VOLTERRA TYPE OPERATORS FROM WEIGHTED
HARDY SPACES TO BLOCH SPACES

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Abstract. Let $H(\mathbb{D})$ denote the space of all analytic functions on the unit disk $\mathbb{D}$ of $\mathbb{C}$. In this paper we consider the following Volterra type operator

$$J_g(f)(z) = \int_0^z f(\xi)g'(\xi)\,d\xi, \quad f \in H(\mathbb{D}),\ z \in \mathbb{D}.$$

The boundedness and compactness of the operator J_g from the weighted Hardy space to a Bloch space are studied.

1. Introduction

Let $\mathbb{D}$ be the unit disk of complex plane $\mathbb{C}$, and $H(\mathbb{D})$ the class of functions analytic in $\mathbb{D}$. Recall that an $f \in H(\mathbb{D})$ is said to belong to the Bloch space $\mathcal{B}$ if

$$\|f\|_b = \sup_{z \in \mathbb{D}}(1 - |z|^2)|f'(z)| < \infty.$$

With the norm $\|f\|_{\mathcal{B}} = |f(0)| + \|f\|_b$, $\mathcal{B}$ is a Banach space. Let $\mathcal{B}_0$ be the space which consists of all $f \in \mathcal{B}$ satisfying

$$\lim_{|z| \to 1}(1 - |z|^2)|f'(z)| = 0.$$

This space is called the little Bloch space.

Throughout this paper, we assume that $\{\beta(n)\}_{n=0}^{\infty}$ is a sequence of positive numbers such that

$$\beta(0) = 1,\ \liminf_{n \to \infty}\beta(n)^{1/n} = 1 \text{ and } \sum_{n=0}^{\infty} 1/(\beta(n))^2 = \infty.$$

The weighted Hardy space, denoted by $H^2(\beta)$, is defined to be the set of all $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D})$ such that

$$\|f\|_{H^2(\beta)}^2 = \sum_{n=0}^{\infty} |a_n|^2 (\beta(n))^2 < \infty.$$

It is clear that $H^2(\beta)$ is a Hilbert space on $\mathbb{D}$ with the inner product given by

$$\langle f, g \rangle = \sum_{n=0}^{\infty} a_n \overline{b_n} (\beta(n))^2,$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ are in $H^2(\beta)$. Some well-known special cases of this type of Hilbert space are, the Hardy space H^2 with weights $\beta(n) \equiv 1$, the Bergman space A^2 with weights $\beta(n) = (n+1)^{-1/2}$, and the Dirichlet space $\mathcal{D}$ with weights $\beta(n) = (n+1)^{1/2}$ for all n. See [4] for more details of the weighted Hardy space.

Suppose that $g \in H(\mathbb{D})$. The integral operator

$$J_g f(z) = \int_0^z f(\xi) g'(\xi) \, d\xi, \quad z \in \mathbb{D}, \tag{1}$$

was introduced by Pommerenke in [12] and is called the Volterra type operator (see [13]).

In [12], Pommerenke showed that J_g is bounded on the Hardy space H^2 if and only if $g \in BMOA$. The boundedness and compactness of J_g between some spaces of analytic functions, as well as their n-dimensional extensions on the unit ball in $\mathbb{C}^n$, were investigated in [1–3,5–9,11,14–20] (see also the related references therein).

In this paper, we study the operator J_g from the weighted Hardy space to Bloch space. Some sufficient and necessary conditions for the operator J_g to be bounded and compact are given.

Throughout the paper, constants are denoted by C, they are positive and may not be the same in every occurrence. The notation $a \asymp b$ means that there is a positive constant C such that $C^{-1}b \le a \le Cb$.

2. Main results and proofs

In this section, we give our main results and their proofs. Before stating these results, we need some auxiliary results, which are incorporated in the lemmas which follow.

LEMMA 1. *Assume that $g \in H(\mathbb{D})$. Then $T_g : H^2(\beta) \to \mathcal{B}$ is compact if and only if $T_g : H^2(\beta) \to \mathcal{B}$ is bounded and for any bounded sequence $\{f_k\}_{k \in \mathbb{N}}$ in $H^2(\beta)$ which converges to zero uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$, we have $\|T_g f_k\|_{\mathcal{B}} \to 0$ as $k \to \infty$.*

The proof of Lemma 1 follows by standard arguments (see, for example, Proposition 3.11 of [4]). Hence, we omit the details.

LEMMA 2. [10] *A closed set K in $\mathcal{B}_0$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1}\sup_{f\in K}(1-|z|^2)|f'(z)| = 0. \tag{2}$$

LEMMA 3. *Let $f \in H^2(\beta)$. Then*

$$|f(z)| \leq \|f\|_{H^2(\beta)}\sqrt{\sum_{n=0}^{\infty}\frac{|z|^{2n}}{\beta^2(n)}}.$$

Proof. For $w \in \mathbb{D}$, define $K_w(z) = \sum_{n=0}^{\infty}\frac{\overline{w}^n}{\beta^2(n)}z^n$. Then $K_w \in H^2(\beta)$. Let $f(z) = \sum_{n=0}^{\infty}a_n z^n$. From [4, p. 16], we see that

$$f(w) = \langle f, K_w\rangle = \sum_{n=0}^{\infty}\frac{a_n w^n}{\beta^2(n)}\beta^2(n) = \sum_{n=0}^{\infty}a_n w^n \tag{3}$$

and

$$\|K_w\|_{H^2(\beta)} = \sqrt{\sum_{n=0}^{\infty}\frac{|w|^{2n}}{\beta^4(n)}(\beta^2(n))} = \sqrt{\sum_{n=0}^{\infty}\frac{|w|^{2n}}{\beta^2(n)}} < \infty. \tag{4}$$

Then the desired result follows from (3) and (4). ∎

Now we are in a position to state and prove our main results.

THEOREM 1. *Assume that $g \in H(\mathbb{D})$. Then the following statements are equivalent.*

(i) The operator $T_g : H^2(\beta) \to \mathcal{B}$ is bounded;

(ii) The operator $T_g : H^2(\beta) \to \mathcal{B}_0$ is bounded;

$$(iii) \qquad M := \sup_{z\in\mathbb{D}}(1-|z|^2)|g'(z)|\sqrt{\sum_{n=0}^{\infty}\frac{|z|^{2n}}{\beta^2(n)}} < \infty. \tag{5}$$

Proof. $(ii) \Rightarrow (i)$. It is obvious.

$(i) \Rightarrow (iii)$. Assume that $T_g : H^2(\beta) \to \mathcal{B}$ is bounded. For $a \in \mathbb{D}$, set

$$f_a(z) = \sum_{n=0}^{\infty}\frac{\overline{a}^n z^n}{\beta^2(n)}\left(\sum_{n=0}^{\infty}\frac{|a|^{2n}}{\beta^2(n)}\right)^{-1/2}. \tag{6}$$

It is easy to see that $f_a \in H^2(\beta)$ and $\sup_{a\in\mathbb{D}}\|f_a\|_{H^2(\beta)} = 1$. We have

$$\infty > \|T_g f_a\|_{\mathcal{B}} = \sup_{z\in\mathbb{D}}(1-|z|^2)|(T_g f_a)'(z)|$$

$$= \sup_{z\in\mathbb{D}}(1-|z|^2)|g'(z)||f_a(z)|$$

$$\geq (1-|a|^2)|g'(a)|\sqrt{\sum_{n=0}^{\infty}\frac{|a|^{2n}}{\beta^2(n)}}, \tag{7}$$

which implies (5).

$(iii) \Rightarrow (ii)$. Assume that (5) holds. Then, for any $f \in H^2(\beta)$, by Lemma 3 we have

$$(1 - |z|^2)|(T_g f)'(z)| = (1 - |z|^2)|g'(z)||f(z)|$$

$$\leq (1 - |z|^2)|g'(z)|\sqrt{\sum_{n=0}^{\infty} \frac{|z|^{2n}}{\beta^2(n)}} \|f\|_{H^2(\beta)}. \qquad (8)$$

Taking the supremum in (8) over $\mathbb{D}$ and using the condition (5), we see that $T_g : H^2(\beta) \to \mathcal{B}$ is bounded.

Since we assume $\sum_{n=0}^{\infty} 1/(\beta(n))^2 = \infty$, (5) implies that $g \in \mathcal{B}_0$. Then, for each polynomial $p(z)$, we have that

$$(1 - |z|^2)|(T_g p)'(z)| = (1 - |z|^2)|g'(z)p(z)| \leq \|p\|_\infty (1 - |z|^2)|g'(z)|,$$

from which it follows that $T_g p \in \mathcal{B}_0$. Since the set of all polynomials is dense in $H^2(\beta)$ (see [4]), we have that for every $f \in H^2(\beta)$ there is a sequence of polynomials $\{p_k\}_{k \in \mathbb{N}}$ such that $\|f - p_k\|_{H^2(\beta)} \to 0$, as $k \to \infty$. Hence

$$\|T_g f - T_g p_k\|_{\mathcal{B}} \leq \|T_g\|_{H^2(\beta) \to \mathcal{B}} \|f - p_k\|_{H^2(\beta)} \to 0$$

as $k \to \infty$, since $T_g : H^2(\beta) \to \mathcal{B}$ is bounded. Since $\mathcal{B}_0$ is a closed subset of $\mathcal{B}$, we obtain $T_g(H^2(\beta)) \subseteq \mathcal{B}_0$. Therefore $T_g : H^2(\beta) \to \mathcal{B}_0$ is bounded. The proof is finished. ∎

THEOREM 2. *Assume that $g \in H(\mathbb{D})$. Then the following statements are equivalent.*

(i) *The operator $T_g : H^2(\beta) \to \mathcal{B}$ is compact;*

(ii) *The operator $T_g : H^2(\beta) \to \mathcal{B}_0$ is compact;*

$$(iii) \qquad \lim_{|z| \to 1} (1 - |z|^2)|g'(z)|\sqrt{\sum_{n=0}^{\infty} \frac{|z|^{2n}}{\beta^2(n)}} = 0. \qquad (9)$$

Proof. $(ii) \Rightarrow (i)$. It is obvious.

$(i) \Rightarrow (iii)$. Assume that $T_g : H^2(\beta) \to \mathcal{B}$ is compact. Let $\{z_k\}_{k \in \mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $\lim_{k \to \infty} |z_k| = 1$ (if such a sequence does not exist (9) is automatically satisfied). Set

$$f_k(z) = \sum_{n=0}^{\infty} \frac{\overline{z_k}^n z^n}{\beta^2(n)} \left(\sum_{n=0}^{\infty} \frac{|z_k|^{2n}}{\beta^2(n)} \right)^{-1/2}, \quad k \in \mathbb{N}. \qquad (10)$$

It is easy to see that $\sup_{k \in \mathbb{N}} \|f_k\|_{H^2(\beta)} < \infty$ and $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$. By Lemma 1,

$$\lim_{k \to \infty} \|T_g f_k\|_{\mathcal{B}} = 0. \qquad (11)$$

In addition, we have

$$\|T_g f_k\|_{\mathcal{B}} = \sup_{z \in \mathbb{D}} (1 - |z|^2)|g'(z)| \, |f_k(z)|$$

$$\geq (1 - |z_k|^2)|g'(z_k)| \sqrt{\sum_{n=0}^{\infty} \frac{|z_k|^{2n}}{\beta^2(n)}}, \tag{12}$$

which together with (11) implies that

$$\lim_{k \to \infty} (1 - |z_k|^2)|g'(z_k)| \sqrt{\sum_{n=0}^{\infty} \frac{|z_k|^{2n}}{\beta^2(n)}} = 0.$$

This proves that (9) holds.

$(iii) \Rightarrow (ii)$. Assume that (9) holds. From Theorem 1, we see that $T_g : H^2(\beta) \to \mathcal{B}_0$ is bounded. Let $f \in H^2(\beta)$. From the proof of Theorem 1, we have that

$$(1 - |z|^2)|(T_g f)'(z)| \leq (1 - |z|^2)|g'(z)| \sqrt{\sum_{n=0}^{\infty} \frac{|z|^{2n}}{\beta^2(n)}} \|f\|_{H^2(\beta)}.$$

Taking the supremum in the above inequality over all $f \in H^2(\beta)$ such that $\|f\|_{H^2(\beta)} \leq 1$, then letting $|z| \to 1$, by (9) it follows that

$$\lim_{|z| \to 1} \sup_{\|f\|_{H^2(\beta)} \leq 1} (1 - |z|^2)|(T_g f)'(z)| = 0.$$

From this and by employing Lemma 2, we see that $T_g : H^2(\beta) \to \mathcal{B}_0$ is compact. The proof is completed. ∎

Let $\beta(n) = (n+1)^{-1/2}$. Then

$$\sum_{n=0}^{\infty} \frac{|z|^{2n}}{\beta^2(n)} = \sum_{n=0}^{\infty} (n+1)|z|^{2n} \asymp \frac{1}{(1 - |z|^2)^2}.$$

From Theorems 1–2 and the last formula, we have the following two corollaries.

COROLLARY 3. *Assume that $g \in H(\mathbb{D})$. Then the following statements are equivalent.*

(i) The operator $T_g : A^2 \to \mathcal{B}$ is bounded;

(ii) The operator $T_g : A^2 \to \mathcal{B}_0$ is bounded;

(iii) $\sup_{z \in \mathbb{D}} |g'(z)| < \infty$.

COROLLARY 4. *Assume that $g \in H(\mathbb{D})$. Then the following statements are equivalent.*

(i) The operator $T_g : A^2 \to \mathcal{B}$ is compact;

(ii) The operator $T_g : A^2 \to \mathcal{B}_0$ is compact;

(iii) g is a constant.

Let $\beta(n) \equiv 1$. Then

$$\sum_{n=0}^{\infty} \frac{|z|^{2n}}{\beta^2(n)} = \sum_{n=0}^{\infty} |z|^{2n} = \frac{1}{1-|z|^2}.$$

From Theorems 1–2 and the last formula, we have the following two corollaries.

COROLLARY 5. *Assume that $g \in H(\mathbb{D})$. Then the following statements are equivalent.*

(i) *The operator $T_g : H^2 \to \mathcal{B}$ is bounded;*

(ii) *The operator $T_g : H^2 \to \mathcal{B}_0$ is bounded;*

(iii) $\sup_{z \in \mathbb{D}} \sqrt{1-|z|^2}|g'(z)| < \infty.$

COROLLARY 6. *Assume that $g \in H(\mathbb{D})$. Then the following statements are equivalent.*

(i) *The operator $T_g : H^2 \to \mathcal{B}$ is compact;*

(ii) *The operator $T_g : H^2 \to \mathcal{B}_0$ is compact;*

(iii) $\lim_{|z| \to 1} \sqrt{1-|z|^2}|g'(z)| = 0.$

Let $\beta(n) = (n+1)^{1/2}$. Then

$$\sum_{n=0}^{\infty} \frac{|z|^{2n}}{\beta^2(n)} = \sum_{n=0}^{\infty} \frac{|z|^{2n}}{n+1} \asymp \ln \frac{e}{1-|z|^2}.$$

From Theorems 1–2 and the last formula, we have the following two corollaries.

COROLLARY 7. *Assume that $g \in H(\mathbb{D})$. Then the following statements are equivalent.*

(i) *The operator $T_g : \mathcal{D} \to \mathcal{B}$ is bounded;*

(ii) *The operator $T_g : \mathcal{D} \to \mathcal{B}_0$ is bounded;*

$$(iii) \qquad \sup_{z \in \mathbb{D}} (1-|z|^2)|g'(z)| \sqrt{\ln \frac{e}{1-|z|^2}} < \infty.$$

COROLLARY 8. *Assume that $g \in H(\mathbb{D})$. Then the following statements are equivalent.*

(i) *The operator $T_g : \mathcal{D} \to \mathcal{B}$ is compact;*

(ii) *The operator $T_g : \mathcal{D} \to \mathcal{B}_0$ is compact;*

$$(iii) \qquad \lim_{|z| \to 1} (1-|z|^2)|g'(z)| \sqrt{\ln \frac{e}{1-|z|^2}} = 0.$$

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