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Multigraded Modules

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Abstract. Let $R = k[x_1, \dots, x_n]$ be a polynomial ring over a field k. We present a characterization of multigraded R-modules in terms of the minors of their presentation matrix. We describe explicitly the second syzygies of any multigraded R-module.

Contents

1. Introduction

In what follows k is a field and $R = k[x_1, \dots, x_n]$ is the polynomial ring in n variables over k. Traditionally, monomial ideals of R have been the focus of intense study: Apart from providing a wide basis for examples, their theory is accessible and beautiful, they reflect various extremal properties of general ideals, and they form the link to combinatorial commutative algebra, [Ei94].

A natural generalization of the notion of monomial ideals is the notion of multigraded modules. These are the modules that stay graded with respect to any grading of the indeterminates. Such modules were considered in [Sa90], [Ch91], [BrHe95], [Mi99], [Rö99], [Ya99], [Sb00]. In the first section of this note we present a necessary and sufficient condition for a module to be multigraded in terms of the minors of its presentation matrix.

In the second section of this note we disscuss the multigraded generators of the second syzygy of a multigraded module and discuss a generalization of the well known Taylor resolution for multigraded modules.

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2. Deciding when a module is multigraded

First we note that $R = \oplus R_i$ where $i = (i_1, \dots, i_d)$, and $R_i = kx_1^{i_1}x_2^{i_2}\dots x_d^{i_d}$. Moreover $R_iR_j \subset R_{i+j}$. An R-module M is *multigraded* if there is a collection of additive subgroups $M_j = M_{j_1,j_2,\dots,j_d}$ with the properties that $M \cong \oplus M_j$ and $R_iM_j \subset M_{i+j}$. A nonzero element of M_j has *multidegree* j while the multidegree of the zero element is undefined. If $F = Re_1 \oplus \cdots \oplus Re_n$ and e_j has multidegree r_j then F is multigraded and $F_i = R_{k_1}e_1 \oplus \cdots \oplus R_{k_n}e_n$ where $k_j + r_j = i$. A homomorphism, $\varphi : M \longrightarrow N$, between two multigraded modules is a multigraded homomorphism if there exists a multidegree $r = (r_1, r_2, \dots r_d)$, such that $\varphi(M_j) \subset N_{r+j}$. Any monomial ideal of R is multigraded as are quotients of multigraded modules and the syzygies in multigraded resolutions. Whenever $\varphi : M \longrightarrow N$ is a multigraded homomorphism, by adjusting the degrees of the generators of M and N if necessary, we can assume that φ has multidegree 0. If $F_1 = \oplus_{i=1}^n R\epsilon_i$, $F_2 = \oplus_{j=1}^m Re_j$, ϵ_i, e_j have multidegrees s_i and r_j respectively, and $\varphi(\epsilon_j) = \sum_t a_{tj}e_t$ is a multigraded homomorphism of degree 0, then $a_{tj} = \lambda_{tj}x^{\alpha_{tj}}$ where $\alpha_{tj} = s_j - r_t$. Conversely the matrix $(\lambda_{tj}x^{\alpha_{tj}})$ is multigraded if there exists s_j and r_t such that

$$\mathbf{s_j} = \boldsymbol{\alpha_{tj}} + \mathbf{r_t}$$

for $\{i = 1, \dots, m\}$, $\{j = 1, \dots, n\}$, and $\lambda_{tj} \neq 0$, [Ch90].

Example 1. A is multigraded while B is not:

$$A = \begin{pmatrix} xy^2 & xyz & 0 \\ yz^2 & 0 & x^2z \\ 0 & yz^2 & x^2y \end{pmatrix}, \qquad B = \begin{pmatrix} x & y & 0 \\ z & 0 & x \\ 0 & z & y \end{pmatrix}.$$

In Theorem 2 we give a criterion for the matrix A to be multigraded that involves computing all size minors of the matrix A. One could state the same criterion in terms of the sums of the multidegrees along all permutations possible of any collection of rows and columns of A. Below we set our notation. If $A = (a_{ij})$ is a monomial matrix and A' is the $k \times k$ submatrix of A formed by the rows $i_1, \dots, i_k$ and columns $j_1, \dots, j_k$, the determinant of A' is the sum: $det(A') = \sum_{\sigma \in S_k} \text{sign}(\sigma) a_{i_1\sigma(j_1)}a_{i_2\sigma(j_2)}\dots a_{i_n\sigma(j_k)}$ and the multidegree of the nonzero σ-term is the sum $\alpha_{i_1\sigma(j_1)} + \alpha_{i_2\sigma(j_2)} \dots + \alpha_{i_n\sigma(j_k)}$. The following theorem shows that the consistency of the system of equations is equivalent to the condition that the nonzero terms of $\det(A')$ have the same multidegree, for each square submatrix A'.

Theorem 2. *Let $A = (a_{ij})$ be an $m \times n$ monomial matrix. A is multigraded if and only if the nonzero terms of the determinant of any square submatrix of A have the same multidegree.*

Proof. First we note that the necessity of this criterion follows at once since there is a functor from modules to exterior powers which respects grading. For a direct proof, one can assume that A is a $k \times k$ multigraded matrix and σ and τ are two elements of S_k that correspond to nonzero terms of $\det A$. The corresponding multidegrees are $\sum \alpha_{i,\sigma(i)}$ and $\sum \alpha_{i,\tau(i)}$. Since A is multigraded,

$$\sum \alpha_{i,\sigma(i)} + \sum r_i = \sum s_{\sigma(i)} = \sum s_{\tau(i)} = \sum \alpha_{i,\tau(i)} + \sum r_i,$$

and $\sum \alpha_{i,\sigma(i)} = \sum \alpha_{i,\tau(i)}$.

Next we prove the sufficiency of our criterion. Note that A is multigraded if and only if for each sequence $\{(\alpha_{i_1,j_1},\alpha_{i_1,j_2}),\dots,(\alpha_{i_{k-1},j_{k-1}},\alpha_{i_{k-1},j_k}),\,(\alpha_{i_k,j_k})\}$, then

$$\begin{aligned}s_{j_1} &= \alpha_{i_1,j_1} - r_{i_1} = \alpha_{i_1,j_1} - \alpha_{i_1,j_2} + s_{j_2} = \cdots = \\ &= \alpha_{i_1,j_1} - \alpha_{i_1,j_2} + \alpha_{i_2,j_2} - \cdots + \alpha_{i_k,j_k} - r_{i_k} \\ &= (\alpha_{i_1,j_1} + \alpha_{i_2,j_2} + \cdots + \alpha_{i_k,j_k}) - (\alpha_{i_1,j_2} + \alpha_{i_2,j_3} + \cdots + \alpha_{i_{k-1},j_k}) - r_{i_k}.\end{aligned}$$

If A is not multigraded then for some $\{j_1, i_k\}$ there are two sets of indices

$$\{(i_1,j_1),(i_2,j_2),\dots,(i_k,j_k)\} \quad \text{and} \quad \{(i'_1,j_1),(i'_2,j'_2),\dots,(i'_l,j'_l)\},$$

where $i'_l = i_k$ and

$$\begin{aligned}&(\alpha_{i_1,j_1} + \alpha_{i_2,j_2} + \cdots + \alpha_{i_k,j_k}) - (\alpha_{i_1,j_2} + \alpha_{i_2,j_3} + \cdots + \alpha_{i_{k-1},j_k}) - r_{i_k} \\ &\qquad \neq (\alpha_{i'_1,j_1} + \alpha_{i'_2,j'_2} + \cdots + \alpha_{i_k,j'_l}) - (\alpha_{i'_1,j'_2} + \alpha_{i'_2,j'_3} + \cdots + \alpha_{i'_{l-1},j'_l}) - r'_{i_l}.\end{aligned}$$

Since $r_{i_k} = r'_{i_l}$, this implies that

$$\begin{aligned}&(\alpha_{i_1,j_1} + \alpha_{i_2,j_2} + \cdots + \alpha_{i_k,j_k}) + (\alpha_{i'_1,j'_2} + \alpha_{i'_2,j'_3} + \cdots + \alpha_{i'_{l-1},j'_l}) \\ &\qquad \neq (\alpha_{i_1,j_2} + \alpha_{i_2,j_3} + \cdots + \alpha_{i_{k-1},j_k}) + (\alpha_{i'_1,j_1} + \alpha_{i'_2,j'_2} + \cdots + \alpha_{i_k,j'_l}).\end{aligned}$$

Next we consider the submatrix A' of A, formed by taking the rows $i_1,\dots,$ i_k,$i'_1,\dots,$ i'_{l-1} and the columns $j_1,\dots,$ j_k, $j'_2,\dots,$ j'_l. The two sums above represent two nonzero terms of its determinant with different multidegree. □

It is clear that if the criterion of Theorem 2 is satisfied then all minors of A are generated by monomials. In [De99] we show that Theorem 2 is actually equivalent to the condition that all minors of A are generated by monomials.

If A satisfies Theorem 2, one can apply the following procedure to obtain multidegrees r_i and s_j by moving along the columns and rows of A, [De99]:

1. Set $r_1 := 0$.
2. If r_i but not s_j has an assigned value, and $a_{ij} \neq 0$, then $s_j := \alpha_{ij} + r_i$.
3. If s_j but not r_i has an asssigned value, and $a_{ij} \neq 0$, then $r_i := s_j - \alpha_{ij}$.
4. If r_i does not have an assigned value, and for every j such that $a_{ij} \neq 0$ s_j also does not have a value, then $r_i := 0$.

By adding appropriate positive vectors, the multidegrees can be adjusted so that the components of r_i, s_j are greater than or equal to zero for all i and j.

Example 3. Let A be the matrix of Example 1. The multidegrees of its rows and columns are: $r_1 = (0,0,0)$, $s_1 = (1,2,0)$, $s_2 = (1,1,1)$, $r_2 = (1,1,-2)$, $s_3 = (3,1,-1)$, and $r_3 = (1,-1,-1)$. By adding $(0,1,2)$ to the above we get a set of positive multidegrees for the rows and columns of A.

Remark 4. The columns of a multigraded matrix may not be a Gröbner basis (with respect to the usual monomial gradings), for the space they generate, as is the case with the matrix A.

Remark 5. Any multigraded module M can be lifted to a multigraded squarefree module $\mathcal{M}$ over a polynomial ring $\mathcal{R}$ so that if $\mathcal{F}$ is a projective resolution of $\mathcal{M}$ over $\mathcal{R}$ then $\mathcal{F} \otimes R$ is a projective resolution of M over R, [BrHe95]. By squarefree we mean that the multigraded presentation matrix of $\mathcal{M}$ consists of squarefree monomials. With the notation as above, suppose that the square of some variable

y divides an entry of M. To find $\mathcal{M}$ with respect to y, we concentrate to the columns with the highest y-degree, and if a_{ij} is divisible by y, we replace it by $y_k \dfrac{a_{ij}}{y}$. We repeat this procedure (and increase the index of y_k), until we reach $\mathcal{M}$. For example the z-squarefree multigraded matrix that corresponds to the matrix A of Example 1 is:

$$A' = \begin{pmatrix} xy^2 & xyz_2 & 0 \\ yzz_1 & 0 & x^2z \\ 0 & yz_1z_2 & x^2y \end{pmatrix}$$

3. Resolving a $m \times n$ multigraded matrix

In this section we describe the second syzygies of $M = \operatorname{coker}\phi$, where ϕ corresponds to the multigraded $m \times n$ matrix $A = (a_{ij})$: $R^n \xrightarrow{\phi} R^m \longrightarrow M \longrightarrow 0$. We note that in [De99], we describe the explicit minimal resolution for all modules when $n = 4$, $m = 2$. Let $\epsilon_1, \dots, \epsilon_n$ be the generators of R^n. Moreover suppose that $I_k(A) \neq 0$ while $I_{k+1}(A) = 0$. We will describe the second syzygies of M as elements of the free module R^t with generators e_J^I, where $t = \binom{n}{k+1}\binom{m}{k}$, and J and I are ordered sets of length $k+1$ and k respectively.

We let I be the ordered set $\{i_1, \dots, i_k\}$, J be the ordered set $\{j_1, \dots, j_{k+1}\}$, J_l be the ordered set $(J, \hat{j_l})$, $M_{J_l}^I$ be the determinant of the $k \times k$ submatrix of A formed by considering the rows indexed by I and the columns indexed by J_l, and we let g_J^I be the monic monomial which is the greatest common divisor of $M_{J_l}^I$.

We define the homomorphism $\varphi_2 : R^t \longrightarrow R^n$, by

$$\varphi_2(e_J^I) = \sum (-1)^{l+1} \frac{M_{J_l}^I}{g_J^I} \epsilon_{j_l}.$$

We note that φ_2 is multigraded and that the y-degree of e_J^I equals the maximum of y-degrees of ϵ_{j_l}. Let B be the matrix of φ_2. Next we show that the image of φ_2 generates the second syzygies of M.

Theorem 6. *Let A be a multigraded $m \times n$ matrix corresponding to the homomorphism φ. The following sequence is exact:*

$$R^t \xrightarrow{\varphi_2} R^n \xrightarrow{\varphi} R^m \longrightarrow M \longrightarrow 0.$$

Proof. The entries (AB_{ij}) of the product AB are equal to zero: If j is the column of B that corresponds to the image of e_J^I then AB_{ij} is the determinant of a $(k+1) \times (k+1)$ matrix with two equal rows when $i \in I$, while when $i \notin I$ then $AB_{ij} \in I_{k+1}(A)$ and is zero by the hypothesis on the minors of A. It follows that $\varphi(\varphi_2) = 0$. To prove the theorem, it is enough to show that $\ker(\varphi) \subset \operatorname{im}(\varphi_2)$. Let A be any multigraded $m \times n$ matrix and let $\mathcal{A}$ be the corresponding squarefree matrix, $\mathcal{M}_J^I$, γ_J^I be the corresponding determinants and greatest common divisors of $\mathcal{A}$. Since $\mathcal{M}_J^I \otimes R = M_I^J$, $\gamma_J^I \otimes R = g_J^I$ it is enough to show that the theorem holds for squarefree multigraded matrices.

Let A be a multigraded squarefree $m \times n$ matrix. We will use induction on the sum of the total degrees of the monomial entries of A.

Suppose that this degree is 0. Without loss of generality we can reorder the rows and columns of A so that $M_{1,\dots,k}^{1,\dots,k} \neq 0$. We let I denote the set $\{1, \dots, k\}$. We consider the $k \times k$ submatrix A' of A which consists of the first k rows and k

columns of A and we let C be the $m \times m$ invertible matrix which is the direct sum of $\mathrm{adj}(A')$ and the $(m-k)\times(m-k)$ identity matrix. Since C is invertible the nullspaces of A and CA are the same. The product CA is the matrix

$$\left(\begin{array}{cccc|ccc} M_I^I & 0 & \dots & 0 & -M^I_{I,\hat{1},k+1} & \dots & -M^I_{I,\hat{1},n} \\ 0 & M_I^I & \dots & 0 & M^I_{I,\hat{2},k+1} & \dots & M^I_{I,\hat{2},n} \\ \vdots & & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & M_I^I & (-1)^k M^I_{I,\hat{k},k+1} & \dots & (-1)^k M^I_{I,\hat{k},n} \\ \hline a_{k+1,1} & \dots & & a_{k+1,k} & a_{k+1,k+1} & \dots & a_{k+1,n} \\ \vdots & \ddots & & \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & \dots & & a_{m,k} & a_{m,k+1} & \dots & a_{m,n} \end{array}\right).$$

The row reduced form of A (and CA) consists of a $k\times k$ identity block. Therefore it is equal to

$$\frac{1}{M_I^I}\left(\begin{array}{cccc|ccc} M_I^I & 0 & \dots & 0 & -M^I_{I,\hat{1},k+1} & \dots & -M^I_{I,\hat{1},n} \\ 0 & M_I^I & \dots & 0 & M^I_{I,\hat{2},k+1} & \dots & M^I_{I,\hat{2},n} \\ \vdots & & \ddots & & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & M_I^I & (-1)^k M^I_{I,\hat{k},k+1} & \dots & (-1)^k M^I_{I,\hat{k},n} \\ \hline 0 & \dots & & 0 & 0 & \dots & 0 \\ \vdots & \ddots & & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & & 0 & 0 & \dots & 0 \end{array}\right).$$

Therefore the kernel of φ is generated by elements of the form $\sum(-1)^{l+1}M^I_{I,\hat{l},s}\epsilon_l$, a subset of our proposed set of syzygies.

Suppose now that the sum of the total degrees of the monomial entries of A is greater than or equal to 1. Let t be a minimal multigraded syzygy: $t = \sum t_{i_r}\epsilon_{i_r}$. The monomials t_{i_r} are squarefree and so are the terms $M^I_{J_l}/g^I_J$. We assign multidegrees to the columns of A. Moreover for any variable y, the y-degree of $M^I_{J_l}/g^I_J$ is 1 if and only if the y-degree of ϵ_{i_s} is 1 for $i_s \in J_l$ while the y-degree of e_l is 0. Let x be a variable that divides a nonzero entry of A, and let S be the multiplicative set $S = \{1, x, x^2, \dots\}$. We consider $S^{-1}R$, and $S^{-1}M$, and $\varphi' = S^{-1}\varphi$. The matrix A' of φ' consists of the entries of A evaluated at $x = 1$: $A' = (a'_{ij})$ where $a'_{ij} = a_{ij}|_{x=1}$, and is multigraded squarefree. Next we consider $t' = \sum t'_{i_r}\epsilon'_{i_r}$ where $t'_j = t_j|_{x_l=1}$ and ϵ'_j are the generators of $S^{-1}R^n$. Since t' is a second syzygy of $\mathrm{coker}(\varphi')$, and the induction hypotheses are satisfied, $t' = \sum b_{J,I}\left(\sum(-1)^{l+1}\frac{M^I_{J_l}}{g^I_J}\epsilon'_{j_l}\right)$. If the x-degree of t is 0, then it is easy to see that $t = \sum b'_{J,I}\left(\sum(-1)^{l+1}\frac{M^I_{J_l}}{g^I_J}\epsilon_{j_l}\right)$. If the x-degree of t is 1, we let $b_{J,I} = xb'_{J,I}$ when the x-degree of ϵ_{j_l} is 0 for all $j_l \in J$ and otherwise we let $b_{J,I} = b'_{J,I}$. The element $g = \sum b_{J,I}\left(\sum(-1)^{l+1}\frac{M^I_{J_l}}{g^I_J}\epsilon_{j_l}\right)$ has the same multidegree as t and $S^{-1}g = t'$, therefore $g = t$. □

In particular, when $m = 2$ and $I_2(A) \neq 0$, the syzygies of M are of the form $\left\{ \frac{M_{jk}}{g_{ijk}}\epsilon_i - \frac{M_{ik}}{g_{ijk}}\epsilon_j + \frac{M_{ij}}{g_{ijk}}\epsilon_k \right\}$ where $M_{ij} = M_{ij}^{12}$ and $g_{ijk} = g_{ijk}^{12}$.

We note that Theorem 6 agrees with the first step in the Buchsbaum-Rim complex whenever the latter one is exact. We also note that Theorem 6 in general does not provide a minimal set of second syzygies. For example suppose that $I_2(A) = 0$, and let $(T_\bullet, \theta_\bullet)$ be the Taylor resolution on the entries of any row, r, [Ei94], $a_{r1}, a_{r2}, \ldots, a_{rn}$, and let $(F_\bullet, d_\bullet)$ be the complex where $F_0 = R^2$, $F_1 = R^n$, the map d_1 is given by the matrix A, and $(F_i, d_i) = (T_i, \theta_i)$ for all $i \geq 2$. In [De99] we remarked the following:

Theorem 7. *Let M be a multigraded module with $I_2(A) = 0$. Then $(F_\bullet, d_\bullet)$ is a free resolution of M.*

Proof. Since $I_2(A) = 0$, $d_2 d_1 = 0$ and $F_\bullet$ is a complex and exact for $i \geq 1$. Moreover the grade of $I_1(A) \geq 1$ and by the Buchsbaum-Eisenbud criterion $F_\bullet$ is exact. □

In this case, the matrix of Theorem 6 is equal to m copies the matrix of d_2.

We also remark that the first part of the proof of Theorem 6 shows that if $M_J^I \neq 0$, while $M_{J,j}^{I'} = 0$ for all possible I', then there is a syzygy involving the generators indexed by J and j. It can be shown that this syzygy can be expressed in terms of the syzygies corresponding to the maximal nonzero minors.

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