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Weighted Ergodic Theorems Along Subsequences of Density Zero

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Abstract. We consider subsequence versions of weighted ergodic theorems, and show that for a wide class of subsequences along which a.e. convergence of Cesaro averages has been established, we also have a.e. convergence for the subsequence Cesaro weighted averages, when the weights are obtained from uniform sequences produced by a connected apparatus.

Contents

1. Introduction

Let $(X, \mathcal{F}, \mu)$ be a probability space. For T a linear contraction of $L_p(X, \mathcal{F}, \mu) = L_p$, $p \geq 1$, various ergodic theorems consider the a.e. convergence of the averages $\frac{1}{N}\sum_{k=1}^{N} T^k f(x)$ for every $f \in L_p$. More generally, for $\{n_k\}$ an increasing sequence of positive integers, various authors have considered the a.e. convergence of averages of the form $\frac{1}{N}\sum_{k=1}^{N} T^{n_k} f(x)$. When $\{n_k\}$ has positive density, this convergence can be represented (e.g., [3]) as convergence of weighted averages $\frac{1}{N}\sum_{k=1}^{N} a(k)T^k f(x)$, with $\{a(k)\}$ a 0-1 sequence. We will be interested in subsequence versions of these weighted averages. That is, for a sequence $\{a(k)\}$ of complex numbers for which the weighted averages converge, we will be interested in studying the almost everywhere convergence of subsequence averages of the form

$$\frac{1}{N}\sum_{k=1}^{N} a(n_k) T^{n_k} f. \tag{1}$$

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The types of integer sequences $\{n_k\}$ which we will be interested in are those for which we have a.e. convergence of the unweighted Cesaro averages in the measure preserving case, that is, for fixed $p > 1$, sequences for which the averages $\frac{1}{N}\sum_{k=1}^{N} f(\tau^{n_k}x)$ converge a.e. for all f in L_p, and for all measure preserving transformations τ. Such sequences include the sequence $\{n_k\}$ where $n_k = k^2$, or, more generally, $n_k = k^t$, t a positive integer, or $n_k = k$-th prime, or any of the sequences studied in [5] or [20], as well as a variety of other sequences. We will call such sequences *good universal in* L_p. If $\{n_k\}$ is good universal in L_p, and also has the property that for every measure preserving transformation τ on a non-atomic probability space, the maximal operator $f^\star(x) = \sup_N \frac{1}{N}\sum_{k=1}^{N} |f(\tau^{n_k}x)|$ is strong type (p,p) (that is, $\|f^\star\|_p \le c_p\|f\|_p$ for every $f \in L_p$), then we will say that the sequence $\{n_k\}$ is *strongly good universal in* L_p.

The types of operators we will consider are those induced by measure preserving point transformations (i.e., $Tf = f \circ \tau$, where τ is a measure preserving point transformation of X), Dunford-Schwartz operators (i.e., linear operators of L_p, all p, $1 \le p \le \infty$ such that $\|T\|_\infty \le 1$ and $\|T\|_1 \le 1$), and positively dominated contractions of L_p, p fixed, $1 < p < \infty$ (i.e., an operator T of L_p such that there is a positive operator S of L_p norm less than or equal to one that takes non-negative functions to non-negative functions and $|Tf(x)| \le S|f|(x)$ a.e.).

When the limit of the averages given in (1) exists for all $f \in L_p$ for a particular sequence $\{n_k\}$, a particular sequence of weights $\{a(k)\}$, and all T in some class $\mathcal{C}$ of operators of L_p, we will say that $\{a(k)\}$ is a *good weight sequence along* $\{n_k\}$ *for* $\mathcal{C}$ *on* L_p. In this terminology, we know that the sequence $\{a(k) = 1\}$ is a good weight sequence along $\{n_k\}$ for $\mathcal{C}$ when $\{n_k\}$ is good universal in L_p, $p > 1$, and $\mathcal{C}$ is the class of measure preserving transformations. Moreover, for the sequences $\{n_k\}$ mentioned earlier and $p > 1$, we can enlarge the class $\mathcal{C}$ to include the operators mentioned above ([4], [5], [20], [11], [13]). We will investigate how much of this is true for some other previously considered sequences of weights $\{a(k)\}$, which are good weights along $\{n_k = k\}$.

2. Besicovitch Weights

For a sequence of complex numbers $\{a(k)\}$, define for $1 \le p < \infty$, the *p semi-norms*

$$\|\{a(k)\}\|_p = (\limsup_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} |a(k)|^p)^{\frac{1}{p}}.$$

If $\{a(k)\}$ is defined by $a(k) = \sum_{j=1}^{m} b_j\lambda_j^k$, where λ_j, $j = 1, \ldots, m$ are complex numbers of modulus one and b_j are complex numbers , we call $\{a(k)\}$ a *trigonometric polynomial.* The p-Besicovitch sequences will be the closure in the p-semi-norm of the trigonometric polynomials. Besicovitch sequences, as good sequences of weights, have already been extensively studied. We give just a few of the references that contain some of the results we will need ([17], [12], [15], [19], [3]).

Note that the p-semi-norm of a bounded sequence does not change if the values of the sequence are changed, in a bounded way, on a subsequence of the integers of density zero. It is clear that a set of trigonometric polynomials can be used to approximate bounded functions that exhibit any behavior whatever along a sequence of density zero. Therefore, we cannot in general expect a Besicovitch sequence $\{a(k)\}$ to be a good weight sequence along good universal sequences $\{n_k\}$ of density zero.

In [15], Besicovitch sequences defined only on subsets of the integers are introduced. In the terminology introduced there, the class $B_{p,\{n_k\}}$ of p, $\{n_k\}$-Besicovitch sequences is defined to be the closure of the trigonometric polynomials in the p semi-norm defined by

$$\|\{a(k)\}\|_{p,\{n_k\}}^p = \limsup_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} |a(n_k)|^p .$$

Since the integers are an abelian group, the closure of the trigonometric polynomials is the same as the closure of the almost periodic functions on the integers (see [15]).

Unfortunately, when we consider good universal sequences of *zero density*, the measures on the integers induced by such sequences $\{n_k\}$, i.e., the measures μ_N that give the measure $\frac{1}{N}$ to the first N terms of the sequence $\{n_k\}$ and zero to the rest of the integers, are not ergodic. Hence, most of the results of [15] will not apply. We do have, however, that for a fixed $\{n_k\}$, all the $B_{p,\{n_k\}}$ classes contain the same bounded sequences, that is, $B_{p,\{n_k\}} \cap \ell_\infty = B_{1,\{n_k\}} \cap \ell_\infty$ for all p, $1 \le p < \infty$ ([15], Theorem 2.1). We will refer to this class as bounded $\{n_k\}$-Besicovitch sequences.

Routine arguments give the following results.

Theorem 2.1. *Fix p, $1 \le p < \infty$. If $\{n_k\}$ is a strongly good universal sequence in L_p, then the bounded $\{n_k\}$-Besicovitch sequences are good weight sequences along $\{n_k\}$ for all Dunford-Schwartz operators on L_p.*

Proof. We only sketch the proof. More details of the argument can be found in the proof of Theorem 1.2 in [12]. By (the proof of) Theorem 4.1 in [11], the constant sequence $\{a(k) = 1\}$ is a good weight sequence along any strongly good universal sequence, for Dunford-Schwartz operators in L_p (p is fixed). Thus the constant sequence is a good weight sequence along $\{n_k\}$ for operators of the form λT, where λ is a complex number with $|\lambda| = 1$, since these operators are Dunford-Schwartz as well. We then have convergence a.e. for the averages given by (1) when $\{a(k)\}$ is a trigonometric polynomial.

Let $\{a(k)\}$ be a bounded $\{n_k\}$-Besicovitch sequence. Fix $f \in L_\infty$. If $\{b(k)\}$ is a trigonometric polynomial with $\|a(k) - b(k)\|_{1,\{n_k\}} < \epsilon$, then we have a.e.

$$\limsup_{N\to\infty} \left|\frac{1}{N}\sum_{k=1}^{N} a(n_k)T^{n_k} f - \frac{1}{N}\sum_{k=1}^{N} b(n_k)T^{n_k} f\right| \le \epsilon \|f\|_\infty.$$

Since $\epsilon > 0$ is arbitrary, we obtain a.e. convergence of the weighted averages given by (1) for f bounded, using the convergence for trigonometric polynomials. Since $\{a(k)\}$ is bounded and $\{n_k\}$ is good universal for L_p, we have for $f \in L_p$

$$\left|\sup_N \frac{1}{N}\sum_{k=1}^{N} a(n_k)T^{n_k} f\right| \le \|\{a(k)\}\|_\infty \sup_N \frac{1}{N}\sum_{k=1}^{N} |T|^{n_k}|f|$$

which is finite a.e. An application of the Banach principle completes the proof. □

Remarks. 1. The proof required only that $\{n_k\}$ be good universal in L_p, with the constant sequence $\{a(k) = 1\}$ a good weight along $\{n_k\}$ for all Dunford-Schwartz operators on L_p. For $p = 1$ and $\{n_k = k\}$, this is satisfied with $\{n_k\}$ not strongly good universal.

2. A. Bellow [2] proved that for any fixed $p > 1$, there are subsequences $\{n_k\}$ which are good universal sequences in L_p, but not in L_r with $1 \leq r < p$.

3. In Theorem 4.1 of [11], the following lemma is implicitly applied to sequences which are good universal in L_p for every $1 < p < \infty$.

Lemma 2.2. *Let $\{n_k\}$ be a good universal sequence in L_p, for all p in an open interval (r,s), $1 \leq r < s$. Then $\{n_k\}$ is a strongly good universal sequence in L_p for every $p \in (r,s)$.*

Proof. Let τ be an ergodic measure preserving transformation. By Sawyer's theorem ([18] or [10]), the a.e. convergence of the averages $\frac{1}{N}\sum_{k=1}^{N} f(\tau^{n_k}x)$ for every $f \in L_p$ implies that the corresponding maximal operator is of weak type (p,p), for every $p \in (r,s)$. Let $r < p_1 < p < p_2 < s$. Then this maximal operator is of weak types (p_1,p_1) and (p_2,p_2), so by (a special case of) the Marcinkiewicz interpolation theorem [8], it is of strong type (p,p). By Corollary 2.2 of [11], the maximal operator along $\{n_k\}$ of any positively dominated contraction of L_p, particularly of any measure preserving transformation, is of strong type (p,p). Since $\{n_k\}$ is good universal, it is strongly good universal. □

Theorem 2.3. *Let $\{n_k\}$ be a good universal sequence in L_p for all $1 < p < \infty$. For a fixed p, if the sequence $\{a(k) = 1\}$ is a good weight sequence along $\{n_k\}$ for all positive [positively dominated] contractions of L_p, then the $r, \{n_k\}$-Besicovitch sequences with $r > p/(p-1)$ are good weight sequences along $\{n_k\}$ for positive [positively dominated] contractions of L_p.*

Proof. Again we only sketch the proof. More details can be found in the proof of Theorem 2.4 of [12]. By the previous lemma, $\{n_k\}$ is *strongly* good universal in L_p for every p, $1 < p < \infty$. By Corollary 2.4 of [11], for any positively dominated contraction T of L_p, $1 < p < \infty$, the maximal operator $\sup_N \frac{1}{N}\sum_{k=1}^{N} |T^{n_k} f|$ is strong type (p,p).

Fix p such that $\{a(k) = 1\}$ is a good weight sequence along $\{n_k\}$ for all positively dominated contractions of L_p, which means we have a.e. convergence of the averages along $\{n_k\}$ for these operators. If T is a positively dominated contraction of L_p, so is the operator λT when λ is a complex number of absolute value 1, so we have a.e. convergence of its averages along $\{n_k\}$, which is convergence in (1) for $\{a(k) = \lambda^k\}$.

We now look at the case that $\{a(k) = 1\}$ is a good weight sequence along $\{n_k\}$ only for *positive* contractions of L_p. Following [17], for T a positive contraction of $L_p(X)$ we take the product space of the unit circle with X, and define $P[g(z)f(x)] = g(\lambda z)Tf(x)$. Then P extends to a positive contraction of L_p of the product space, and applying to P the assumed convergence for positive operators, with $g(z) = z$ and $f \in L_p(X)$, we obtain a.e. convergence in (1) for $\{a(k) = \lambda^k\}$.

We now prove the part of the theorem when $\{a(k) = 1\}$ is a good weight sequence along $\{n_k\}$ for all positively dominated contractions of L_p; the restricted case of positive contractions is obtained by putting $S = T$ in the proof. Let $q = p/(p-1)$ be the dual index of p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$, fix $r > q$, and let T be dominated by a positive contraction S on L_p. By [1], there exists a larger L_p', a positive isometric embedding D of L_p into L_p', a conditional expectation operator E and a positive invertible isometry Q such that for each $n \in Z^+$ we have $DS^n f = EQ^n Df$. Since Q

can be written in the form $Q^n f(x') = w_n(x') f(\tau x')$, where τ is a non-singular point transformation, $|Q^n f(x)|^s = R^n |f|^s$ where R is an $L'_{p/s}$ isometry for $s = r/(r-1) < p$. Since the maximal operator $\sup_N \frac{1}{N} \sum_{k=1}^N R^{n_k}|f|$ is strong type $(\frac{p}{s}, \frac{p}{s})$, we have that the maximal operator $\sup_N \frac{1}{N} \sum_{k=1}^N [S^{n_k}|f|]^s$ is strong type (p,p) and hence is finite a.e. Thus, if $\{b_j(k)\}$ is a sequence of trigonometric polynomials that approach $\{a(k)\}$ in the $\| \cdot \|_{r,\{n_k\}}$ semi-norm, then for a.e. x Hölder's inequality shows that the sequence $\{a(k)T^k f(x)\}$ will converge in the $\| \cdot \|_{1,\{n_k\}}$ semi-norm. □

Corollary 2.4. *Fix p, $1 < p < \infty$. Let $n_k = k^t$ (fixed $t \in \mathbf{N}$), or let n_k denote the k-th prime, then the $r, \{n_k\}$-Besicovitch sequences, for $r > p/(p-1)$, are good weight sequences along $\{n_k\}$ for Dunford-Schwartz operators in L_p and for positive contractions of L_p.*

Proof. Recall that Bourgain [6] has proved that the sequence $\{k^t\}$ is strongly good universal in L_p, for every $p > 1$. Wierdl [20] has proved that the sequence of primes is good universal in L_p, $p > 1$. In [13] it is shown that in both cases the constant sequence is a good weight sequence along $\{n_k\}$ for positive contractions of L_p, so the previous theorem yields the result for positive contractions of L_p. For T Dunford-Schwartz the proof of the previous theorem applies, since a.e. convergence in (1) for $\{a(k) = \lambda^k\}$ holds by the first part of the proof of Theorem 2.1. □

3. Uniform Sequences

In this section we will consider the uniform sequences of Brunel-Keane [7]. These are bounded Besicovitch sequences with some further restrictions. We will also consider good averaging sequences $\{n_k\}$ such that for every irrational $\theta \in [0,1)$, $\{n_k\theta\}$ is uniformly distributed (*mod* 1). We will show that in this case every uniform sequence produced by an apparatus with a connected space is in $B_{1,\{n_k\}}$. Since a uniform sequence is bounded, this means that those uniform sequences will then also belong to $B_{p,\{n_k\}}$ for all $p > 1$, and we will be able to apply the results of the previous section to the uniform sequences $\{a(k)\}$ along the sequences $\{n_k\}$.

We first give the construction of the uniform sequences of Brunel and Keane [7], the details of which we will need. Let Ω be a compact metric space, $\mathcal{B}$ the collection of Borel subsets of Ω, and ϕ a homeomorphism of Ω such that $\{\phi^n\}_{n\geq 0}$ is an equicontinuous family of mappings. The system (Ω, ϕ) is then called *uniformly L stable*. We assume that Ω possesses a dense orbit. It then follows (see [7] or [17]) that there exists a unique ϕ invariant probability measure on $(\Omega, \mathcal{B})$, denoted by ν. Then for any $w \in \Omega$, and any continuous function f on Ω,

$$\lim_n \frac{1}{n} \sum_{t=0}^{n-1} f(\phi^t w) = \int f \, d\nu \, .$$

Such a system $(\Omega, \mathcal{B}, \nu, \phi)$ is called *strictly L stable*.

If $(\Omega, \mathcal{B}, \nu, \phi)$ is strictly L stable, $Y \in \mathcal{B}$ with $\nu(Y) > 0$, $\nu(\partial Y) = 0$ and $y \in \Omega$, the sequence $\{a_k(y)\} = \{\mathcal{X}_Y(\phi^k y)\}$ is called a *uniform sequence of weights*. The entire collection $\{(\Omega, \mathcal{B}, \nu, \phi), y, Y\}$ is called the *apparatus* producing the uniform sequence of weights. The apparatus is said to be *connected* if Ω is connected. It is clear that a uniform sequence is a "return times" sequence. In fact, we will be interested in the sequences $\{a_k(y)\}$ for *all* $y \in \Omega$. This should be contrasted with the usual situation for return times weights, where one considers only y in a subset Ω' of Ω, where $\nu(\Omega') = 1$. The fact that uniform sequences of weights are bounded Besicovitch is proved in [17], p. 149. V. Losert has shown us (private communication) that uniform sequences need not be weakly almost periodic.

Theorem 3.1. *Let $\{n_k\}$ be a good universal sequence in L_p, $p > 1$ fixed. For $\theta \in [0,1)$ irrational such that $\{n_k\theta\}$ is uniformly distributed* mod 1*, let T be induced by the measure preserving transformation $\tau x = \theta + x \mod 1$. Then for all $f \in L_p[0,1)$, we have*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} T^{n_k} f = \int f \quad a.e. \tag{2}$$

Proof. We first note that if (2) holds for a dense class of functions in L_p, then we are done: given $\epsilon > 0$ and $f \in L_p[0,1)$ we can choose f' in our dense class such that $\|f - f'\|_p < \epsilon$. Putting $A_N = \frac{1}{N}\sum_{k=1}^{N} T^{n_k}$, we then have

$$\begin{aligned}\|A_N f - \int f\|_p &\leq \|A_N f - A_N(f') + A_N(f') - \int f' + \int f' - \int f\|_p \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}.\end{aligned}$$

Thus $\|A_N f - \int f\|_p \to 0$. Since the a.e. convergence of the sequence $\{A_N f\}$ is assumed, the limit must be $\int f$.

To see the dense class, we just note that for characteristic functions of intervals, by the assumption that $(n_k\theta)$ is uniformly distributed, we have convergence to the integral. Hence it is true for finite linear combinations of characteristic functions of intervals, and these are dense. □

Definition. Let τ be a measure preserving point transformation. We say that τ is *totally ergodic* if the transformations $\tau^n, n = 1, 2, ...$ are all ergodic.

We can now extend the previous theorem to totally ergodic transformations as opposed to irrational rotations of the circle.

Theorem 3.2. *Let $\{n_k\}$ be a good universal sequence in L_p, for every $1 < p < \infty$, such that $\{n_k\theta\}$ is uniformly distributed* mod 1 *for all $\theta \in [0,1)$ irrational, τ a totally ergodic measure preserving point transformation of a probability space X, $f \in L_p(X)$, $p > 1$. Then for a.e. x we have*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} f(\tau^{n_k} x) = \int f.$$

Proof. Let $\lambda = e^{2\pi i\theta}$ be a complex number of modulus one that is not a root of unity. By Theorem 3.1, for the function $f(z) = z$ defined on the unit circle, we have for a.e. z

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} \lambda^{n_k} z = \int_{\{z:|z|=1\}} z = 0.$$

Hence

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} \lambda^{n_k} = 0.$$

We know that for $f \in L_p$, $p > 1$,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} f(\tau^{n_k}x)$$

exists for a.e. x, and hence also in L_p norm. For $f \in L_2$, we have

$$\frac{1}{N}\sum_{k=1}^{N} (f \circ \tau^{n_k}, f) = \frac{1}{N}\sum_{k=1}^{N} \int_{\{\lambda : |\lambda|=1\}} \lambda^{n_k}\, dE_f(\lambda)$$

where $dE_f(\lambda)$ is the spectral measure of the linear operator on L_2 defined by $Tf = f \circ \tau$. We have shown that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} \lambda^{n_k} = 0$$

unless λ belongs to the countable set of the roots of unity. But since τ is totally ergodic, no root of unity $\neq 1$ is an eigenvalue of T on L_2. Hence, no root of unity except 1 is an atom of the spectral measure of T, so for $f \in L_2$,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} f(\tau^{n_k}x) = \int f\ .$$

Since $L_2 \cap L_p$ is dense in L_p, $p > 1$, the theorem follows as in the proof of Theorem 3.1. □

Even for strongly good universal sequences, the requirement that τ be totally ergodic is necessary. In fact, consider the strongly good universal sequence of the primes. Let λ be a primitive r-th root of unity, and let X be r point space with each point having measure $\frac{1}{r}$. Let τ be any cyclic permutation of all the points, and let f be defined by $f(x) = 1$ for one particular x and 0 otherwise. Then it is easy to see that for some x

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} f(\tau^{n_k}x) = 0 \neq \int f$$

and λ is an eigenvalue for the operator induced by τ, which is ergodic.

Lemma 3.3. *If* $\{(\Omega, \mathcal{B}, \nu, \phi), y, Y\}$ *is a connected apparatus producing a uniform sequence, then* ϕ *is totally ergodic.*

Proof. Suppose ϕ is not totally ergodic. Then some power of ϕ has a non-constant invariant function, which implies that the operator S defined by $Sf = f \circ \phi$ has an eigenfunction in L_2 with an associated eigenvalue that is a root of unity.

Let R be the operator S restricted to $C(\Omega)$. Then R is almost periodic, so for all λ with $|\lambda| = 1$ also $\overline{\lambda}R$ is almost periodic. Consequently, the averages

$$\frac{1}{N}\sum_{k=1}^{N} \overline{\lambda}^k R^k f$$

converge uniformly to the projection of f onto the eigenspace of R associated with λ.

If this projection is non-zero, which will happen if and only if λ is an eigenvalue for S (since $C(\Omega)$ is dense in $L_2(\Omega)$), R (and hence S) will have a continuous eigenfunction g associated with the eigenvalue λ. Then g assumes only the values $g(x_0)$, $g(\phi x_0), \ldots, g(\phi^{r-1}x_0)$, where x_0 has a dense orbit. Since g is continuous and Ω is connected, this is a contradiction. □

Theorem 3.4. *Let $\{n_k\}$ be a good universal sequence in L_∞, such that for every irrational $\theta \in [0,1)$ the sequence $\{n_k\theta\}$ is uniformly distributed* mod 1, *and let $\{a(k)\}$ be a uniform sequence produced by a connected apparatus $\{(\Omega, \mathcal{B}, \nu, \phi), y, Y\}$. Then $\{a(k)\}$ is $\{n_k\}$-Besicovitch.*

Proof. By Lemma 3.3, ϕ is totally ergodic. Let $g \in C(\Omega)$. Since $\{n_k\}$ is a good universal sequence, we have from Theorem 3.2

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} g(\phi^{n_k}y) = \int g \tag{3}$$

for a.e. $y \in \Omega$. Since g is uniformly continuous, $\{\phi^n\}_{n\geq 0}$ is an equicontinuous family, and open sets have positive measure, we have that (3) holds for *all* $y \in \Omega$.

Let g_1 and g_2 be continuous functions such that $g_1(y) \leq \mathcal{X}_Y(y) \leq g_2(y)$ for all $y \in \Omega$ and $\int g_2 - \int g_1 < \epsilon$, where $\epsilon > 0$ is arbitrary (see [7, 17]). We then have, for all $y \in \Omega$,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} g_1(\phi^{n_k}y) \leq \liminf_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} a(n_k)$$

$$\leq \limsup_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} a(n_k) \leq \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} g_2(\phi^{n_k}y)$$

Fix y. Then $\{g_2(\phi^n y)\}$ is almost periodic, and using (3) we obtain

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} |g_2(\phi^{n_k}y) - a(n_k)| \leq$$

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} [g_2(\phi^{n_k}y) - g_1(\phi^{n_k}y)] = \int (g_2 - g_1) < \epsilon.$$

Since ϵ is arbitrary, $\{a(k)\}$ is $\{n_k\}$-Besicovitch.

Furthermore, for any $\epsilon > 0$ we also have

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} a(n_k) - \liminf_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} a(n_k) < \epsilon\,.$$

Since ϵ is arbitrary, $\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} a(n_k)$ exists (and equals $\nu(Y)$). □

Remark. To get a better picture of the class of sequences $\{a(k)\}$ considered in the theorem, we note that Halmos and von-Neumann proved (see Theorem 3 of [17]) that every strictly L stable system is isomorphic to a rotation by a generator of a compact metric monothetic group. Thus, we may assume that Ω is a compact metric (connected) monothetic group, ν its Haar measure, and $\phi(x) = x + \alpha$ with

$\{\alpha^n\}$ dense in Ω. The referee has remarked that this yields an alternative proof of Lemma 3.3.

Combining the previous theorem with the results of the previous section, we obtain the following corollaries.

Corollary 3.5. *Fix p, $1 \leq p < \infty$. Let $\{n_k\}$ be a strongly good universal sequence in L_p, such that $\{n_k\theta\}$ is uniformly distributed* mod 1 *for every irrational $\theta \in [0,1)$. Then any uniform sequence $\{a(k)\}$ produced by a connected apparatus is a good weight sequence along $\{n_k\}$ for Dunford-Schwartz operators in L_p.*

Corollary 3.6. *Let $\{n_k\}$ be a good universal sequence in L_s for every $1 < s < \infty$, such that $\{n_k\theta\}$ is uniformly distributed* mod 1 *for every irrational $\theta \in [0,1)$. If for a fixed p the constant sequences are good weight sequences along $\{n_k\}$ for positive [positively dominated] contractions of L_p, then any uniform sequence $\{a(k)\}$ produced by a connected apparatus is a good weight sequence along $\{n_k\}$ for positive [positively dominated] contractions of L_p.*

Corollary 3.7. *If $n_k = k^t$ (for fixed $t \in \mathbf{N}$), or if n_k denotes the k-th prime, and T is a Dundord-Schwartz operator or a positive contraction of $L_p(X)$, $p > 1$, then for any uniform sequence $\{a(k)\}$ produced by a connected apparatus and for all $f \in L_p(X)$, we have*

$$\lim_{n\to\infty} \frac{1}{N} \sum_{k=1}^{N} a(n_k) T^{n_k} f$$

exists a.e.

Proof. Bourgain [6] has established that for fixed $t \in \mathbf{N}$, $\{n_k = k^t\}$ is a strongly good universal sequence in L_p for all $p > 1$. Weyl's theorem ([14], p. 27) says that for θ irrational, $\{k^t\theta\}$ is uniformly distributed mod 1. For $\{n_k\}$ the sequence of primes, Wierdl [20] has established that it is good universal in L_p for every $p > 1$, and the uniform distribution of $\{n_k\theta\}$ for irrational θ follows from [9], Theorem 9.8. Hence the hypotheses of Theorem 3.4 are satisfied in both cases, and Corollary 2.4 yields the result. □

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