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On the Braiding on a Hopf Algebra in a Braided Category

Peter Schauenburg

Abstract. By definition, a bialgebra H in a braided monoidal category $(\mathcal{C}, \tau)$ is an algebra and coalgebra whose multiplication and comultiplication (and unit and counit) are compatible; the compatibility condition involves the braiding τ.

The present paper is based upon the following simple observation: If H is a Hopf algebra, that is, if an antipode exists, then the compatibility condition of a bialgebra can be solved for the braiding. In particular, the braiding $\tau_{HH} : H \otimes H \to H \otimes H$ is uniquely determined by the algebra and coalgebra structure, if an antipode exists. (The notions of algebra and coalgebra (and antipode) need only the monoidal category structure of $\mathcal{C}$.)

We list several applications. Notably, our observation rules out that any nontrivial examples of commutative (or cocommutative) Hopf algebras in non-symmetric braided categories exist. This is a rigorous proof of a version of Majid's observation that commutativity is too restrictive a condition for Hopf algebras in braided categories.

Hopf algebras in braided categories are generalizations of ordinary Hopf algebras. For the definition of a k-Hopf algebra one needs the tensor product of vector spaces and the canonical flip of tensor factors $V \otimes W \cong W \otimes V$, used in the compatibility condition between multiplication and comultiplication. The structure of a monoidal category formalizes tensor products and the structure of a braiding formalizes the flip of tensor factors. An older formalization of the properties of flipping tensor factors in a tensor product of vector spaces is the notion of a symmetric monoidal category. The key difference is that in a symmetric monoidal category, flipping factors twice acts as the identity. Braided monoidal categories are obtained from symmetric ones by omitting this one axiom. Through relations to the braid groups they are endowed with a rich topological flavor and have applications in knot and manifold theory. They are also related strongly to quantum group theory. General references to these relations are [1, 5]. The notion of a Hopf algebra in a symmetric category is well known [4]. The definition of Hopf algebras in braided categories is no different from that in the symmetric case. Apart from being interesting objects in themselves, they arise naturally in the structure theory of ordinary Hopf algebras

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through Radford's theorem [3]. A survey of the theory of Hopf algebras in braided categories is [2].

Let us review some definitions. A prebraided monoidal category $(\mathcal{C},\tau)$ is a monoidal category (whose tensor product, denoted $\otimes$, we assume to be strictly associative with a strict unit object I) equipped with a prebraiding τ, that is, a natural morphism $\tau=\tau_{XY}:X\otimes Y\to Y\otimes X$ satisfying $\tau_{X\otimes Y,Z}=(\tau_{XZ}\otimes Y)(X\otimes\tau_{YZ})$, $\tau_{X,Y\otimes Z}=(Y\otimes\tau_{XZ})(\tau_{XY}\otimes Z)$ and $\tau_{XI}=\tau_{IX}=\mathrm{id}_X$. A braiding is a prebraiding which is an isomorphism. A symmetry is a (pre)braiding satisfying $\tau_{XY}\tau_{YX}=1$.

A bialgebra (H,∇,Δ) in a prebraided category is an algebra (H,∇) with unit $\eta:I\to H$ and a coalgebra (H,Δ) with counit $\epsilon:H\to I$ such that

$$\Delta\nabla=(\nabla\otimes\nabla)(H\otimes\tau_{HH}\otimes H)(\Delta\otimes\Delta),$$

$\Delta\eta=\eta\otimes\eta$ and $\epsilon\nabla=\epsilon\otimes\epsilon$ hold. A Hopf algebra is a bialgebra that has an antipode, that is, an inverse for id_H in the convolution monoid $\mathrm{Mor}(H,H)$, that is, a morphism $S:H\to H$ with $\nabla(S\otimes H)\Delta=\nabla(H\otimes S)\Delta=\eta\epsilon$.

Let H be a bialgebra in the prebraided monoidal category $(\mathcal{C},\tau)$. A left-right Hopf module over H is an object M of $\mathcal{C}$ that is a left H-module (with module structure map $\mu\colon H\otimes M\to M$) as well as a right H-comodule (with structure map $\rho\colon M\to M\otimes H$) such that

$$\rho\mu=(\mu\otimes\nabla)(H\otimes\tau\otimes H)(\Delta\otimes\rho)\colon H\otimes M\to M\otimes H.$$

In particular, H is canonically a left-right Hopf module over itself.

Theorem. *Let (H,∇,Δ) be a Hopf algebra in the prebraided monoidal category $(\mathcal{C},\tau)$. Then*

$$\tau_{HH}=(\nabla\otimes\nabla)(S\otimes\Delta\nabla\otimes S)(\Delta\otimes\Delta).$$

More generally, if M is a left-right Hopf module over H, then

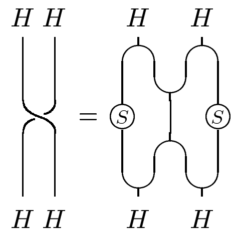
$$\tau_{HM}=(\mu\otimes\nabla)(S\otimes\rho\mu\otimes S)(\Delta\otimes\rho).$$

Proof. Of course, we need only prove the more general statement on Hopf modules, which was pointed out by the referee:

$$\begin{aligned}
(\mu\otimes\nabla)&(S\otimes\rho\mu\otimes S)(\Delta\otimes\rho)\\
&=(\mu\otimes\nabla)\big(S\otimes(\mu\otimes\nabla)(H\otimes\tau_{HM}\otimes H)(\Delta\otimes\rho)\otimes S\big)(\Delta\otimes\rho)\\
&=(\mu\otimes\nabla)(H\otimes\mu\otimes\nabla\otimes H)(S\otimes H\otimes\tau_{HM}\otimes H\otimes S)\\
&\qquad\circ(H\otimes\Delta\otimes\rho\otimes H)(\Delta\otimes\rho)\\
&=(\mu\otimes\nabla)\big(\nabla(S\otimes H)\Delta\otimes\tau_{HM}\otimes\nabla(H\otimes S)\Delta\big)(\Delta\otimes\rho)\\
&=(\mu\otimes\nabla)(\eta\epsilon\otimes\tau_{HM}\otimes\eta\epsilon)(\Delta\otimes\rho)\\
&=\tau_{HM}
\end{aligned}$$

□

Remark. In the widely used graphical calculus for braided categories (many examples of which are found in [2]), the picture



represents the equation obtained above for the braiding on a Hopf algebra in a braided monoidal category.

Let $\mathcal{C},\mathcal{D}$ be monoidal categories. Recall that a monoidal functor $(\mathcal{F},\xi,\zeta):\mathcal{C}\to\mathcal{D}$ consists of an ordinary functor $\mathcal{F}$, an isomorphism $\zeta:\mathcal{F}(I)\to I$, and an isomorphism $\xi:\mathcal{F}(X\otimes Y)\to\mathcal{F}(X)\otimes\mathcal{F}(Y)$ which is natural in $X,Y\in\mathcal{C}$, such that the diagrams

$$\begin{array}{ccc} \mathcal{F}(X\otimes Y\otimes Z) & \xrightarrow{\xi} & \mathcal{F}(X)\otimes\mathcal{F}(Y\otimes Z) \\ {\scriptstyle\xi}\downarrow & & \downarrow{\scriptstyle 1\otimes\xi} \\ \mathcal{F}(X\otimes Y)\otimes\mathcal{F}(Z) & \xrightarrow{\xi\otimes 1} & \mathcal{F}(X)\otimes\mathcal{F}(Y)\otimes\mathcal{F}(Z) \end{array}$$

commute and both

$$\mathcal{F}(X\otimes I)\overset{\xi}{\to}\mathcal{F}(X)\otimes\mathcal{F}(I)\xrightarrow{1\otimes\zeta}\mathcal{F}(X)\otimes I \quad\text{and}$$

$$\mathcal{F}(I\otimes X)\overset{\xi}{\to}\mathcal{F}(I)\otimes\mathcal{F}(X)\xrightarrow{\zeta\otimes 1}I\otimes\mathcal{F}(X)$$

are identities. Given such a monoidal functor, an algebra (A,∇) and a coalgebra (C,Δ) in $\mathcal{C}$, we obtain an algebra $(\mathcal{F}(A),\mathcal{F}(\nabla)\xi^{-1})$ and a coalgebra $(\mathcal{F}(C),\xi\mathcal{F}(\Delta))$ in $\mathcal{D}$. The Theorem implies immediately that a monoidal functor between prebraided monoidal categories that maps, in this way, a Hopf algebra H to a Hopf algebra, necessarily preserves the braiding on $H\otimes H$. More precisely:

Corollary 1. *Let $(\mathcal{C},\tau)$ and $(\mathcal{D},\sigma)$ be prebraided monoidal categories, and $(\mathcal{F},\xi,\zeta):\mathcal{C}\to\mathcal{D}$ a monoidal functor. Let (H,∇,Δ) be a Hopf algebra in $(\mathcal{C},\tau)$.*
If $\big(\mathcal{F}(H),\mathcal{F}(\nabla)\xi^{-1},\xi\mathcal{F}(\Delta)\big)$ is a Hopf algebra in $(\mathcal{D},\sigma)$, then

$$\begin{array}{ccc} \mathcal{F}(H\otimes H) & \xrightarrow{\xi} & \mathcal{F}(H)\otimes\mathcal{F}(H) \\ {\scriptstyle\mathcal{F}(\tau)}\downarrow & & \downarrow{\scriptstyle\sigma} \\ \mathcal{F}(H\otimes H) & \xrightarrow{\xi} & \mathcal{F}(H)\otimes\mathcal{F}(H) \end{array}$$

commutes.

A special case of this occurs in the study of Hopf algebras in categories built upon the category of vector spaces and their tensor product. Examples are the category of modules over quasitriangular Hopf algebras, comodules over coquasitriangular Hopf algebras, or categories of Yetter-Drinfeld-B-modules over a k-bialgebra B. A Yetter-Drinfeld-module $V\in\mathcal{YD}_B^B$ is by definition a right B-module and right B-comodule satisfying the compatibility condition

$$v_{(0)}\leftharpoonup h_{(1)}\otimes v_{(1)}h_{(2)}=(v\leftharpoonup h_{(2)})_{(0)}\otimes h_{(1)}(v\leftharpoonup h_{(2)})_{(1)}$$

for all $v \in V$, $h \in H$, where we make free use of Sweedler's notation for comodule structures and comultiplications, leaving out the summation symbols. The category $\mathcal{YD}_B^B$ is prebraided with

$$\tau : V \otimes W \ni v \otimes w \mapsto w_{(0)} \otimes v \leftharpoonup w_{(1)} \in W \otimes V$$

for $V, W \in \mathcal{YD}_B^B$. The prebraiding is a braiding if B^{op} is a Hopf algebra, in particular if B is a Hopf algebra with bijective antipode. Hopf algebras in the category $\mathcal{YD}_B^B$ arise naturally in the context of Radford's theorem on Hopf algebras with a projection [3].

Corollary 2. *Let B be a k-bialgebra and H a Hopf algebra in the category $\mathcal{YD}_B^B$. If H is a k-Hopf algebra, then τ_{HH} is the usual flip of vector spaces, that is*

$$g_{(0)} \otimes h \leftharpoonup g_{(1)} = g \otimes h$$

holds for all $g, h \in H$, where $g_{(0)} \otimes g_{(1)} \in H \otimes B$ is the image of g under the B-comodule structure on H.

Let $\mathcal{C}$ be a monoidal category and H an object of $\mathcal{C}$. Then one can construct the full monoidal subcategory $\langle H \rangle$ of $\mathcal{C}$ generated by H. If $\mathcal{C}$ is prebraided, then so is $\langle H \rangle$. If $\mathcal{C}$ is k-linear abelian, we denote by $\langle H \rangle_k$ the full monoidal k-linear abelian subcategory generated by H. Note that if $\mathcal{C} = {}_B\mathcal{M}$ is the category of left B-modules over a k-bialgebra B, then there is a quotient bialgebra $\overline{B}$ of B such that $\langle H \rangle_k \cong {}_{\overline{B}}\mathcal{M}$, and if $\mathcal{C} = \mathcal{M}^B$ then there is a subbialgebra B' of B such that $\langle H \rangle_k \cong \mathcal{M}^{B'}$. Explicitly, B' is the subalgebra of B generated by the subcoalgebra $C = \{\langle \varphi, h_{(0)} \rangle h_{(1)} | h \in H, \varphi \in H^*\}$.

Corollary 3. *Let B be a k-bialgebra and H a Hopf algebra in the category $\mathcal{YD}_B^B$. If H is a k-Hopf algebra, then there exists a subbialgebra $B' \subset B$ such that the B'-comodule structure of H takes values in $H \otimes B'$, and the action of B' on H is trivial.*

In fact, in the notations preceding the Corollary, it suffices to show that C acts trivially on H. Now

$$h \leftharpoonup \langle \varphi, g_{(0)} \rangle g_{(1)} = (\varphi \otimes \mathrm{id})(\tau_{HH}(h \otimes g)) = \langle \varphi, g \rangle h = \epsilon(\langle \varphi, g_{(0)} \rangle g_{(1)}) h.$$

By definition, a coquasitriangular k-bialgebra is a bialgebra B equipped with a map $R : B \otimes B \to k$ such that

$$V \otimes W \ni v \otimes w \mapsto w_{(0)} \otimes v_{(0)} R(v_{(1)} \otimes w_{(1)}) \in W \otimes V$$

for $V, W \in \mathcal{M}^B$ defines a braiding for $\mathcal{M}^B$. Any subbialgebra of a coquasitriangular bialgebra is again coquasitriangular. A quasitriangular k-bialgebra is a bialgebra B equipped with an element $R = \sum r_i \otimes s_i \in B \otimes B$ such that

$$V \otimes W \ni v \otimes w \mapsto \sum s_i w \otimes r_i v \in W \otimes V$$

for $V, W \in {}_B\mathcal{M}$ defines a braiding on ${}_B\mathcal{M}$. Any quotient bialgebra of a quasitriangular bialgebra is again quasitriangular.

Corollary 4. *Let (B, R) be a quasitriangular (resp. coquasitriangular) bialgebra and H a Hopf algebra in the braided monoidal category ${}_B\mathcal{M}$ (resp. $\mathcal{M}^B$). Assume that H is also a k-Hopf algebra. Then there is a quotient bialgebra $\overline{B}$ of B (resp. a* *subbialgebra B' of B) such that $H \in {}_{\overline{B}}\mathcal{M}$ (resp. $H \in \mathcal{M}^{B'}$) and the image of R in $\overline{B} \otimes \overline{B}$ is $\overline{R} = 1$ (resp. the restriction of R to $B' \otimes B'$ is $R|_{B' \otimes B'} = \epsilon$.)*

Finally, our Theorem rules out rigorously a naive notion of commutative (or cocommutative) Hopf algebra in braided categories. Majid was the first to observe that the condition $\nabla = \nabla\tau$ on the multiplication of a Hopf algebra H in a braided monoidal category appears to be too strong to admit interesting examples. We can now *prove* that commutative Hopf algebras in this sense can *only* arise in symmetric categories. More precisely, we will show that $\tau^2_{HH} = \mathrm{id}$ holds for a commutative Hopf algebra H in a braided category $(\mathcal{C}, \tau)$. This implies that $\langle H \rangle$ is symmetric (and so is $\langle H \rangle_k$ if $\mathcal{C}$ is k-linear abelian).

Corollary 5. *Let $(\mathcal{C}, \tau)$ be a braided monoidal category and (H, ∇, Δ) a commutative (resp. cocommutative) Hopf algebra in $(\mathcal{C}, \tau)$. Then $\tau^2_{H,H} = 1$.*

In particular, if H is a commutative or cocommutative Hopf algebra in the category ${}_B\mathcal{M}$ (resp. $\mathcal{M}^B$) of modules (resp. comodules) over a quasitriangular (resp. coquasitriangular) bialgebra, then there is a triangular quotient bialgebra $\overline{B}$ (resp. cotriangular subbialgebra B') of B such that H is contained in ${}_{\overline{B}}\mathcal{M}$ (resp. $\mathcal{M}^{B'}$).

For the proof observe that $(\mathcal{C}, \tau^{-1})$ is a braided monoidal category, where $\tau^{-1}_{XY} = (\tau_{YX})^{-1}$. One can check that $(H, \nabla\tau^{-1}, \Delta)$ is a Hopf algebra in $(\mathcal{C}, \tau^{-1})$ whenever (H, ∇, Δ) is a Hopf algebra in $(\mathcal{C}, \tau)$. Now if $\nabla = \nabla\tau$, then $\tau_{H,H} = \tau^{-1}_{H,H}$ follows directly from the Theorem, or one can apply Corollary 1 to the identical functor on $\mathcal{C}$.

References

[1] Christian Kassel, *Quantum Groups*, Graduate Texts in Mathematics, no. 155, Springer-Verlag, Berlin, 1995, MR 96e:17041.

[2] Shahn Majid, *Algebras and Hopf algebras in braided categories*, Advances in Hopf algebras (J. Bergen and S. Montgomery, eds.), Lecture Notes in Pure and Applied Math., Marcel Dekker, New York, 1994, pp. 55–105, MR 95d:18004.

[3] David E. Radford, *The structure of Hopf algebras with a projection*, J. Algebra **92** (1985), 322–347, MR 86k:16004.

[4] Neantro Saavedra Rivano, *Catègories Tannakiennes*, Lecture Notes in Mathematics, no. 265, Springer-Verlag, Berlin, 1972, MR 49 #2769.

[5] Vladimir G. Turaev, *Quantum Invariants of Knots and* 3*-Manifolds*, de Gruyter Studies in Mathematics, no. 18, W. de Gruyter, Berlin, 1994, MR 95k:57014.

Mathematisches Institut der Universität München, Theresienstr. 39, 80333 München, Germany

schauen@rz.mathematik.uni-muenchen.de

http://www.mathematik.uni-muenchen.de/personen/schauenburg.html

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