be two fuzzy bi-ideals of S, then by Theorem 2, we have $f_1 \wedge f_2 = f_1 \wedge f_2 \wedge f_S = f_1 \circ f_2 \circ f_S \wedge f_2 \circ f_S \circ f_1 \wedge f_S \circ f_1 \circ f_2$, implies $f_1 \circ f_2 \circ f_S \wedge f_2 \circ f_S \circ f_1 \wedge f_S \circ f_1 \circ f_2 \leq f_1 \wedge f_2$. By hypothesis, f_1 and f_2 are strongly prime fuzzy bi-ideals of S, so is $f_1 \wedge f_2$. Then $f_1 \leq f_1 \wedge f_2$ or $f_2 \leq f_1 \wedge f_2$ or $f_S \leq f_1 \wedge f_2$. Thus $f_1 \leq f_2$ or $f_2 \leq f_1$. Hence the set of fuzzy bi-ideals of S is totally ordered by inclusion. Conversely, assume that each fuzzy bi-ideal of S is idempotent and the set of fuzzy bi-ideals of S is totally ordered by inclusion. We show that each fuzzy bi-ideal of S is strongly prime. Let f is an arbitrary fuzzy bi-ideal of S and f_1, f_2, f_3 be any fuzzy bi-ideals of S such that $f_1 \circ f_2 \circ f_3 \wedge f_2 \circ f_3 \circ f_1 \wedge f_3 \circ f_1 \circ f_2 \leq f$. By Theorem 2, we have $f_1 \wedge f_2 \wedge f_3 = f_1 \circ f_2 \circ f_3 \wedge f_2 \circ f_3 \circ f_1 \wedge f_3 \circ f_1 \circ f_2 \leq f$ $\ldots$ $\ldots$ (**i**). Since the set of fuzzy bi-ideals of S is totally ordered by inclusion, so for f_1, f_2, f_3 we have the following six possibilities:

(**ii**) $f_1 \leq f_2 \leq f_3$	(**iii**) $f_1 \leq f_3 \leq f_2$	(**iv**) $f_2 \leq f_3 \leq f_1$
(**v**) $f_2 \leq f_1 \leq f_3$	(**vi**) $f_3 \leq f_1 \leq f_2$	(**vii**) $f_3 \leq f_2 \leq f_1$.

In these cases we have

(**ii**) $f_1 \wedge f_2 \wedge f_3 = f_1$	(**iii**) $f_1 \wedge f_2 \wedge f_3 = f_1$	(**iv**) $f_1 \wedge f_2 \wedge f_3 = f_2$
(**v**) $f_1 \wedge f_2 \wedge f_3 = f_2$	(**vi**) $f_1 \wedge f_2 \wedge f_3 = f_3$	(**vii**) $f_1 \wedge f_2 \wedge f_3 = f_3$.

Thus (**i**) gives, either $f_1 \leq f$ or $f_2 \leq f$ or $f_3 \leq f$. Hence f is strongly prime. ∎

THEOREM 4. *If the set of fuzzy bi-ideals of a regular ternary semigroup S is totally ordered, then each fuzzy bi-ideal of S is idempotent if and only if each fuzzy bi-ideal of S is prime.*

Proof. Suppose that each fuzzy bi-ideal of S is idempotent and let f be an arbitrary fuzzy bi-ideal of S and f_1, f_2, f_3 be any fuzzy bi-ideals of S such that $f_1 \circ f_2 \circ f_3 \leq f$. As the set of fuzzy bi-ideals of S is totally ordered, then for f_1, f_2, f_3 we have the following six possibilities:

(**i**) $f_1 \leq f_2 \leq f_3$	(**ii**) $f_1 \leq f_3 \leq f_2$	(**iii**) $f_2 \leq f_3 \leq f_1$
(**iv**) $f_2 \leq f_1 \leq f_3$	(**v**) $f_3 \leq f_1 \leq f_2$	(**vi**) $f_3 \leq f_2 \leq f_1$.

For (**i**) and (**ii**) we have $f_1^3 = f_1 \circ f_1 \circ f_1 \leq f_1 \circ f_2 \circ f_3 \leq f$, implies $f_1 \leq f$, as f_1 is idempotent. Similarly for other possibilities we have $f_2 \leq f$ or $f_3 \leq f$. Conversely, suppose that each fuzzy bi-ideal of S is prime, so is semiprime, by Proposition 4. Thus by Theorem 2, each fuzzy bi-ideal of S is idempotent. ■

PROPOSITION 8. *If the set of fuzzy bi-ideals of a ternary semigroup S is totally ordered, then the concept of primeness and strongly primeness coincides.*

Proof. Let f be a prime fuzzy bi-ideal of S and let f_1, f_2, f_3 be any fuzzy bi-ideals of S such that $f_1 \circ f_2 \circ f_3 \wedge f_2 \circ f_3 \circ f_1 \wedge f_3 \circ f_1 \circ f_2 \leq f$. As the set of fuzzy bi-ideals of a ternary semigroup S is totally ordered, then for f_1, f_2, f_3 we have the following six possibilities:

(i)	$f_1 \leq f_2 \leq f_3$	**(ii)**	$f_1 \leq f_3 \leq f_2$	**(iii)**	$f_2 \leq f_3 \leq f_1$
(iv)	$f_2 \leq f_1 \leq f_3$	**(v)**	$f_3 \leq f_1 \leq f_2$	**(vi)**	$f_3 \leq f_2 \leq f_1$.

For (i) and (ii) we have $f_1^3 = f_1^3 \wedge f_1^3 \wedge f_1^3 \leq f_1 \circ f_2 \circ f_3 \wedge f_2 \circ f_3 \circ f_1 \wedge f_3 \circ f_1 \circ f_2 \leq f$, implies $f_1 \leq f$, as f is a prime fuzzy bi-ideal of S. Similarly for other possibilities we have $f_2 \leq f$ or $f_3 \leq f$. This shows that f is a strongly prime fuzzy bi-ideal of S. Thus every prime fuzzy bi-ideal of S is a strongly prime fuzzy bi-ideal of S. Also every strongly prime fuzzy bi-ideal of S is a prime fuzzy bi-ideal of S by Proposition 3. Thus the concept of primeness and strongly primeness coincides. ■

THEOREM 5. *For a ternary semigroup S, the following assertions are equivalent:*

(1) *The set of fuzzy bi-ideals of S is totally ordered by set inclusion.*

(2) *Each fuzzy bi-ideal of S is strongly irreducible.*

(3) *Each fuzzy bi-ideal of S is irreducible.*

Proof. (1) $\Rightarrow$ (2) Let the set of fuzzy bi-ideals of S is totally ordered by set inclusion. Assume that f is an arbitrary fuzzy bi-ideal of S and f_1, f_2, f_3 be any fuzzy bi-ideals of S such that $f_1 \wedge f_2 \wedge f_3 \leq f$. Since the set of fuzzy bi-ideals of S is totally ordered by set inclusion, therefore either $f_1 \wedge f_2 \wedge f_3 = f_1$ or f_2 or f_3. Thus either $f_1 \leq f$ or $f_2 \leq f$ or $f_3 \leq f$. So f is strongly irreducible. Hence each fuzzy bi-ideal of S is strongly irreducible.

(2) $\Rightarrow$ (3) Suppose each fuzzy bi-ideal of S is strongly irreducible and let f is an arbitrary fuzzy bi-ideal of S and f_1, f_2, f_3 be any fuzzy bi-ideals of S such that $f_1 \wedge f_2 \wedge f_3 = f$. This implies $f \leq f_1$ or $f \leq f_2$ or $f \leq f_3$. On the other hand, by hypothesis we have $f_1 \leq f$ or $f_2 \leq f$ or $f_3 \leq f$. Hence either $f_1 = f$ or $f_2 = f$ or $f_3 = f$. Thus f is an irreducible fuzzy bi-ideal of S. Hence each fuzzy bi-ideal of S is irreducible.

(3) $\Rightarrow$ (1) Let each fuzzy bi-ideal of S is irreducible. To show that the set of fuzzy bi-ideals of S is totally ordered by set inclusion, let f_1 and f_2 be any two fuzzy bi-ideals of S, then by Lemma 4, $f_1 \wedge f_2$ is also a fuzzy bi-ideal of S and so is irreducible fuzzy bi-ideal of S. Since $f_1 \wedge f_2 \wedge f_S = f_1 \wedge f_2$, implies $f_1 = f_1 \wedge f_2$ or $f_2 = f_1 \wedge f_2$ or $f_S = f_1 \wedge f_2$. Hence either $f_1 \leq f_2$ or $f_2 \leq f_1$ or $f_1 = f_2 = f_S$. Thus the set of fuzzy bi-ideals of S is totally ordered by set inclusion. ■

Definition 19. Let S be a ternary semigroup, β be the set of all fuzzy bi-ideals of S and ρ be the set of all strongly prime proper fuzzy bi-ideals of S. Define for each $f \in \beta$

$$\theta_f = \{g \in \rho : f \nleq g\} \qquad \tau(\rho) = \{\theta_f : f \in \beta\}.$$

Theorem 6. *$\tau(\rho)$ forms a topology.*

Proof. Since $\{0\}$ is a bi-ideal of S, therefore $f_{\{0\}}$ is a fuzzy bi-ideal of S, by Lemma 3. Then $\theta_{f_{\{0\}}} = \left\{g \in \rho : f_{\{0\}} \nleq g\right\} = \Phi$ (the empty set) $\in \tau(\rho)$. Also S is a bi-ideal of S, therefore f_S is a fuzzy bi-ideal of S, by Lemma 3. Then $\theta_{f_S} = \{g \in \rho : f_S \nleq g\} = \rho \in \tau(\rho)$. Thus Φ, $\rho \in \tau(\rho)$. Now we show that intersection of finite number of members of $\tau(\rho)$ belongs to $\tau(\rho)$. For this, let θ_{f_1}, $\theta_{f_2} \in \tau(\rho)$, then we have to show that $\theta_{f_1} \cap \theta_{f_2} \in \tau(\rho)$. For this we show that $\theta_{f_1} \cap \theta_{f_2} = \theta_{f_1 \wedge f_2} \in \tau(\rho)$. Let $g \in \theta_{f_1} \cap \theta_{f_2}$, implies $g \in \theta_{f_1}$ and $g \in \theta_{f_2}$. So $g \in \rho$ and $f_1 \nleq g$ and $f_2 \nleq g$, by the definition of θ_{f_1} and θ_{f_2}. Thus $f_1 \wedge f_2 \nleq g$, implies $g \in \theta_{f_1 \wedge f_2}$. Hence $\theta_{f_1} \cap \theta_{f_2} \subseteq \theta_{f_1 \wedge f_2}$. Now let $g \in \theta_{f_1 \wedge f_2}$, this implies $g \in \rho$ and $f_1 \wedge f_2 \nleq g$, by the definition of $\theta_{f_1 \wedge f_2}$. Implies $f_1 \nleq g$ and $f_2 \nleq g$. So $g \in \theta_{f_1}$ and $g \in \theta_{f_2}$, implies $g \in \theta_{f_1} \cap \theta_{f_2}$. Thus $\theta_{f_1 \wedge f_2} \subseteq \theta_{f_1} \cap \theta_{f_2}$. Hence $\theta_{f_1} \cap \theta_{f_2} = \theta_{f_1 \wedge f_2} \in \tau(\rho)$. Now we show that union of any number of members of $\tau(\rho)$ belong to $\tau(\rho)$. For this let $\{\theta_{f_\alpha} : \alpha \in I\} \subseteq \tau(\rho)$. Then we have to show that $\bigcup_\alpha \theta_{f_\alpha} \in \tau(\rho)$. It follows from

$$\begin{aligned}\bigcup_\alpha \theta_{f_\alpha} &= \{g \in \rho : f_\alpha \nleq g \text{ for some } \alpha \in I\} \\ &= \left\{g \in \rho : \bigvee_\alpha^{\Lambda} f_\alpha \nleq g\right\} = \theta_{\bigvee_\alpha^{\Lambda} f_\alpha} \in \tau(\rho).\end{aligned}$$

where $\bigvee_\alpha^{\Lambda} f_\alpha$ is the minimum of all fuzzy bi-ideals of S which are greater than or equal to $\bigvee_\alpha f_\alpha$. ■

Remark 4. Each fuzzy subset f of the ternary semigroup S is a fuzzy bi-ideal of S if and only if $f(0) \geq f(x)$ for all $x \in S$.

Proof. Let f be a fuzzy bi-ideal of S, then by definition

$$f(0) = f(x0x0x) \geq f(x) \wedge f(x) \wedge f(x) = f(x) \qquad \text{for all } x \in S.$$

Conversely, suppose that f satisfies $f(0) \geq f(x)$ for all $x \in S$. Then we have to show that f is a fuzzy bi-ideal of S. As $xyz = x$ if x, y, $z \in \{a, b\}$ and $xyz = 0$ if one of x, y, z is zero. Thus $f(xyz) \geq f(x) \wedge f(y) \wedge f(z)$ for all $x, y, z \in S$. Now as $vwxyz = v$ if $v, w, x, y, z \in \{a, b\}$ and $vwxyz = 0$ if one of v, w, x, y, z is zero. Thus $f(vwxyz) \geq f(v) \wedge f(x) \wedge f(z)$ for all $v, w, x, y, z \in S$. ■

Remark 5. As $x^3 = x$ for all $x \in S$, implies S is regular ternary semigroup. Now if f is a fuzzy bi-ideal of S then $f \circ f \circ f \leq f$. Also for $x \in S$, we have $(f \circ f \circ f)(x) = \bigvee_{x=abc} [f(a) \wedge f(b) \wedge f(c)] \geq f(x) \wedge f(x) \wedge f(x) = f(x)$, because $x^3 = x$ for all $x \in S$. So $f \circ f \circ f \geq f$. Thus $f \circ f \circ f = f$, implies f is idempotent. Hence every fuzzy bi-ideal of S is idempotent and so is semiprime. But every fuzzy bi-ideal of S is not prime.

Example 8. Consider the fuzzy bi-ideals f, g, h and k of a ternary semigroup $S = \{0, a, b\}$ given by

$$\begin{array}{lll} k(0) = .7 & k(a) = .6 & k(b) = .4 \\ f(0) = 1 & f(a) = .5 & f(b) = .3 \\ g(0) = .7 & g(a) = .65 & g(b) = .3 \\ h(0) = 1 & h(a) = .5 & h(b) = .3. \\ \text{Then } f \circ g \circ h(0) = .7 & f \circ g \circ h(a) = .5 & f \circ g \circ h(b) = .3 \end{array}$$

Here $f \circ g \circ h \leq k$ but neither $f \leq k$, $g \leq k$ nor $h \leq k$. Hence k is not a prime fuzzy bi-ideal of S.

Example 9. Consider $S = \{0, x, 1\}$. Define the operation of multiplication on S as

$\cdot$	0	x	1
0	0	0	0
x	0	x	x
1	0	x	1

Then $(S, \cdot)$ is a ternary semigroup. It is evident that S is commutative and regular. Bi-ideals of S are $\{0\}$, $\{0, x\}$ and S. All bi-ideals are strongly prime.

Remark 6. Each fuzzy subset f of the ternary semigroup S is a fuzzy bi-ideal of S if and only if $f(0) \geq f(x) \geq f(1)$.

Proof. Let f be a fuzzy bi-ideal of S, then by definition $f(0) = f(a0a0a) \geq f(a) \wedge f(a) \wedge f(a) = f(a)$ for all $a \in S$. Also $f(x) = f(1x1x1) \geq f(1) \wedge f(1) \wedge f(1) = f(1)$. Hence $f(0) \geq f(x) \geq f(1)$.

Conversely, assume that f satisfies $f(0) \geq f(x) \geq f(1)$. Then obviously f is a fuzzy ternary subsemigroup of S. Also $abcde = x$ if $a, b, c, d, e \in \{x, 1\}$ and one of a, b, c, d, e is x. Then $f(abcde) = f(x) \geq f(a) \wedge f(c) \wedge f(e)$ and $abcde = 0$ if one of a, b, c, d, e is 0 implies $f(abcde) = f(0) \geq f(a) \wedge f(c) \wedge f(e)$, and if ($abcde = 1$ if $a = b = c = d = e = 1$) then $f(abcde) = f(1) = f(a) \wedge f(c) \wedge f(e)$. Thus $f(abcde) \geq f(a) \wedge f(c) \wedge f(e)$ for all a, b, c, d, $e \in S$. Hence f is a fuzzy bi-ideal of S. Now we show that every fuzzy bi-ideal of S is not a strongly prime fuzzy bi-ideal of S. Consider the fuzzy bi-ideals f, g, h and k of S given by

$$\begin{array}{lll} k(0) = .7 & k(a) = .6 & k(b) = .4 \\ f(0) = 1 & f(a) = .5 & f(b) = .3 \\ g(0) = .7 & g(a) = .65 & g(b) = .3 \\ h(0) = 1 & h(a) = .5 & h(b) = .3 \end{array}$$

Then

$$f \circ g \circ h(0) = g \circ h \circ f(0) = h \circ f \circ g(0) = .7$$
$$f \circ g \circ h(x) = g \circ h \circ f(x) = h \circ f \circ g(x) = .5$$
$$f \circ g \circ h(0) = g \circ h \circ f(1) = h \circ f \circ g(1) = .3$$

This implies

$$\begin{aligned} f \circ g \circ h\,(0) \wedge g \circ h \circ f\,(0) \wedge h \circ f \circ g\,(0) &= .7 \\ f \circ g \circ h\,(x) \wedge g \circ h \circ f\,(x) \wedge h \circ f \circ g\,(x) &= .5 \\ \text{and} \quad f \circ g \circ h\,(1) \wedge g \circ h \circ f\,(1) \wedge h \circ f \circ g\,(1) &= .3 \end{aligned}$$

Thus $f \circ g \circ h \wedge g \circ h \circ f \wedge h \circ f \circ g \leq k$ but neither $f \leq k$, $g \leq k$ nor $h \leq k$. Hence k is not a strongly prime fuzzy bi-ideal of S. ■

REFERENCES

[**1**] J. Ahsan, K.Y. Li, M. Shabir, *Semigroups characterized by their fuzzy bi-ideals*, J. Fuzzy Systems Math. **9** (1995), 441–449.

[**2**] A.H. Clifford, G.B. Preston, *The Algebraic Theory of Semigroups*, Part I, American Mathematical Society, 1961.

[**3**] V.N. Dixit, Sarita Dewan, *A note on quasi and bi-ideals in ternary semigroups*, Internat. J. Math. Math. Sci. **18** (1995), 501–508.

[**4**] J. Kavikumar, A.B. Khamis, *Fuzzy ideals and fuzzy quasi-ideals in ternary semirings*, IAENG Intern. J. Appl. Math. **37** (2007), 102–106.

[**5**] J. Kavikumar, A. Khamis, Y.B. Jun, *Fuzzy Bi-ideals in ternary semirings*, Intern. J. Math. Stat. Sci. **1** (2009), 54–58.

[**6**] D.H. Lehmer, *A ternary analogue of abelian groups*, Amer. J. Math. (1932), 329–338.

[**7**] J.N. Mordeson, D.S. Malik, N. Kuroki, *Fuzzy Semigroups*, Springer, Berlin, 2003.

[**8**] O. Steinfeld, *Quasi-ideals in rings and semigroups*, Akademia Kiado, Budapest, 1978.

[**9**] F.M. Sioson, *Ideal theory in ternary semigroups*, Math. Japonica **10** (1965), 63–84.

[**10**] M. Shabir, N. Kanwal, *Prime bi-ideals of semigroups*, Southeast Asian Bull. Math. **31** (2007), 757–764.

[**11**] M. Shabir, M. Bano, *Prime bi-ideals in ternary semigroups*, Quasigroups and Related Systems **16** (2008), 239–256.

[**12**] M. Shabir, S. Bashir, *Prime ideals in ternary semigroups*, Asian-European J. Math. **2** (2009), 141–154.

[**13**] M. Shabir, Y.B. Jun, M. Bano, *On prime fuzzy bi-ideals of semigroups*, submitted.

[**14**] M.A. Rezvi, J. Mahmood, *A class of fuzzy ideals in ternary semigroups*, submitted.

[**15**] L.A. Zadeh, *Fuzzy sets*, Information and Control **8** (1965), 338–353.

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