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Subspaces of L_p for $0 \le p < 1$ that are admissible as kernels

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Abstract. In L_p for $0 \le p < 1$, we classify a large collection of subspaces as admissible kernels, meaning that each subspace is the kernel of some continuous linear automorphism on L_p for $0 \le p < 1$. We then show that this result eliminates those subspaces as potential rigid subspaces.

Contents

1. Introduction

The aim of this paper is to classify a large collection of subspaces of L_p, $0 \le p < 1$. Because we focus on those p-Banach spaces, unless otherwise specified, a space L_p will be a p-Banach space $L_p([0,1])$ with $0 \le p < 1$. If X is a subspace of L_p, then we will call X an *admissible kernel* if there is a continuous linear automorphism $T \in \mathcal{L}(L_p)$ such that $X = \ker T$. Even though a large collection of well-behaved subspaces will be shown to be admissible kernels, we will also demonstrate that there are nice subspaces that are not admissible kernels. The first step will be to establish an important property of admissible kernels, and then this property will be used to classify a collection of admissible kernels.

On a side note, it will be shown that looking at admissible kernels will also help in the search for a classical example of a rigid space, that is, a space whose only continuous linear automorphisms are constant multiples of the identity operator. In 1977, J. Roberts, following a construction of L. Waelbroeck [Wae77], constructed a closed, infinite-dimensional, linear subspace of $L_0 = L_0[0,1]$ that was rigid (the original construction went unpublished but an enhanced version embedded in L_p was published in [KR81]). Although the rigid subspace was a subspace of the space

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of measurable functions (or alternately a subspace of any L_p), the subspace itself was not classical in nature.

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2. A property and the classification

Obviously, the trivial subspace is an admissible kernel. It also turns out that most simple subspaces are. For example, consider a one-dimensional subspace $X = \langle f \rangle$ of L_p. Since L_p is transitive, there is an operator $Q \in \mathcal{L}(L_p)$ so that $Qf = 1$. One can actually say that $Qg = 1$ if and only if $g = f$ a.e. (see for example [KPR84] p. 126). That means that the image under Q of X is the constant functions. Now consider the operators $S : L_p([0,1]) \to L_p([0,1]^2)$ defined by $Sf(x,y) = f(x) - f(y)$ and $R : L_p([0,1]^2) \to L_p([0,1])$ defined by $Rf(x) = f(0,x)$. We can see that the kernel of the composition RS is the set of constant functions. Then the kernel of the composition $T = RSQ$ will be exactly X, and hence any one-dimensional subspace of L_p will be an admissible kernel.

Now that we know of some admissible kernels, it is helpful to be able to combine them to make new ones.

Theorem 1. *Let $\{X_n\}_{n=1}^{\infty}$ be a collection of admissible kernels in L_p. Then $\cap_{n=1}^{\infty} X_n$ is an admissible kernel.*

Proof. For each n, X_n is an admissible kernel so let T_n be the continuous linear operator whose kernel is X_n. Now construct a sequence of nonsingular, measurable maps, $\{\sigma_n\}_{i=1}^{\infty}$ where $\sigma_n(x) = 2^n x - 1$. Each of these is a linear, order-preserving dilation of $\left[\frac{1}{2^n}, \frac{1}{2^{n-1}}\right)$ onto $[0,1)$.

Next define the operator T by

$$Th = \sum_{n=1}^{\infty} K_n C_{\sigma_n} T_n h$$

where $K_n = \chi_{\left[\frac{1}{2^n}, \frac{1}{2^{n-1}}\right)}$ and $C_{\sigma_n} h = h \circ \sigma_n$. The series will converge since the domains of the terms are pairwise disjoint. Further, $Th = 0$ if and only if Th is zero on each of the intervals $\left[\frac{1}{2^n}, \frac{1}{2^{n-1}}\right)$, which happens if and only if $T_n h = 0$ for each n, so $\ker T = \cap X_n$. □

Our aim is to show that the closed linear span of a collection of linearly independent random variables will be an admissible kernel. We will therefore start with a result that will help deal with a basis.

Theorem 2. *Let X be a subspace of L_p with a basis $\{f_n\}$, and let S_n be the partial sum operators for the basis. If each S_n can be extended to an operator on L_p in such a way that $S_n g \to g$ for all $g \in L_p$, then X is an admissible kernel.*

Proof. For each n, let $X_n = (S_n - S_{n-1})^{-1}(\langle f_n \rangle)$. Notice that although X_n is not necessarily one-dimensional, each one is the inverse image of a one-dimensional subspace and hence can be shown to be an admissible kernel by using a composition of operators. It therefore follows from Theorem 1 that $\cap X_n$ is also an admissible kernel.

Let $f \in X$, then $f = \sum c_n f_n$ for some collection $\{c_n\}$. From that, we know that $(S_n - S_{n-1}) f = c_n f_n$ and hence $f \in (S_n - S_{n-1})^{-1} (\langle f_n \rangle) = X_n$. This is true for all n, so $f \in \cap X_n$.

Conversely, suppose $f \in X_n$ for all n. For simplicity, assume that $S_0 g = 0$ for all $g \in L_p$. With that, $(S_n - S_{n-1}) f = \alpha_n f_n$ for some collection $\{\alpha_n\}$. From this we see that $T_n f = \sum_{i=1}^{n} (S_n - S_{n-1}) f = S_n f - S_0 f = S_n f$ converges to f. At the same time, $T_n f = \sum_{i=1}^{n} \alpha_i f_i$ which means that $T_n f$ converges to $\sum_{i=1}^{\infty} \alpha_i f_i = f$, and $f \in X$.

Thus, $X = \cap X_n$ and by Theorem 1 must be an admissible kernel. □

The last step is to classify the collection of subspaces that are spanned by a collection of independent symmetric random variables with an added property as admissible kernels.

Theorem 3. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of independent symmetric random variables which generate the full σ-algebra of Borel sets. Then $X = \langle f_n \rangle_{n=1}^{\infty}$ is an admissible kernel.*

Proof. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence defined with the above properties and let λ be the chosen Borel probability measure on $\mathbb{R}$. We can assume that each random variable is symmetric about 0. For each n, define $\mu_n(B) = \lambda\left(f_n^{-1}(B)\right)$ for all Borel sets B. Then each μ_n is a Borel probability measure. Since the collection $\{f_n\}$ generates the Borel sets and each μ_n is a probability measure, the space $L_p\left(\Pi_n\left(\mathbb{R}, \mu_n\right)\right)$ is isomorphic to $L_p\left(\mathbb{R}, \lambda\right)$ and hence can be used in its place.

With that in mind, the original random variables f_n can be realized by the simpler form $f_n\left(x_1, x_2, \dots\right) = x_n$. Define a sequence of maps $\sigma_n : \Pi_n \mathbb{R} \to \Pi_n \mathbb{R}$ by

$$\sigma_n(x_1, x_2, \dots) = (x_1, x_2, \dots, x_n, -x_{n+1}, -x_{n+2}, \dots).$$

Each σ_n is measure preserving and for any $g \in L_0$, $g \circ \sigma_n$ converges to g.

Define a sequence of operators S_n by $S_n g = \frac{1}{2}\left(g + g \circ \sigma_n\right)$. Since every $f \in X$ is symmetric, $S_n f = \displaystyle\sum_{i=1}^{n} \alpha_i f_i$, where the α_i are the coefficients of the f_i for f. That is, the S_n act as partial sums for the elements of X. On the other hand, since $g \circ \sigma_n \to g$ for any $g \in L_0$, we also have $S_n g$ converges to g for all $g \in L_0$. Therefore by Theorem 2, X is an admissible kernel. □

So, for example, the closed linear span of the Radamacher functions is an admissible kernel, but, as we will see later, H_p, $0 \le p < 1$ is not.

3. Consequences

A subspace X is called *strictly transitive* if given any sequence $x_1, x_2, \dots, x_n \in X$ which is linearly independent and any sequence $y_1, y_2, \dots, y_n \in X$, there is a continuous linear automorphism T which maps each x_i to the corresponding y_i. A subspace that is strictly transitive can not be rigid. The following theorem says that subspaces that are admissible kernels are strictly transitive and hence can not possibly be rigid.

Theorem 4. *Let X be a subspace of L_p. If X is an admissible kernel, then L_p/X is strictly transitive.*

Proof. Let $T \in \mathcal{L}(L_p)$ be the operator with $\ker T = X$. It will suffice to show that if $f_1, f_2, \ldots, f_n$ are independent with respect to X (i.e., if $\sum a_i f_i \in X$, then $a_i = 0$ for all i), then $Tf_i \neq 0$ for all i and $\{Tf_i\}$ are independent.

The independence of $\{f_i\}$ with respect to X implies that for each i, $f_i \notin X = \ker T$, so $Tf_i \neq 0$. Now suppose that there are $\{a_i\}_{i=1}^n$ not all zero so that $\sum a_i T f_i = 0$. Then $0 = \sum a_i T f_i = T\left(\sum a_i f_i\right)$, which implies that $\sum a_i f_i \in X$, and this is a contradiction. □

This means that the subspaces from Theorem 3, which include many classical spaces, can not be rigid. It also tells us that an unconditional basis is not necessarily sufficient to be an admissible kernel, i.e., that the requirements in Theorem 3 can not be relaxed too far and still hold. For example, consider the subspace H_p of L_p for $p < 1$. H_p has an unconditional basis [KPR84], yet since every operator $T : L_p/H_p \to L_p$ is zero, H_p can not be an admissible kernel.

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