МАТЕМАТИЧКИ ВЕСНИК
45 (1993), 61–64

UDC 517.984.4
оригинални научни рад
research paper

GENERALIZED EIGENVECTOR EXPANSION FOR WEAKLY PERTURBATED DISCRETE OPERATORS

M. Dostanić

Abstract. In this paper we consider the expansion theorem in generalized eigenvectors of the operator $A = L + T$, where L is a discrete, positive selfadjoint operator in a separable Hilbert space, and T is a closed operator which is subordinated to L in a certain sense.

Let $\mathcal{H}$ be a separable Hilbert space over $\mathbf{C}$ and let L be a discrete, positive selfadjoint operator on $\mathcal{H}$. Vector $x \neq 0$ is a generalized eigenvector (for the eigenvalue λ) if for some $k \geqslant 1$ $(\lambda - L)^k x = 0$. Denote by $N(\cdot)$ the eigenvalue distribution function of L. Let $\mathcal{D}(L)$ and $\mathcal{D}(T)$ denote the domain of the operators L and T, respectively.

In this paper we consider the expansion theorem for the operator $A = L + T$, where T is a closed operator which is subordinated to L in a certain sense.

In the case when T is a bounded operator, $L = L^*$ is a discrete operator and $\lambda_{n+1}(L) - \lambda_n(L) \to \infty$ $(n \to \infty)$ the problem was solved in [3].

THEOREM 1. *Suppose that T is a closed operator on $\mathcal{H}$, $L = L^*$ is a positive discrete operator, $\mathcal{D}(L) \subset \mathcal{D}(T)$, $A = L + T$,*

$$\|Tx\| \leqslant C\|L^\beta x\|, \quad x \in \mathcal{D}(L), \tag{1}$$

and numbers α and β satisfy one of the following two conditions:
a) $0 < \beta < 1$, $0 < \alpha < \frac{2}{3}(1-\beta)$ and $N(t) = C_0 t^\alpha (1 + o(1))$ $(t \to +\infty)$;
b) $0 < \beta < 1$, $0 < \alpha < 1 - \beta$ and $N(t) = C_0 t^\alpha (1 + O(t^{-\delta}))$, $\alpha < \delta < 1$ $(t \to +\infty)$.
Then for every $f \in \mathcal{D}(L)$ we have

$$f = \sum_{k=1}^{\infty} \left(\sum_{s=1}^{n_k} c_{ks} x_{ks} \right), \tag{2}$$

where x_{ks} are generalized eigenvectors of A and $c_{ks} \in \mathbf{C}$.

Proof. Suppose that $\{e_n\}_{n=1}^{\infty}$ is the system of eigenvectors of L $(Le_n = \lambda_n e_n)$. Since $L = L^*$, $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis of $\mathcal{H}$. Then

$$(L - \lambda)^{-1} = \sum_{n=1}^{\infty} \frac{(\,\cdot\,, e_n) e_n}{\lambda_n - \lambda}$$

and

$$T(L-\lambda)^{-1}=\sum_{n=1}^{\infty}\frac{(\,\cdot\,,e_n)Te_n}{\lambda_n-\lambda}. \tag{3}$$

From (1) and (3), applying Cauchy's inequality, we conclude that

$$\|T(L-\lambda)^{-1}\|\leqslant C^{1/2}\Big(\sum_{n=1}^{\infty}\frac{\lambda_n^{2\beta}}{|\lambda-\lambda_n|^2}\Big)^{1/2}. \tag{4}$$

By the following Lemma, the righthandside of this inequality tends to zero if λ belongs to a certain sequence of circles with radii tending to infinity.

LEMMA. *If either of the conditions a) and b) of the Theorem 1 is satisfied, then there exists a sequence of circles* $\Gamma_k=\{\,\lambda:|\lambda|=r_k\,\}$, $\lim_{k\to\infty}r_k=\infty$, *such that*

$$\lim_{k\to\infty}\max_{\lambda\in\Gamma_k}\Big(\sum_{\nu=1}^{\infty}\frac{\lambda_\nu^{2\beta}}{|\lambda-\lambda_\nu|^2}\Big)=0. \tag{5}$$

Since $\lim_{n\to\infty}\max_{\lambda\in\Gamma_n}\|T(\lambda-L)^{-1}\|=0$ (follows from (4) and the Lemma), it follows from $(\lambda-A)^{-1}=(\lambda-L)^{-1}(I-T(\lambda-L)^{-1})^{-1}$ that the operator A is discrete and

$$\lim_{k\to\infty}\max_{\lambda\in\Gamma_k}\|(\lambda-A)^{-1}\|=0. \tag{6}$$

From (6) and Naymark's theorem [4] we obtain the relation (2), for all $f\in\mathcal{D}(L)$, where x_{ks}, $s=1,2,\ldots,n_k$, are the generalized eigenvectors corresponding to eigenvalues lying in the ring $\{\,\lambda:r_k<|\lambda|<r_{k+1}\,\}$. ∎

REMARK. In the case when in each interval I of the fixed length l the number of eigenvalues λ of A with property $\operatorname{Re}\lambda\in I$ is uniformly bounded, the Riesz basis property of the generalized eigenvectors system was proved in [1] (under some aditional conditions).

Proof of the Lemma. Case a). It follows from $N(t)=C_0t^\alpha(1+o(1))$ that $\lambda_n=C_0^{-1/\alpha}n^{1/\alpha}(1+o(1))$. Let q be a real number such that

$$0<\alpha q<C_0^{-1/\alpha}. \tag{7}$$

Denote by S the set of natural numbers n such that $\lambda_{n+1}-\lambda_n\geqslant qn^{1/\alpha-1}$. Suppose that S is finite, i.e. $S=\{n_1,n_2,\ldots,n_s\}$. Then we have $\lambda_{n+1}-\lambda_n<qn^{1/\alpha-1}$ for all $n>n_s+1$ and

$$\lambda_{N+1}-\lambda_{n_s+1}<q\sum_{\nu=n_s+1}^{N}\nu^{1/\alpha-1}<q\int_{n_s+1}^{N+1}x^{1/\alpha-1}dx=\alpha q[(N+1)^{1/\alpha}-(n_s+1)^{1/\alpha}],$$

i.e.

$$\frac{\lambda_{N+1}-\lambda_{n_s+1}}{N^{1/\alpha}}\leqslant\alpha q\frac{(N+1)^{1/\alpha}-(n_s+1)^{1/\alpha}}{N^{1/\alpha}}$$

for each $N>n_s$. When $N\to\infty$ we obtain $C_0^{-1/\alpha}\leqslant\alpha q$, i.e. a contradiction with (7). So, it follows that S is an infinite set.

Let $\Gamma_\nu = \{\,\lambda\,:\,|\lambda| = r_\nu = \frac{1}{2}(\lambda_{n_\nu+1} + \lambda_{n_\nu})\,\}$. We will prove now the realtion (5). If $\lambda \in \Gamma_k$, then

$$\sum_{\nu=1}^{\infty} \frac{\lambda_\nu^{2\beta}}{|\lambda - \lambda_\nu|^2} \leqslant \sum_{\nu=1}^{\infty} \frac{\lambda_\nu^{2\beta}}{(r_k - \lambda_\nu)^2}$$
$$= \sum_{\nu=1}^{n_k-1} \frac{\lambda_\nu^{2\beta}}{(r_k - \lambda_\nu)^2} + \sum_{\nu=n_k+2}^{\infty} \frac{\lambda_\nu^{2\beta}}{(r_k - \lambda_\nu)^2} + \frac{\lambda_{n_k}^{2\beta}}{(r_k - \lambda_{n_k})^2} + \frac{\lambda_{n_k+1}^{2\beta}}{(r_k - \lambda_{n_k+1})^2}.$$

As we have $0 < \alpha < \frac{2}{3}(1-\beta)$, by direct computation we get

$$\lim_{k\to\infty} \left[\frac{\lambda_{n_k}^{2\beta}}{(r_k - \lambda_{n_k})^2} + \frac{\lambda_{n_k+1}^{2\beta}}{(r_k - \lambda_{n_k+1})^2} \right] = 0. \tag{8}$$

Since the function $\varphi(x) = x^\beta/(r_k - x)$ is nondecreasing on $[0, r_k)$, we obtain

$$\sum_{\nu=1}^{n_k-1} \frac{\lambda_\nu^{2\beta}}{(r_k - \lambda_\nu)^2} \leqslant \text{const}\cdot n_k \frac{\lambda_{n_k}^{2\beta}}{(r_k - \lambda_{n_k})^2} \leqslant \frac{\text{const}}{n_k^{\frac{2}{3}-3-\frac{2\beta}{\alpha}}} \to 0\ (k\to\infty). \tag{9}$$

Since

$$\sum_{\nu=n_k+2}^{\infty} \frac{\lambda_\nu^{2\beta}}{(r_k - \lambda_\nu)^2} = \int_{\lambda_{n_k+1}}^{\infty} \frac{t^{2\beta}}{(r_k - t)^2}\, dN(t)$$
$$= \frac{n_k \lambda_{n_k}^{2\beta}}{(r_k - \lambda_{n_k+1})^2} - \int_{\lambda_{n_k+1}}^{\infty} N(t) \left(\frac{t^{2\beta}}{(r_k - t)^2} \right)' dt,$$

it is enough to prove that

$$\lim_{k\to\infty} \int_{\lambda_{n_k+1}}^{\infty} t^\alpha \left(\frac{t^{2\beta}}{(r_k - t)^2} \right)' dt = 0. \tag{10}$$

The function $G(x) = \int_x^\infty [(\beta-1)u - \beta]/(u-1)^3\, du$ $(x > 1)$ has the following asymptotical behavior in the neighborhood of $x = 1$: $G(x) \sim \frac{1}{2}(x-1)^{-2}$. Then (10) follows from

$$\int_{\lambda_{n_k+1}}^{\infty} t^\alpha \left(\frac{t^{2\beta}}{(r_k - t)^2} \right)' dt = 2 r_k^{\alpha+2\beta-2} G(c_k) \sim \frac{r_k^{\alpha+2\beta}}{(\lambda_{n_k+1} - r_k)^2} \to 0\ (k\to\infty),$$

where $c_k = \lambda_{n_k+1}/r_k$ $(\to 1)$. From (8), (9) and (10) we obtain (5).

Case b). It follows from b) that

$$\lambda_n = C_0^{-1/\alpha} n^{1/\alpha}(1 + O(n^{-\delta/\alpha})). \tag{11}$$

Let $\mu_n = C_0^{-1/\alpha} n^{1/\alpha}$ and $\Gamma_n = \{\,\lambda\,:\,|\lambda| = r_n = \frac{1}{2}(\mu_n + \mu_{n+1})\,\}$. From (11) we get

$$\sup_{n,\nu} \left| \frac{\lambda_\nu - \mu_\nu}{r_n - \lambda_\nu} \right| < \infty. \tag{12}$$

If $\lambda \in \Gamma_n$, then from (12) we obtain

$$\sum_{\nu=1}^{\infty} \frac{\lambda_\nu^{2\beta}}{|\lambda - \lambda_\nu|^2} \leqslant \text{const} \sum_{\nu=1}^{\infty} \frac{\mu_\nu^{2\beta}}{(r_n - \mu_\nu)^2}.$$

As in the case a) it can be proved that

$$\sum_{\nu=1}^{\infty} \frac{\mu_\nu^{2\beta}}{(r_n - \mu_\nu)^2} \to 0 \quad (n \to \infty)$$

for $0 < \alpha < 1 - \beta$. The Lemma is proved. ■

EXAMPLE. Suppose m, n and r are integers, $m \geqslant 1$, $n \geqslant 2$, $0 < r < m$, Ω is a bounded domain in $\mathbf{R}^n$ with sufficiently smooth boundary, L is a formal selfadjoint eliptic differential expression

$$L = (-1)^{m/2} \sum_{|k|=m} a_k(x) D^k$$

with smooth coefficients and T is a linear differential expression

$$T = \sum_{|k| \leqslant r} b_k(x) D^k$$

with smooth complex functions b_k. Let $A\colon \mathcal{D}(A) \to L^2(\Omega)$ $(\mathcal{D}(A) = W_2^m \cup \overset{\circ}{W}{}_2^{m/2})$ be a differential operator defined by $A = L + T$. Then we get

THEOREM 2. *If $n/m < \frac{2}{3}(1 - r/m)$, the for $f \in \mathcal{D}(A)$ the expansion theorem in generalized eigenvectors of the operator A holds.*

Proof. The statement of the theorem is obtained from Theorem 1 for $\alpha = n/m$, $\beta = r/m$ (see [2]). ■

REFERENCES

[1] В. Г. Долголаптев, *О базисности корневых векторов слабо возмущеных операторов*, Мат. заметкы, **34**, 6 (1983), 867–872

[2] В. Э. Кацнельсон, *О сходимости и суммируемости рядов по корневым векторам некоторых классов несамосопряженных операторов*, Канд. дис, Харков 1967

[3] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, 1966

[4] R. D. Richtmyer, *Principles of Advanced Mathematical Physics*, Vol. I, Springer-Verlag 1978

(received 21 06 1993)