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A Simple Functional Operator

Ice B. Risteski, Kostadin G. Trenčevski, and Valéry C. Covachev

Abstract. In this paper a new linear operator Ψ is defined such that $\Psi \circ \Psi = 0$. The general analytic solution of the vector functional equation $\Psi f = 0$ is given.

Contents

1. Main Results

Definition 1.1. Let $\mathcal{V}$ and $\mathcal{V}'$ be complex vector spaces. For an arbitrary mapping $f : \mathcal{V}^{n-1} \mapsto \mathcal{V}'$ $(n > 1)$ we define a mapping $\Psi f : \mathcal{V}^n \mapsto \mathcal{V}'$ by

$$(\Psi f)(\mathbf{Z}_1, \dots, \mathbf{Z}_n) = (-1)^{n-1} f(\mathbf{Z}_1, \dots, \mathbf{Z}_{n-1}) - f(\mathbf{Z}_2, \dots, \mathbf{Z}_n) + \sum_{i=1}^{n-1} (-1)^{i+1} f(\mathbf{Z}_1, \dots, \mathbf{Z}_i + \mathbf{Z}_{i+1}, \dots, \mathbf{Z}_n). \tag{1}$$

If $n = 1$, we define $\Psi f = 0$.

Remark 1.2. The definition of the operator Ψ is a variation on the formula giving the differential of the bar construction.

Lemma 1.3. *For an arbitrary mapping $f : \mathcal{V}^{n-1} \mapsto \mathcal{V}'$ we have*

$$(\Psi \circ \Psi) f(\mathbf{Z}_1, \dots, \mathbf{Z}_{n+1}) = 0. \tag{2}$$

Proof. This follows by a straightforward calculation similar to that giving the identity $d^2 = 0$, where d is the differential in the bar construction (see [7, Chapter IV, formula (5.8)]). □

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This lemma shows that the kernel of the operator Ψ contains all mappings of the form Ψf. The next theorem provides a complete description of this kernel.

Theorem 1.4. *The general solution of the operator equation*

$$(\Psi f)(\mathbf{Z}_1,\dots,\mathbf{Z}_{n+1}) = 0 \tag{3}$$

in the set of analytic functions $f : \mathcal{V}^n \mapsto \mathcal{V}'$ $(n \geq 1)$ *is given by*

$$f(\mathbf{Z}_1,\dots,\mathbf{Z}_n) = (\Psi F)(\mathbf{Z}_1,\dots,\mathbf{Z}_n) + L(\mathbf{Z}_1,\dots,\mathbf{Z}_n), \tag{4}$$

where $F : \mathcal{V}^{n-1} \mapsto \mathcal{V}'$ *is an arbitrary analytic function and* L *is an arbitrary linear mapping:* $\mathcal{V}^n \mapsto \mathcal{V}'$ $(n \geq 1)$.

Proof. First note that if $n = 1$, the equation $(\Psi f)(\mathbf{Z}_1,\mathbf{Z}_2) = 0$ is the Cauchy functional equation

$$f(\mathbf{Z}_1 + \mathbf{Z}_2) - f(\mathbf{Z}_1) - f(\mathbf{Z}_2) = 0.$$

The general analytic solution of this equation is $f(\mathbf{Z}) = A\mathbf{Z}$, where A is an $(s \times r)$ matrix with arbitrary complex constant entries ($r = \dim \mathcal{V}$ and $s = \dim \mathcal{V}'$). About the solution of the Cauchy matrix functional equation see [2] and [6].

Now let $n \geq 2$. The operator equation (3) is equivalent to

$$(-1)^n f(\mathbf{Z}_1,\dots,\mathbf{Z}_n) - f(\mathbf{Z}_2,\dots,\mathbf{Z}_{n+1}) + \sum_{i=1}^{n}(-1)^{i+1} f(\mathbf{Z}_1,\dots,\mathbf{Z}_i + \mathbf{Z}_{i+1},\dots,\mathbf{Z}_{n+1}) = 0. \tag{5}$$

Note that it is sufficient to prove the theorem if $\dim \mathcal{V}' = 1$ and the general case is just a consequence. So let us assume that $\dim \mathcal{V}' = 1$. Note also that f given by (4) is a solution of (3), but we want to prove that each solution is included in (4).

Let $\dim \mathcal{V} = r$ and let $\mathbf{Z}_i = (z_{i1},\cdots,z_{ir})^T$ $(1 \leq i \leq n+1)$. By differentiating the equation (5) partially with respect to $z_{n+1,\nu}$ $(1 \leq \nu \leq r)$ at $\mathbf{Z}_{n+1} = 0$, we obtain the following system of r equations

$$\frac{\partial}{\partial z_{n\nu}} f(\mathbf{Z}_1,\dots,\mathbf{Z}_n) = -p_\nu(\mathbf{Z}_2,\dots,\mathbf{Z}_n) + \sum_{i=1}^{n-1}(-1)^{i+1} p_\nu(\mathbf{Z}_1,\dots,\mathbf{Z}_i + \mathbf{Z}_{i+1},\dots,\mathbf{Z}_n),$$

$(1 \leq \nu \leq r)$, where

$$\frac{\partial}{\partial t_\nu} f(\mathbf{Z}_1,\dots,\mathbf{Z}_{n-1},\mathbf{Z})\Big|_{\mathbf{Z}=0} = (-1)^n p_\nu(\mathbf{Z}_1,\dots,\mathbf{Z}_{n-1}) \text{ for } \mathbf{Z} = (t_1,\dots,t_r)^T.$$

After integration of this system we obtain

$$f(\mathbf{Z}_1,\dots,\mathbf{Z}_n) = R(\mathbf{Z}_1,\dots,\mathbf{Z}_{n-1}) - P(\mathbf{Z}_2,\dots,\mathbf{Z}_n) + \sum_{i=1}^{n-1}(-1)^{i+1} P(\mathbf{Z}_1,\dots,\mathbf{Z}_i + \mathbf{Z}_{i+1},\dots,\mathbf{Z}_n), \tag{6}$$

where

$$\frac{\partial}{\partial z_{n-1,\nu}} P(\mathbf{Z}_1,\dots,\mathbf{Z}_{n-1}) = p_\nu(\mathbf{Z}_1,\dots,\mathbf{Z}_{n-1}) \qquad (1 \leq \nu \leq r),$$

and R is an arbitrary analytic function with respect to $\mathbf{Z}_1, \ldots, \mathbf{Z}_{n-1}$. We write

$$R(\mathbf{Z}_1, \ldots, \mathbf{Z}_{n-1}) = (-1)^{n-1} P(\mathbf{Z}_1, \ldots, \mathbf{Z}_{n-1}) + Q(\mathbf{Z}_1, \ldots, \mathbf{Z}_{n-1}),$$

so that equality (6) becomes

$$f(\mathbf{Z}_1, \ldots, \mathbf{Z}_n) = (\Psi P)(\mathbf{Z}_1, \ldots, \mathbf{Z}_n) + Q(\mathbf{Z}_1, \ldots, \mathbf{Z}_{n-1}), \tag{7}$$

with Q analytic in $\mathbf{Z}_1, \ldots, \mathbf{Z}_{n-1}$.

If $f(\mathbf{Z}_1, \ldots, \mathbf{Z}_n)$ is a solution of (3), then

$$(\Psi Q)(\mathbf{Z}_1, \ldots, \mathbf{Z}_n) = 0,$$

because $(\Psi \circ \Psi)P = 0$. Thus Q satisfies an equation of the form (3) with n replaced by $n-1$. If $n = 2$, then $Q(\mathbf{Z}) = A\mathbf{Z}$. Otherwise we may assume that Q is given by an equality of the form (7) (n replaced by $n-1$) and complete the proof by induction. $\square$

In other words, the general analytic solution of the functional equation (5) is given by

$$\begin{aligned} f(\mathbf{Z}_1, \ldots, \mathbf{Z}_n) = {} & (-1)^{n-1} F(\mathbf{Z}_1, \ldots, \mathbf{Z}_{n-1}) - F(\mathbf{Z}_2, \ldots, \mathbf{Z}_n) \\ & + \sum_{i=1}^{n-1} (-1)^{i+1} F(\mathbf{Z}_1, \ldots, \mathbf{Z}_i + \mathbf{Z}_{i+1}, \ldots, \mathbf{Z}_n) \\ & + L(\mathbf{Z}_1, \ldots, \mathbf{Z}_n), \end{aligned} \tag{8}$$

where F is an arbitrary analytic function and L is a linear mapping.

Remark 1.5. The equality $\Psi \circ \Psi = 0$ permits the construction of a cohomology theory, which we intend to develop in a subsequent paper. Theorem 1.4 plays a role analogous to the Poincaré Lemma for differential forms.

2. Some Particular Cases

As particular cases of operator equation (3), we consider the following functional equations given in [5, 8, pp. 230–231].

1^o. If $n = 2$, then the functional equation (5) becomes

$$f(\mathbf{Z}_1, \mathbf{Z}_2) - f(\mathbf{Z}_2, \mathbf{Z}_3) + f(\mathbf{Z}_1 + \mathbf{Z}_2, \mathbf{Z}_3) - f(\mathbf{Z}_1, \mathbf{Z}_2 + \mathbf{Z}_3) = 0.$$

According to (8), the general analytic solution of this functional equation is given by

$$f(\mathbf{Z}_1, \mathbf{Z}_2) = F(\mathbf{Z}_1 + \mathbf{Z}_2) - F(\mathbf{Z}_1) - F(\mathbf{Z}_2) + L(\mathbf{Z}_1, \mathbf{Z}_2).$$

2^o. If $n = 3$, the functional equation (5) is

$$\begin{aligned} & - f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) - f(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4) + f(\mathbf{Z}_1 + \mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4) \\ & \qquad - f(\mathbf{Z}_1, \mathbf{Z}_2 + \mathbf{Z}_3, \mathbf{Z}_4) + f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3 + \mathbf{Z}_4) = 0. \end{aligned}$$

The general analytic solution of this equation is given by

$$\begin{aligned} f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) = {} & F(\mathbf{Z}_1 + \mathbf{Z}_2, \mathbf{Z}_3) + F(\mathbf{Z}_1, \mathbf{Z}_2) - F(\mathbf{Z}_1, \mathbf{Z}_2 + \mathbf{Z}_3) \\ & - F(\mathbf{Z}_2, \mathbf{Z}_3) + L(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3). \end{aligned}$$

3^o. If $n = 4$, the functional equation (5) takes on the form

$$f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4) - f(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_5) + f(\mathbf{Z}_1 + \mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_5) - f(\mathbf{Z}_1, \mathbf{Z}_2 + \mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_5) + f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3 + \mathbf{Z}_4, \mathbf{Z}_5) - f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4 + \mathbf{Z}_5) = 0.$$

According to (8), the general analytic solution of this functional equation is given by

$$f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4) = F(\mathbf{Z}_1 + \mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4) + F(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3 + \mathbf{Z}_4) - F(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4) - F(\mathbf{Z}_1, \mathbf{Z}_2 + \mathbf{Z}_3, \mathbf{Z}_4) - F(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) + L(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4).$$

In the above examples F is an arbitrary analytic function, and L is an arbitrary linear mapping.

This method for solving functional equations does not appear in the other references [1, 3, 4, 9]. In [5, 8] the solutions of the above functional equations are obtained in a very complicated way. In the literature there is no generalization about the respective functional equations with general n. Moreover, we consider functional equations in a vector form.

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Bratfordska 2/3–9, 91000 Skopje, MACEDONIA

Institute of Mathematics, St. Cyril and Methodius University, P. O. Box 162, 91000 Skopje, MACEDONIA
kostatre@iunona.pmf.ukim.edu.mk

Institute of Mathematics, Bulgarian Academy of Sciences, 8 Acad. G. Bonchev Str., 1113 Sofia, BULGARIA
matph@math.bas.bg

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